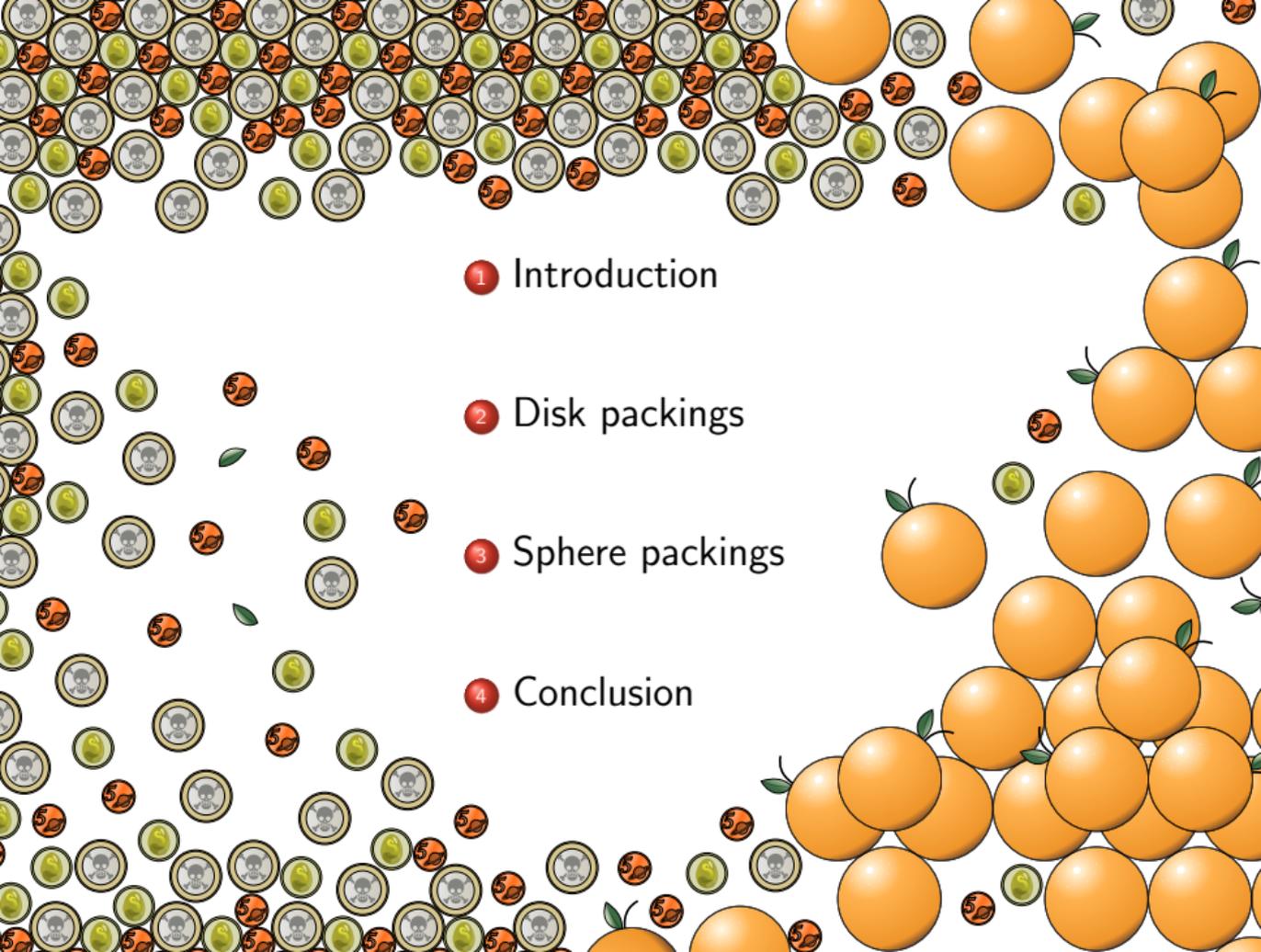
The slide features a decorative border composed of several elements: a top row of oranges, a right-side column of oranges, and a bottom-left corner filled with a dense arrangement of small circular icons. These icons include stylized human faces, green circles, orange circles, and circles with the number '5'.

