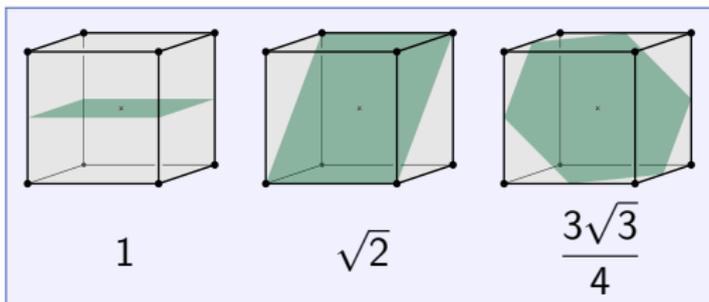
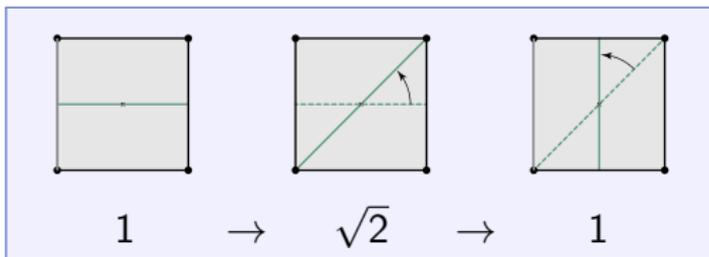


Hypercube sections and the Eulerian numbers

Lionel Pournin, LIPN, Villetaneuse



$$A(d, i) = (d - i)A(d - 1, i - 1) + (i + 1)A(d - 1, i)$$

where $1 \leq i < d$ and

$$\begin{cases} A(d, 1) = 1 \\ A(d, d - 1) = 1 \end{cases}$$

1						
1	1					
1	4	1				
1	11	11	1			
1	26	66	26	1		
1	57	302	302	57	1	
1	120	1191	2416	1191	120	1

Theorem (Hensley 1979, Ball 1986):

$$1 \leq V \leq \sqrt{2} \text{ in all dimensions.}$$

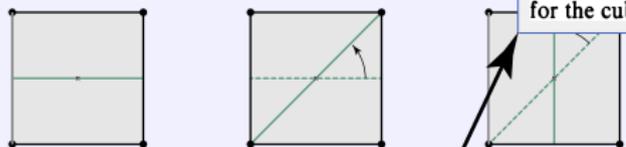
Hypercube sections and the

Lionel Poul

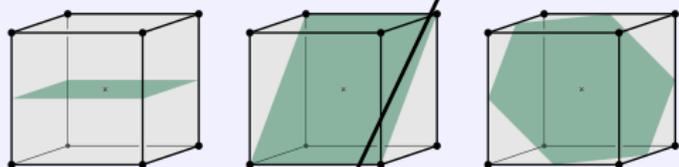
DOUGLAS HENSLEY

ABSTRACT. A cube of dimension n and side 1 is cut by a hyperplane of dimension $n - 1$ through its center. The usual $n - 1$ measure of the intersection is bounded between 1 and M , independent of n . The proof uses an inequality for sums of independent random variables.

1. Introduction. If a square or a cube of side 1 is cut by a line (resp. plane) through its center then the length (area) of the intersection is ≥ 1 . But even for the cube this is not obvious.



1 \rightarrow $\sqrt{2}$ \rightarrow 1



1 $\sqrt{2}$ $\frac{3\sqrt{3}}{4}$

$$+(i + 1)A(d - 1, i)$$

where $1 \leq i < d$ and

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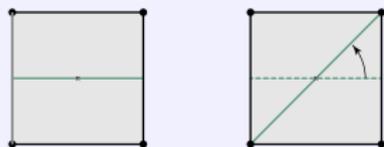
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1. Introduction. If a square or a cube of side 1 is cut by a line (resp. plane) through its center then the length (area) of the intersection is ≥ 1 . But even for the cube this is not obvious.

REMARK. In a discussion of this problem at the Institute for Advanced Study I opined that taking $v = [1, 1, \dots, 1]$ and n large gave a counterexample. Professor Selberg put the problem in its probabilistic form and used the central limit theorem to show that for my purported counterexample, $m(B \cap C) \sim (6/\pi)^{1/2}$ as $n \rightarrow \infty$. This was a surprise to both of us as neither had expected a value close to 1.