Optimal disk and sphere packings

Daria Pchelina
CNRS, LIP, ENS Lyon

Journées de combinatoire de Bordeaux

3 février 2026

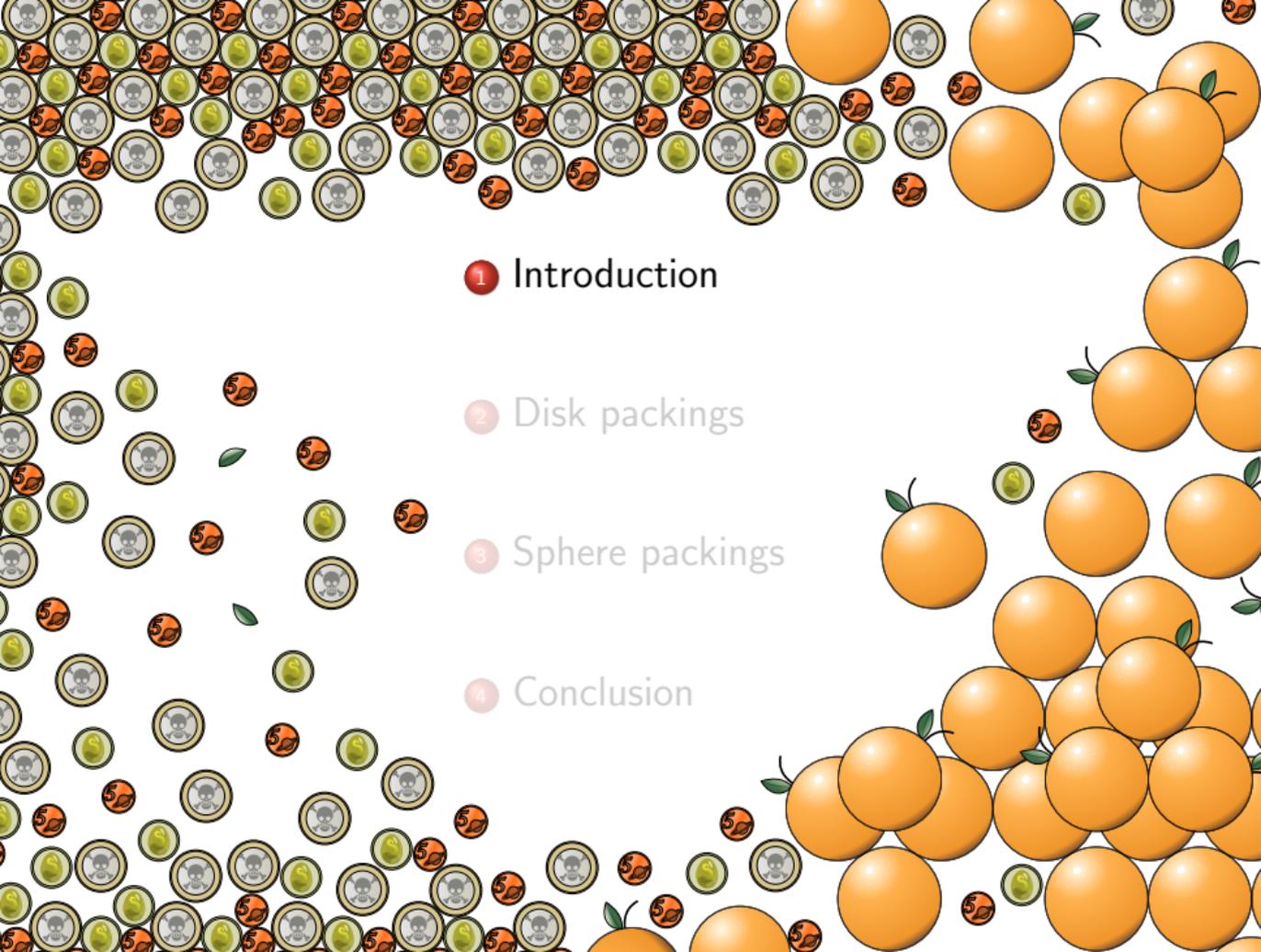


1 Introduction

2 Disk packings

3 Sphere packings

4 Conclusion



1 Introduction

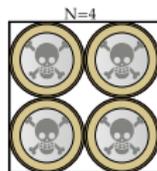
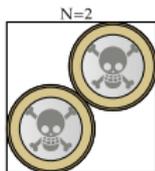
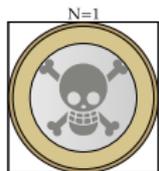
2 Disk packings

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4 Conclusion

Optimal coin packings in containers

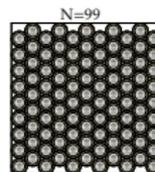
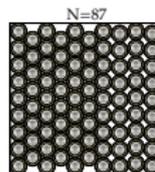
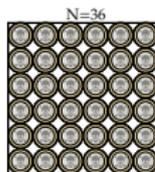
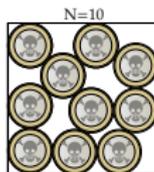
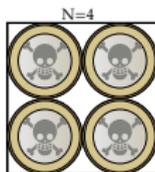
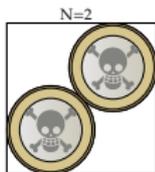
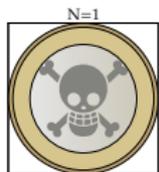
Given N identical coins (☠) ,
place them in a smallest possible square without overlap



Optimal coin packings in containers

Given N identical coins ,

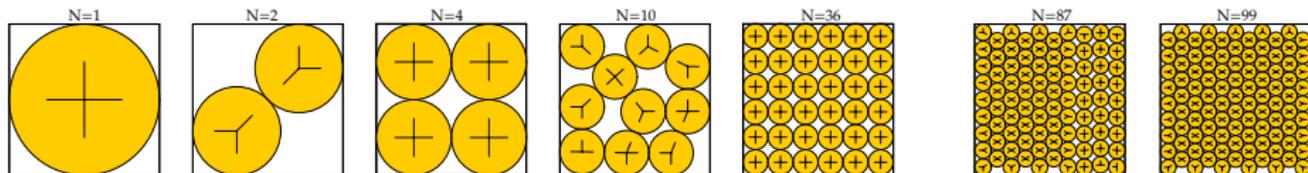
place them in a smallest possible square without overlap solved for 1–33, 36 (1964–2021)



Optimal coin packings in containers

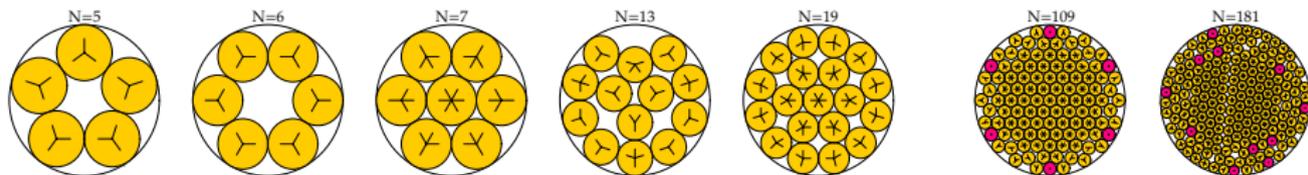
Given N identical coins (\bigcirc) ,

place them in a smallest possible square without overlap solved for 1–33, 36 (1964–2021)



... circle

solved for 1–13, 19 (1967–2003)

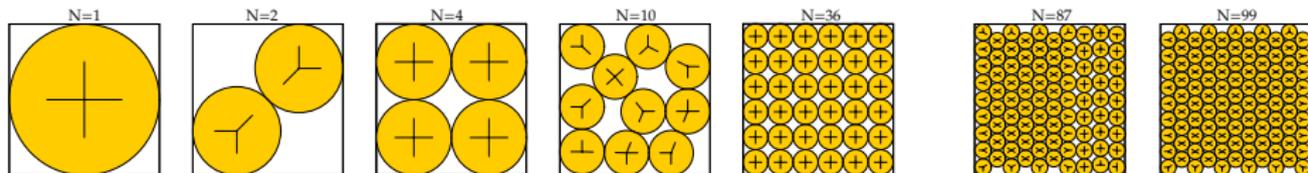


Optimal coin packings in containers

Given N identical coins (\bigcirc) ,

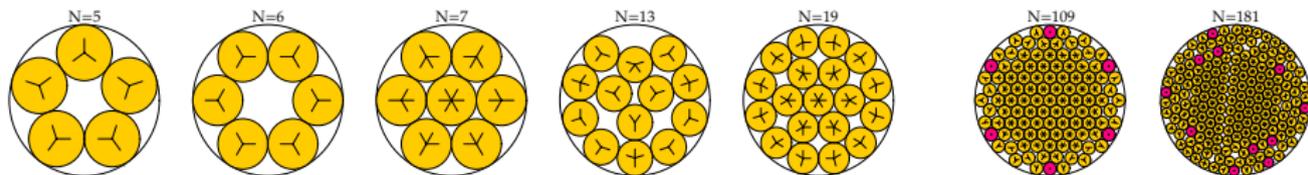
place them in a smallest possible square without overlap

solved for 1–33, 36 (1964–2021)



... circle

solved for 1–13, 19 (1967–2003)



Find candidates: by hand for small n , billiard simulation, perturbation method

Prove optimality: by hand for small n and circles, interval analysis, branch-and-bound

Optimal coin packings on the plane

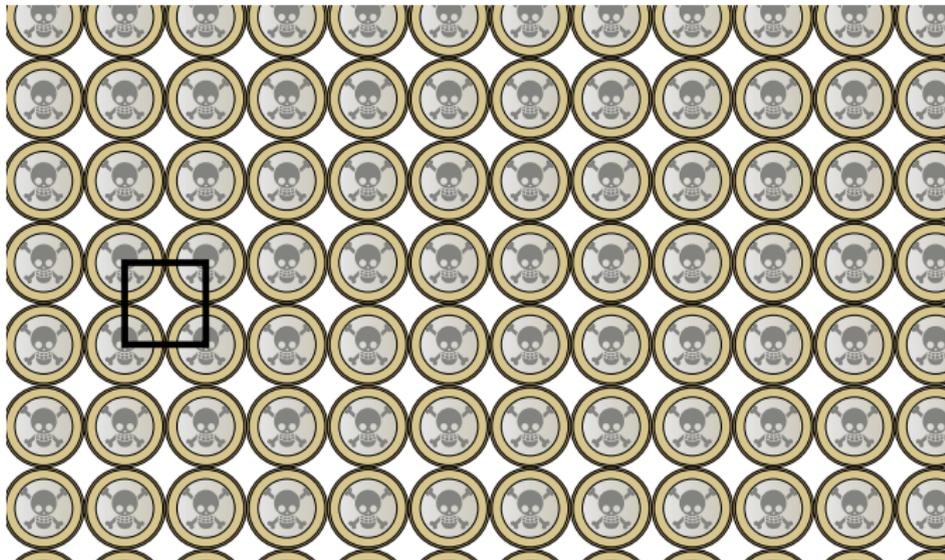
Given infinite number of identical coins $\left(\textcircled{\text{skull}}$ \right),
how to place them on an infinite plane without overlap to maximize the covered surface?

coin packing:



Optimal coin packings on the plane

Given infinite number of identical coins ,
how to place them on an infinite plane without overlap to maximize the covered surface?

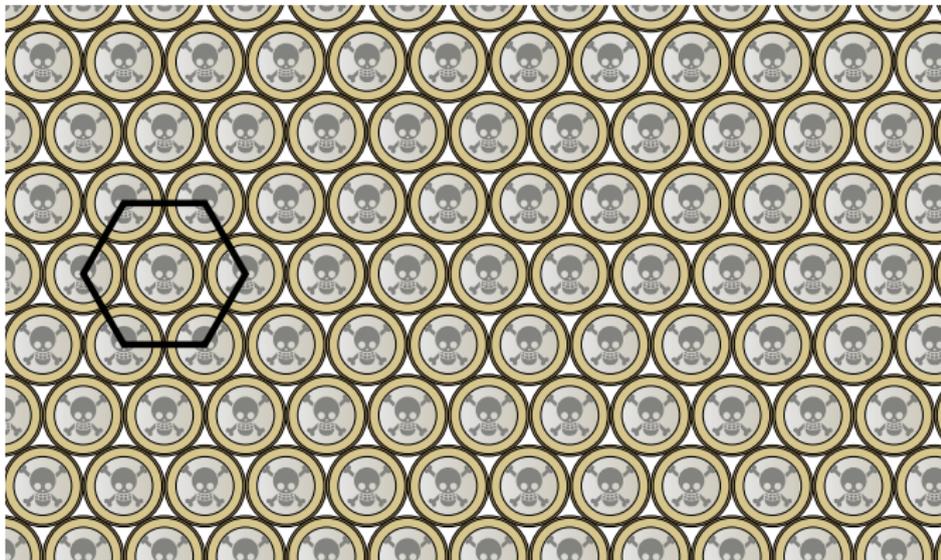


square coin packing:
covers 78% of the surface

Optimal coin packings on the plane

Given infinite number of identical coins ,
how to place them on an infinite plane without overlap to maximize the covered surface?