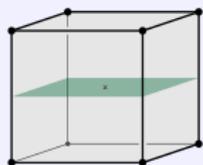


1

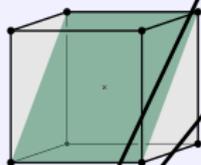
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$\sqrt{2}$

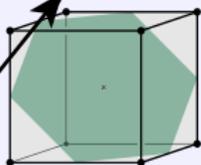
→



1



$\sqrt{2}$



$\frac{3\sqrt{3}}{4}$

$$\left\{ \begin{array}{l} A(d, d-1) = 1 \end{array} \right.$$

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

1 57 302 302 57 1

1 120 1191 2416 1191 120 1

Theorem (Hensley 1979, Ball 1986):

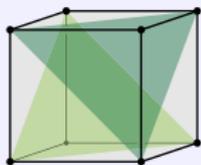
$$1 \leq v \leq \sqrt{2} \text{ in all dimensions.}$$

Summary

- I. Link between the section volumes and the **eulerian numbers**
 - II. A conjecture by Milman and König
 - III. Two expressions for \check{V}
 - III.1. A discrete sum formula for \check{V}
 - III.2. An integral formula for \check{V}
 - IV. Results
 - IV.1. Proof when t is close to the circumradius of $[0, 1]^d$
 - IV.2. Local extremality when t is close to circumradius of $[0, 1]^d$
 - IV.3. Local extremality when t is close to the center of $[0, 1]^d$
 - IV.4. Monotonicity of \check{V} as a function of d
 - IV.5. Convergence rate of the **eulerian numbers** to a normal law
 - V. Some proof sketches
 - V.1. Monotonicity
 - V.2. Local extremality
 - VI. Wrap up on the Milman–König conjecture
- Among the hyperplanes H at distance t from the center of $[0, 1]^d$, when is the volume \check{V} of $H \cap [0, 1]^d$ extremal?

Volumes and the Eulerian numbers

When H is orthogonal to a diagonal of $[0, 1]^d$ and contains a vertex, $H \cap [0, 1]^d$ is a **hypersimplex** (Gabriélov–Gel'fand–Losik 1975).



Remark (Stanley 1977): If $\Delta_{d,k}$ is the hypersimplex obtained by slicing $[0, 1]^d$ along the vertices of 1-norm k , then

$$V = \frac{\sqrt{d}}{(d-1)!} A(d-1, k-1)$$

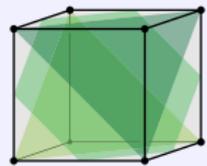
the Eulerian numbers

$$A(d-1, k-1) = \sum_{i=0}^k (-1)^i \binom{d}{i} (k-i)^{d-1}$$

Question: What is the volume of $H \cap [0, 1]^d$ when

$$H : x_1 + x_2 + \dots + x_d = k$$

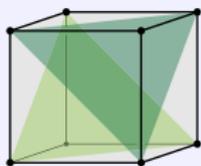
... but k is **not** an integer?



Remark (Deza–Hiriart-Urruty–P 2021): This allows to compute the volume of certain polytopes that appear in sparse optimization.

Volumes and the Eulerian numbers

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Remark (Deza–Hiriart-Urruty–P 2021): If $k \in [0, d]$, then

$$V = \frac{\sqrt{d}}{(d-1)!} \sum_{i=0}^{\lfloor k \rfloor} (-1)^i \binom{d}{i} (k-i)^{d-1}$$

(a consequence of Barrow–Smith 1979)

A conjecture by Milman and König

Question: If H is at distance t from the center of the hypercube, when is V maximal or minimal?

Conjecture (V. Milman): V is maximal when H is orthogonal to a diagonal or a lower-order sub-diagonal.

- A sub-diagonal of order k is the line segment between the centers of two opposite $(d - k)$ -dimensional faces of the hypercube.
- A diagonal is an order d sub-diagonal.

Theorem (Ball 1986): If $t = 0$, then V is maximal exactly when H is orthogonal to an order 2 sub-diagonal.

Theorem (Moody–Stone–Zach–Zvavitch 2013): If $d \geq 3$ and

$$\frac{\sqrt{d-1}}{2} < t < \frac{\sqrt{d}}{2}$$

then V is maximal exactly when H is orthogonal to a diagonal.