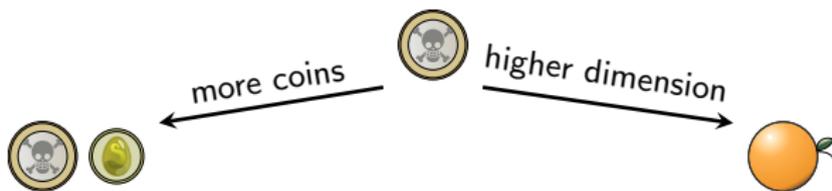
hexagonal coin packing:
covers **90%** of the surface



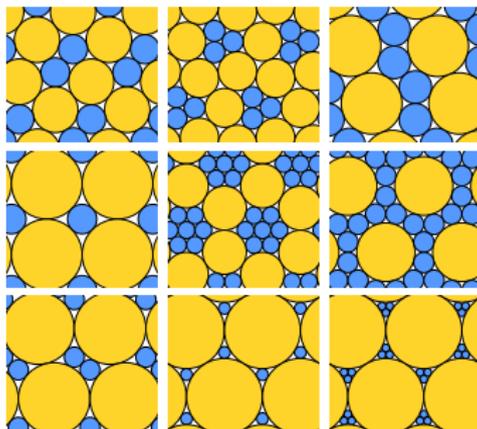
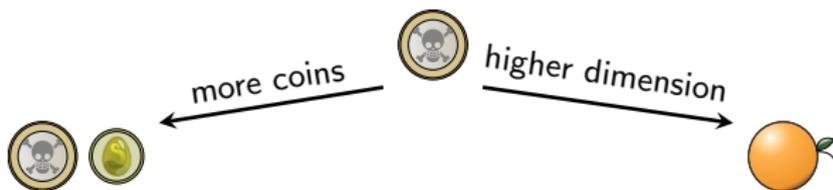
1910–1940

The hexagonal coin packing is optimal.

Introduction



Introduction



(proved optimal in 2000-2022)

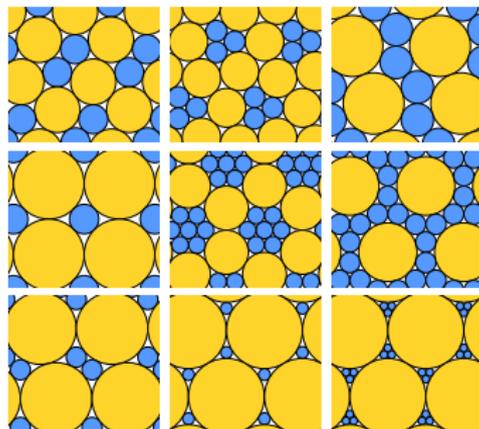
Kepler conjecture, 1611

The "cannonball" packing is optimal:



(proved in 1998-2014)

Introduction



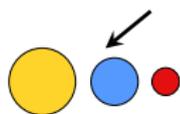
(proved optimal in 2000-2022)

Kepler conjecture, 1611

The "cannonball" packing is optimal:

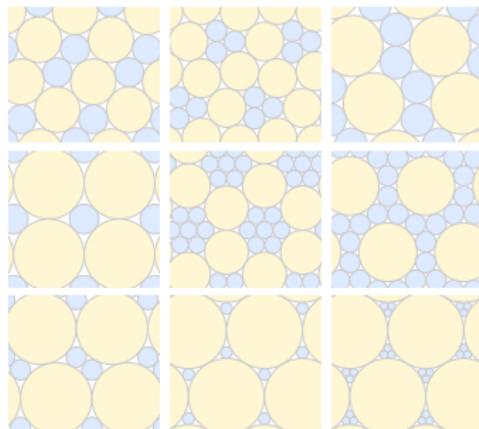
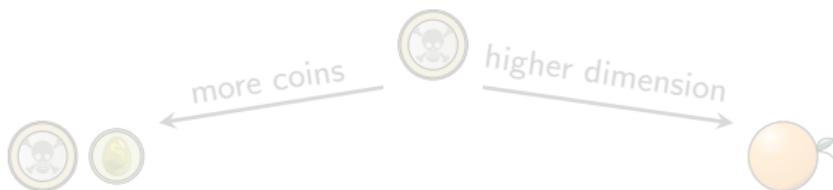


(proved in 1998-2014)



$\mathbb{R}^8, \mathbb{R}^{24}$
(Viazovska, Fields Medal 2022)

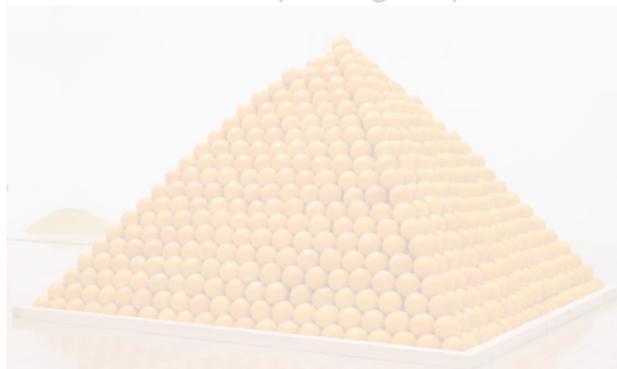
Introduction



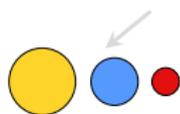
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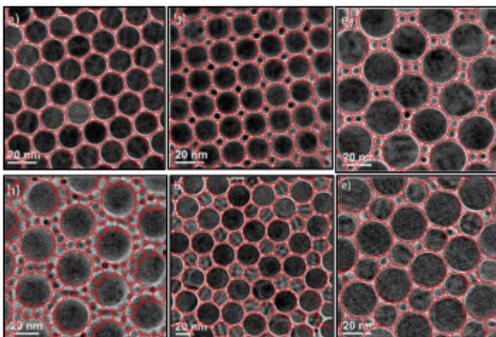
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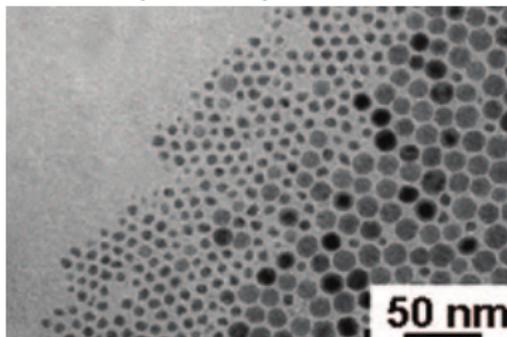
combine different types of nanoparticles
self-assembly

new material



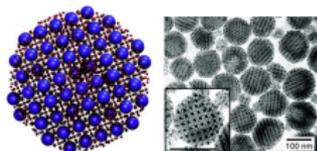
Paik et al 2015

phase separation



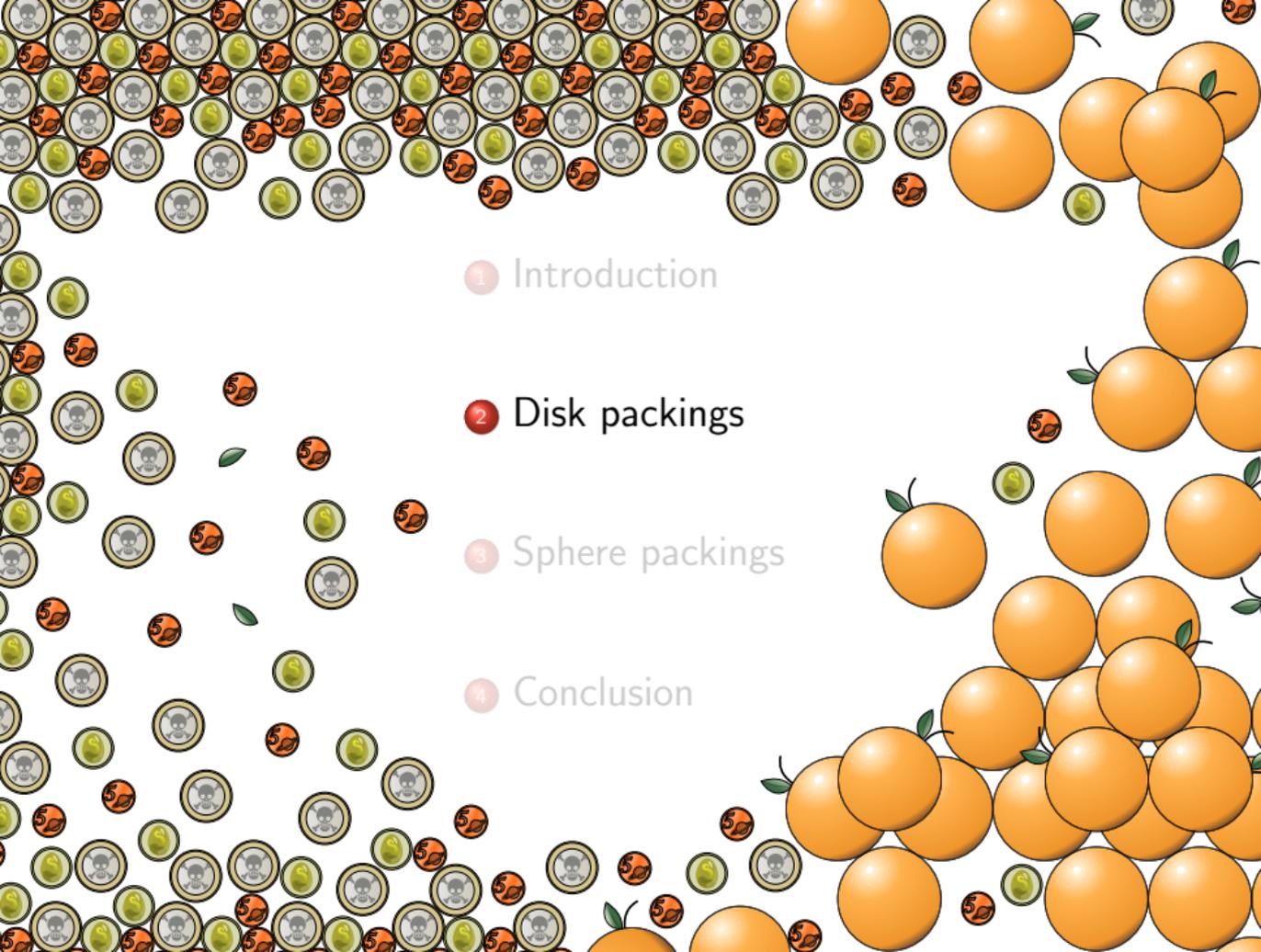
Cheon et al 2006

Also in 3D:



Wu, Fan, Yin 2022

Chemists' question : **which sizes and concentrations allow for new materials?**



1 Introduction

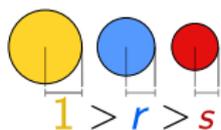
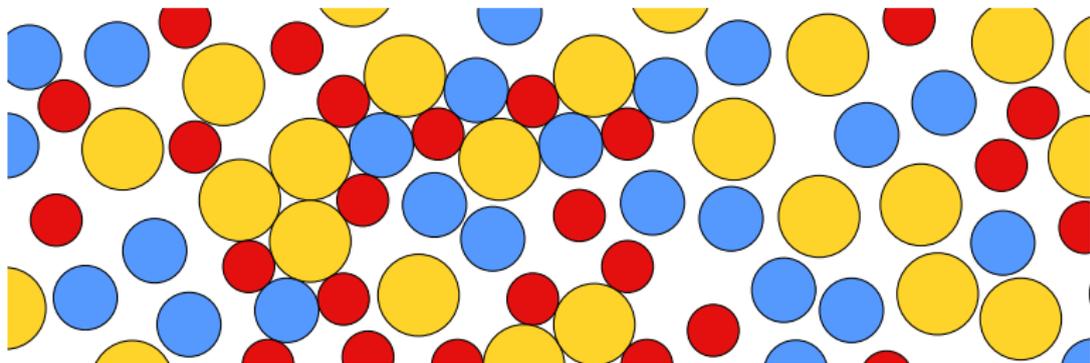
2 Disk packings