Theorem (König 2021): If $d \geq 5$ and

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then V is locally maximal when H is orthogonal to a diagonal.

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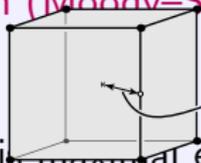
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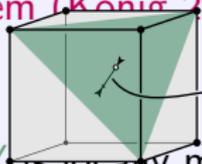
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Two expressions for the volume

Denote $\begin{cases} \sigma(x) \text{ the sum of the coordinates of a point } x \in \mathbb{R}^d \text{ and} \\ \pi(x) \text{ the product of its coordinates.} \end{cases}$

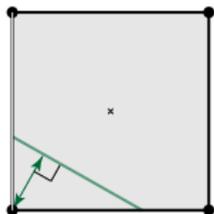
Theorem (P 2023): Denote $H = \{x \in \mathbb{R}^d : a \cdot x = b\}$ where $a \in (\mathbb{R} \setminus \{0\})^d$ and $b \in \mathbb{R}$. If $d \geq 2$, then the volume of $H \cap [0, 1]^d$ is

$$V = \|a\| \sum \frac{(-1)^{\sigma(v)} (b - a \cdot v)^{d-1}}{(d-1)! \pi(a)}$$

where the sum is over the vertices v of $[0, 1]^d$ such that $a \cdot v \leq b$.

V varies like h_0^{d-1}

(up to a multiplicative term that depends on a and d)



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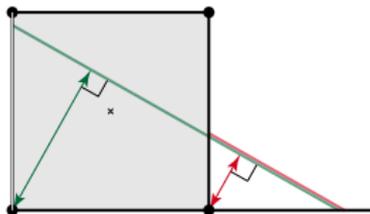
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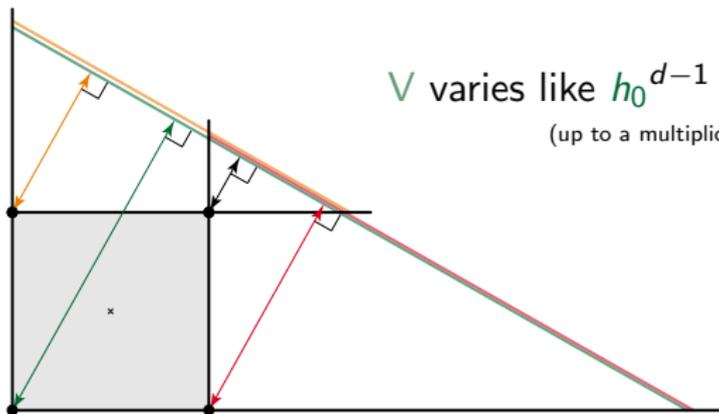
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V varies like $h_0^{d-1} - h_1^{d-1} - h_2^{d-1} + h_3^{d-1} = 0$
(up to a multiplicative term that depends on a and d)

Two expressions for the volume

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where the sum is over the vertices v of $[0, 1]^d$ such that $a \cdot v \leq b$.

However, Ball proved that $V \leq \sqrt{2}$ using a different formula...

GEORG PÓLYA in Budapest.

Ich will das Integral $\int \int \dots \int_{(R_n)} dx_1 dx_2 \dots dx_n$ berechnen, wo das n -dimensionale Gebiet R_n durch die $n+1$ Ungleichungen

$$-a_1 \leq x_1 \leq +a_1, \quad -a_2 \leq x_2 \leq +a_2, \quad \dots, \quad -a_n \leq x_n \leq +a_n,$$

$$-a_{n+1} \leq x_1 + x_2 + \dots + x_n \leq +a_{n+1}$$

begrenzt ist.

Two expressions for the volume

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However, Ball proved that $V \leq \sqrt{2}$ using a different formula...

Theorem (Pólya 1913, Ball 1986): If a has at least two non-zero coordinates, then the volume of $H \cap [0, 1]^d$ is

$$V = \frac{\|a\|}{\pi} \int_{-\infty}^{+\infty} \left(\prod_{i=1}^d \frac{\sin(a_i u)}{a_i u} \right) \cos \left(2 \left[b - \frac{\sigma(a)}{2} \right] u \right) du.$$

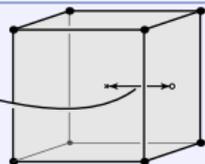

 $\cos(2tu)$

Close to the circumradius of $[0, 1]^d$

Theorem (P 2023): If $d \geq 5$ and

$$\frac{\sqrt{d-2}}{2} < t < \frac{\sqrt{d}}{2}$$

then V is maximal exactly when H is orthogonal to a diagonal.