3 Sphere packings

4 Conclusion

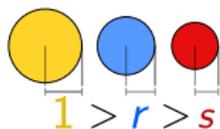
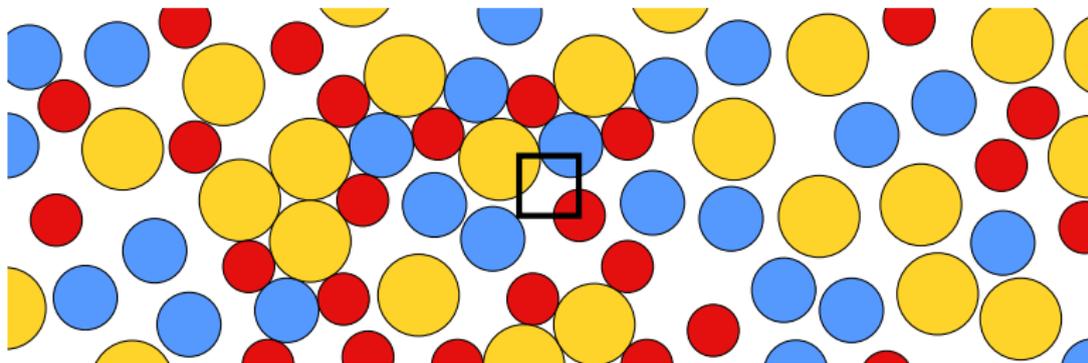
Definitions

Disks:

Packing P :
(in \mathbb{R}^2)

Definitions

Disks:

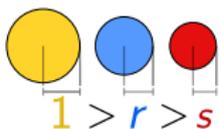
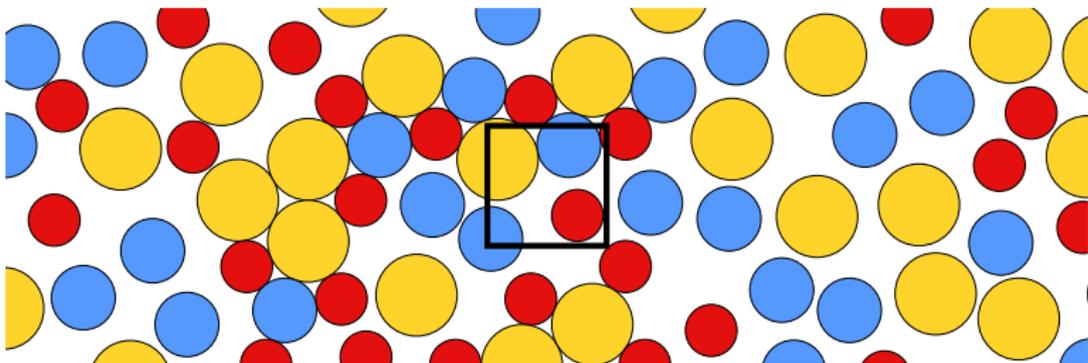
Packing P :
(in \mathbb{R}^2)

Density:

$$\delta \left(\begin{array}{|c|} \hline \text{disk} \\ \hline \end{array} \right) := \frac{\text{area} \left(\begin{array}{|c|} \hline \text{disk} \\ \hline \end{array} \right)}{\text{area} \left(\begin{array}{|c|} \hline \text{square} \\ \hline \end{array} \right)}$$

Definitions

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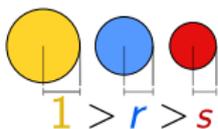
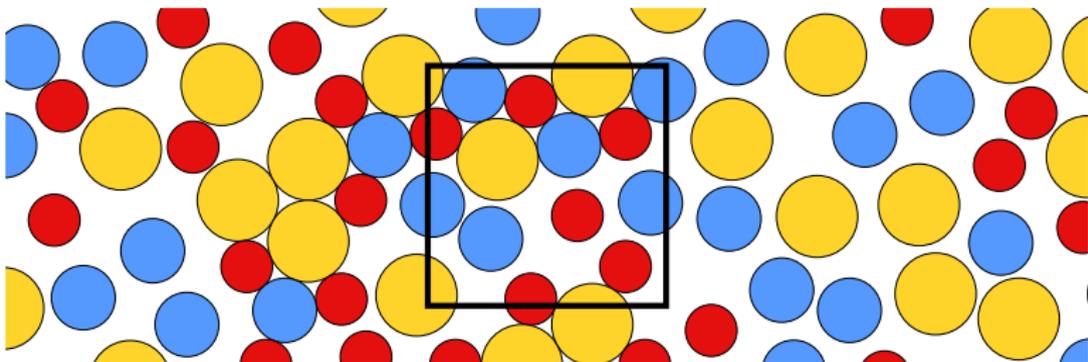
Packing P :
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Density:

$$\delta \left(n \downarrow \blacksquare \cap P \right) := \frac{\text{area} \left(n \downarrow \blacksquare \cap P \right)}{\text{area} \left(n \downarrow \blacksquare \right)}$$

Definitions

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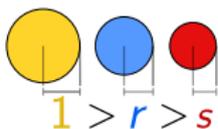
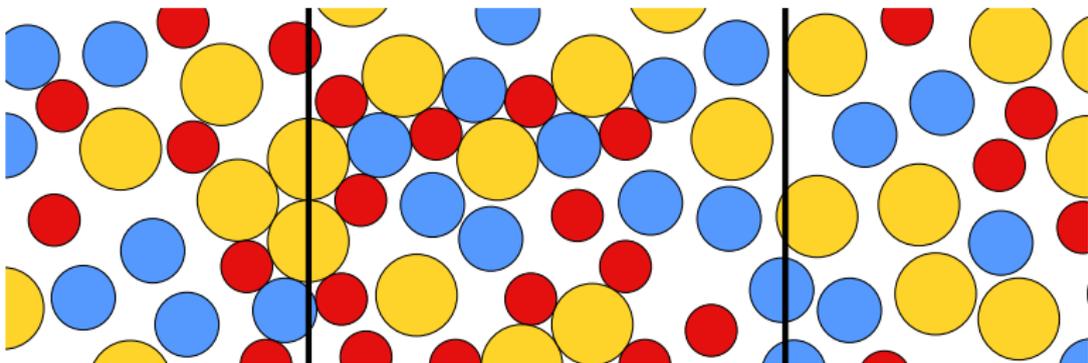
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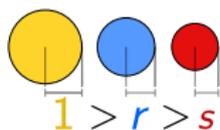
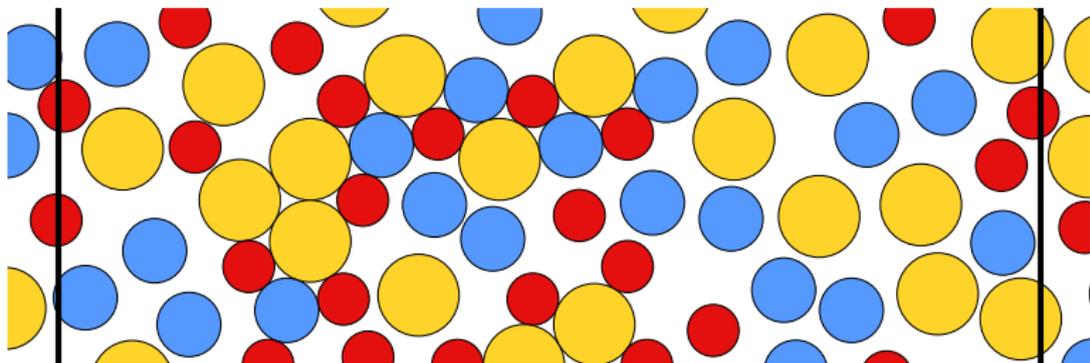
Packing P :
(in \mathbb{R}^2)

Density:

$$\delta \left(n \downarrow \begin{array}{c} \vec{n} \\ \blacksquare \end{array} \cap P \right) := \frac{\text{area} \left(n \downarrow \begin{array}{c} \vec{n} \\ \blacksquare \end{array} \cap P \right)}{\text{area} \left(n \downarrow \begin{array}{c} \vec{n} \\ \blacksquare \end{array} \right)}$$

Definitions

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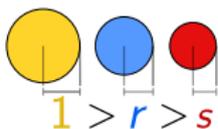
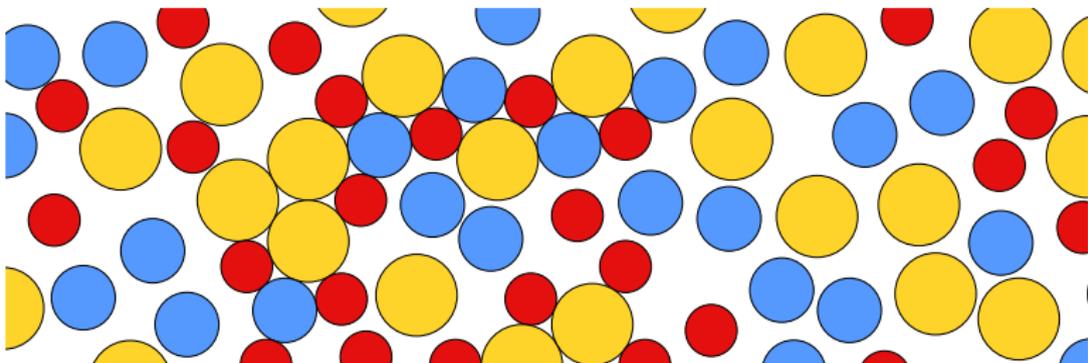
Packing P :
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Definitions

Disks:

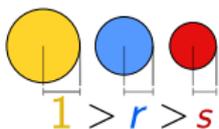
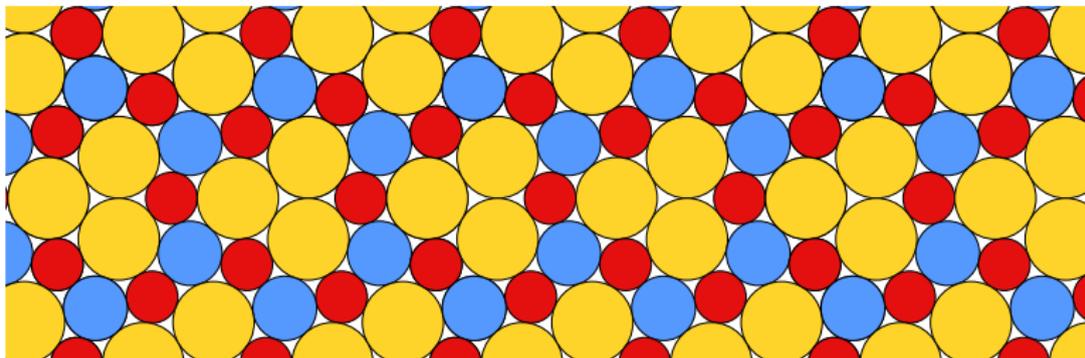
Packing P :
(in \mathbb{R}^2)

Density:

$$\delta(P) := \limsup_{n \rightarrow \infty} \frac{\text{area} \left(n \downarrow \begin{array}{c} \overrightarrow{n} \\ \blacksquare \end{array} \cap P \right)}{\text{area} \left(n \downarrow \begin{array}{c} \overrightarrow{n} \\ \blacksquare \end{array} \right)}$$

Definitions

Disks:

Packing P :
(in \mathbb{R}^2)

Density:

$$\delta^* \approx 90.9\%$$

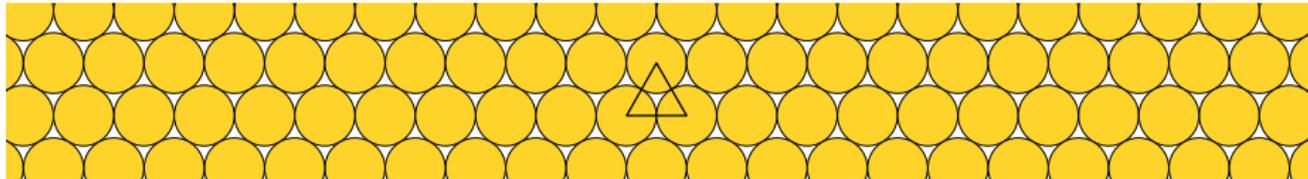
Main Question

Given a finite set of disks (e.g., $\bigcirc \bullet \bullet$),
what is the maximal density δ^* of a packing?