Strategy:

- When t is fixed, b , and therefore V become functions of a .
- Regularity is obtained for V from the improper integral expression.
- Two cases: H separates one or two vertices from the other vertices.

One vertex:

$$\frac{V}{\|a\|} = \frac{b^{d-1}}{(d-1)! \pi(a)}$$

Two vertices:

$$\frac{V}{\|a\|} = \frac{b^{d-1}}{(d-1)! \pi(a)} - \frac{(b-a_1)^{d-1}}{(d-1)! \pi(a)}$$

- Use Lagrange multipliers for $V/\|a\|$ (and constrain $\|a\| = 1$),
- Identify the possible maxima by solving a non-linear system.

If $\frac{\sqrt{d-3}}{2} < t \leq \frac{\sqrt{d-2}}{2}$, then { there are five cases instead of two and the non-linear system is much harder.

Local extremality at diagonals

$$\text{Denote } p_{i,d}(z) = \frac{i(d-i)}{d-1} - \left(\frac{d}{2} - i\right) \frac{z-i}{d-2} + \frac{2d(z-i)^2}{(d-1)(d-2)}$$

$$\text{where } z = \frac{d}{2} - t\sqrt{d}.$$

Theorem (P 2024): Assume that $d \geq 4$. If

$$\sum_{i=0}^{\lfloor z \rfloor} (-1)^i \binom{d}{i} (z-i)^{d-3} p_{i,d}(z) < 0$$

then V has a strict local maximum at diagonals.

Theorem (P 2024): If $d \geq 4$ and

$$\frac{\sqrt{d}}{2} - \frac{1}{\sqrt{d}} \min \left\{ \frac{d-1}{4}, \frac{d^{1/(d-3)}}{d^{1/(d-3)} - 1} \right\} < t < \frac{\sqrt{d}}{2} \sim \frac{\sqrt{d}}{\log(d)}$$

then V has a strict local maximum at diagonals.

Local extremality at diagonals

Theorem (P 2024): If $d \geq 4$ and

$$t < \frac{7 - (17 - 12\sqrt{2})^{1/3} - (17 + 12\sqrt{2})^{1/3}}{24} \approx 0.143856$$

then V has a strict local maximum at diagonals.

Theorem (P 2024): Take any non-negative t . If d is large enough then V has a strict local maximum at diagonals when

$$t < \frac{1}{2}\sqrt{1 - \sqrt{\frac{2}{3}}} \text{ or } t > \frac{1}{2}\sqrt{1 + \sqrt{\frac{2}{3}}}$$

and a strict local minimum at diagonals when

$$0.214186 \approx \frac{1}{2}\sqrt{1 - \sqrt{\frac{2}{3}}} < t < \frac{1}{2}\sqrt{1 + \sqrt{\frac{2}{3}}} \approx 0.673887.$$

Local extremality at diagonals

Theorem (P 2024): If d is at least 136, then for every non-negative t ,

$$\left| v - \sqrt{\frac{6}{\pi}} e^{-6t^2} \left(1 + \frac{p(t)}{20d} + \frac{q(t)}{5600d^2} \right) \right| < \frac{r(t)}{3d^3}$$

where p , q , and r are the polynomials

$$\begin{cases} p(t) = -3 + 72t^2 - 144t^4, \\ q(t) = -65 - 6480t^2 + 96480t^4 - 246528t^6 + 145152t^8, \\ r(t) = 2 + 3t + 2t^2 + 3t^3. \end{cases}$$

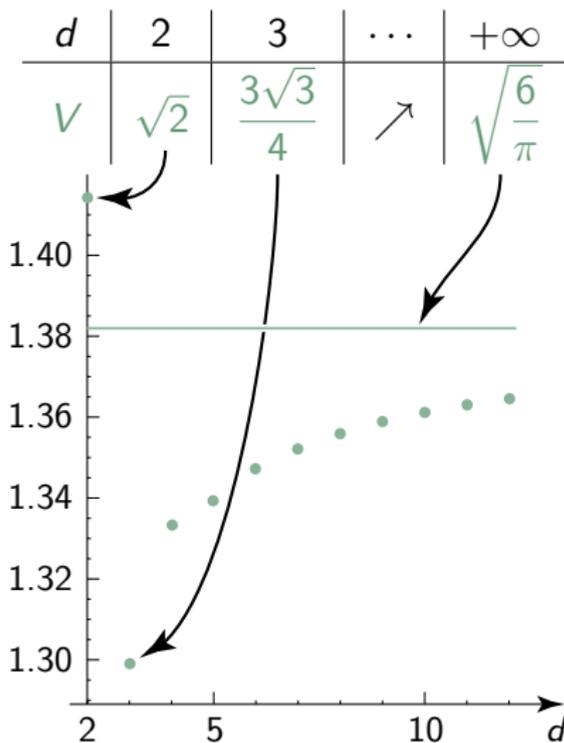
$$t < \frac{1}{2} \sqrt{1 - \sqrt{\frac{2}{3}}} \text{ or } t > \frac{1}{2} \sqrt{1 + \sqrt{\frac{2}{3}}}$$

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The monotonicity of V

Theorem (Bartha–Fodor–González Merino 2021): If $t = 0$, then V is a (strictly) increasing function of d when d is at least 3.



Theorem (P 2024): If

$$t < \frac{1}{2} \sqrt{1 - \sqrt{\frac{2}{3}}} \text{ or } t > \frac{1}{2} \sqrt{1 + \sqrt{\frac{2}{3}}}$$

then V is a (strictly) increasing function of d for large enough d (which means $d \geq 2$ when $0.0706 < t < 0.1441$) and if

$$\frac{1}{2} \sqrt{1 - \sqrt{\frac{2}{3}}} < t < \frac{1}{2} \sqrt{1 + \sqrt{\frac{2}{3}}}$$

then V is a (strictly) decreasing function of d for large enough d (which means $d \geq 2$ when $0.4075 < t \leq 0.5$).

Normality of the Eulerian numbers

The Eulerian numbers are asymptotically normal

Theorem (Pólya 1913):

$$\lim_{d \rightarrow +\infty} V = \sqrt{\frac{6}{\pi}} e^{-6t^2}$$

Theorem (Carlitz–Kurtz–Scoville–Stackelberg 1972): If d is large enough, then for every integer i such that $0 \leq i \leq d/2$,

$$\left| \frac{\sqrt{d}}{(d-1)!} A\left(d, \left\lceil \frac{d}{2} - \sqrt{(d)t} \right\rceil\right) - \sqrt{\frac{6}{\pi}} e^{-6t^2} \right| \leq \frac{K}{d^{1/4}}$$

where

$$t = \frac{\sqrt{d}}{2} - \frac{i}{\sqrt{d}}.$$

Theorem (P 2024): $1/4$ can be replaced by $1/2$ and this is sharp.

Sources:

[Shallow sections of the hypercube](#), Israel Journal of Mathematics 255(2), 685–704 (2023)

[Local Extrema for hypercube sections](#), Journal d'Analyse Mathématique 152(2), 557–594 (2024)

[Deep sections of the hypercube](#), arXiv:2407.04637 (2024)

Proof ideas: monotonicity when $t = 0$

The proof of Bartha–Fodor–González Merino (2021) evaluates

$$V = I_d(0) = \frac{2\sqrt{d}}{\pi} \int_0^{+\infty} \left(\frac{\sin u}{u} \right)^d du$$

precisely enough to show that $I_{d+1}(0) - I_d(0)$ is positive.

Strategy: use the change of variables $(\sin u)/u = e^{-x^2/6}$ which is such that

$$u = x - \frac{x^3}{60} - \frac{13x^5}{151200} + \frac{x^7}{7!} \frac{d^7 u}{dx^7}(\xi)$$

for all x in $[0, 1]$ where $0 \leq \xi \leq x$. Moreover, for all x in $[0, 1]$,

$$\begin{aligned} \left| \frac{du}{dx} \right| &< 1.00001, & \left| \frac{d^3 u}{dx^3} \right| &< 1.10718, \\ \left| \frac{d^5 u}{dx^5} \right| &< 0.02591, & \left| \frac{d^7 u}{dx^7} \right| &< 0.79461. \end{aligned}$$

Proof ideas: monotonicity when $t \geq 0$

The strategy is the same: use the change of variables $(\sin u)/u = e^{-x^2/6}$ in

$$V = I_d(t) = \frac{2\sqrt{d}}{\pi} \int_0^{+\infty} \left(\frac{\sin u}{u} \right)^d \cos(2\sqrt{d}tu) du$$

to determine the sign of $I_{d+1}(t) - I_d(t)$.

However, in order to handle the term

$$\cos\left(2\sqrt{d}t\left(x - \frac{x^3}{60} - \frac{13x^5}{151200} + \frac{x^7}{7!} \frac{d^7 u}{dx^7}(\xi)\right)\right),$$

use the trigonometric identities

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

and then use series expansions of \cos and \sin with Lagrange remainders (all derivatives are bounded by 1).

Proof ideas: monotonicity when $t \geq 0$

The strategy is the same: use the change of variables $(\sin u)/u = e^{-x^2/6}$ in

Theorem (P 2024): If d is at least 136, then for every non-negative t ,

$$\left| V - \sqrt{\frac{6}{\pi}} e^{-6t^2} \left(1 + \frac{p(t)}{20d} + \frac{q(t)}{5600d^2} \right) \right| < \frac{r(t)}{3d^3}$$

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$$\begin{cases} p(t) = -3 + 72t^2 - 144t^4, \\ q(t) = -65 - 6480t^2 + 96480t^4 - 246528t^6 + 145152t^8, \\ r(t) = 2 + 3t + 2t^2 + 3t^3. \end{cases}$$

When d is less than 136: Use the expression

$$V = I_d(t) = \frac{\sqrt{d}}{(d-1)!} \sum_{i=0}^{\lfloor z \rfloor} (-1)^i \binom{d}{i} (z-i)^{d-1}$$

where $z = d/2 - \sqrt{d}t$. This is piecewise polynomial in t and so is the difference $I_{d+1}(t) - I_d(t)$. It suffices to find the roots of (many) polynomials (of large degree) using symbolic computation.

derivatives are bounded by 1).

Proof ideas: local extremality

Theorem (P 2024): If

$$\int_0^{+\infty} \left(\frac{\sin u}{u}\right)^{d-2} \left(2\frac{\sin^2 u}{u^2} - \cos u \frac{\sin u}{u} - 1\right) \cos(2\sqrt{d}tu) du$$

is negative then V is locally maximal at diagonals and if this quantity is positive then V is locally minimal at diagonals.

Theorem (P 2024): If $4 \leq n < d$ while

$$\int_0^{+\infty} \left(\frac{\sin u}{u}\right)^{n-2} \left(2\frac{\sin^2 u}{u^2} - \cos u \frac{\sin u}{u} - 1\right) \cos(2\sqrt{d}tu) du$$

and

$$\int_0^{+\infty} \left(\frac{\sin u}{u}\right)^{n-2} \left(\cos u + \left(\frac{u^2}{3} - 1\right)2\frac{\sin u}{u}\right) \cos(2\sqrt{d}tu) du$$

are both negative then V is locally maximal at order n sub-diagonals and if they are both positive then V is locally minimal at order n sub-diagonals. If they have opposite signs, V is not locally extremal at order n sub-diagonals.

Wrap up: the Milman–König conjecture

Conjecture (V. Milman): \mathcal{V} is maximal when the hyperplane H is orthogonal to a diagonal or a lower-order sub-diagonal of the hypercube.

Conjecture (König): if t is at least some constant, \mathcal{V} is maximal precisely when the hyperplane H is orthogonal to a diagonal of the hypercube.

Theorem (P 2024): For any fixed t , if

$$t \leq \frac{1}{2} \sqrt{1 - \sqrt{\frac{2}{3}}} - \frac{K}{d} \text{ or } \frac{1}{2} \sqrt{1 + \sqrt{\frac{2}{3}}} + \frac{K}{d} \leq t \leq \sqrt{\frac{\log d}{6}}$$

and d is large enough (where K can be taken equal to 76) or if

$$\frac{\sqrt{d}}{2} - \Omega\left(\frac{\sqrt{d}}{\log d}\right) < t < \frac{\sqrt{d}}{2}$$

and d is at least 4, then \mathcal{V} is strictly locally maximal when H is orthogonal to a diagonal of the d -dimensional hypercube. Replacing d by $n < d$ and under the same conditions on n , \mathcal{V} is not locally extremal when H is orthogonal to an order n sub-diagonal of the d -dimensional hypercube.