$$\delta^* := \max_P \delta(P)$$

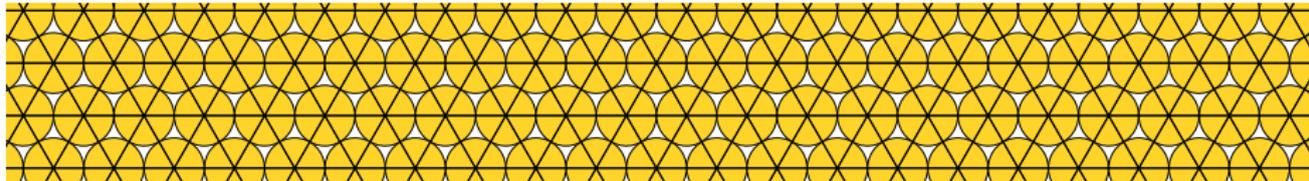
Triangulated packings

A packing is called **triangulated** if its contact graph is a triangulation:



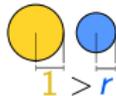
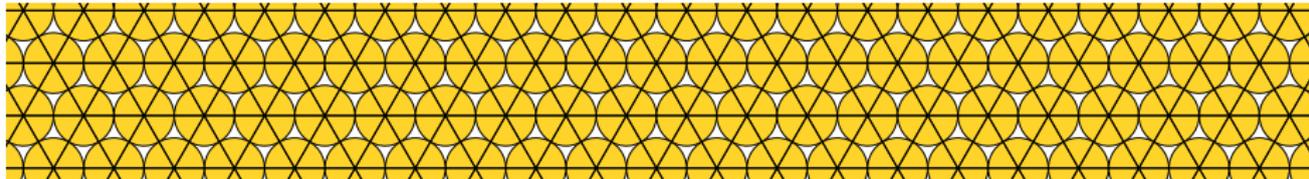
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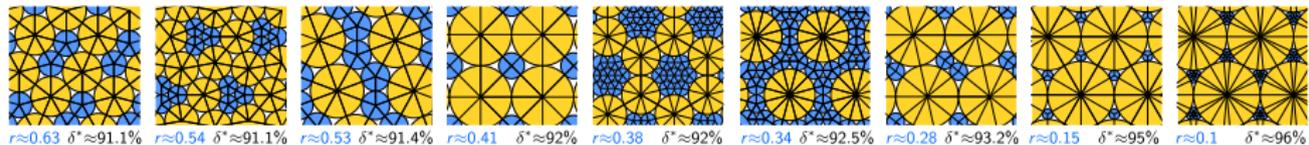
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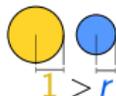
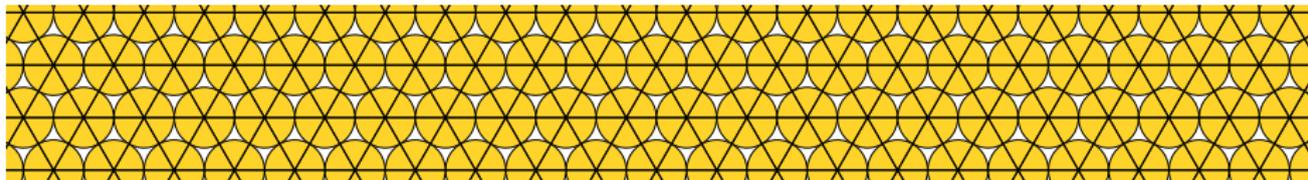
Theorem (Kennedy, 2006)

There are 9 values of r allowing triangulated packings:



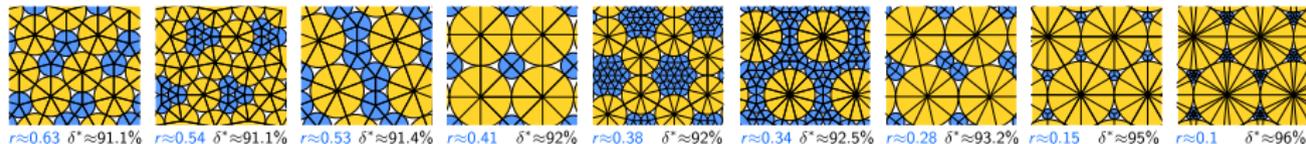
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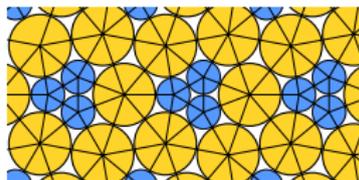


Theorem (Heppes 2000, 2003, Kennedy 2005, Bedaride and Fernique 2022)

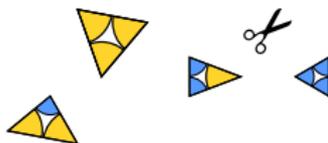
Each of these packings is optimal (densest) for disks of radii 1 and r .

(Packings and tilings)

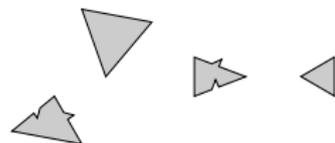
triangulated packings



~



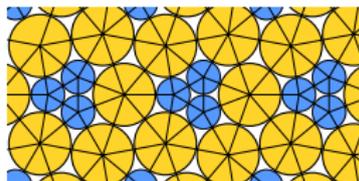
tilings by triangles
with local rules



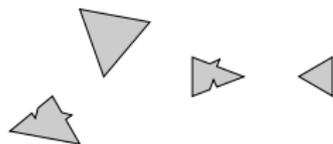
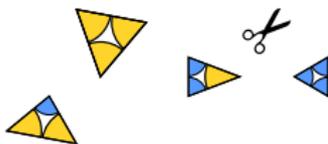
density = weighted proportion of tiles

(Packings and tilings)

triangulated packings



~

tilings by triangles
with local rules

density = weighted proportion of tiles

Triangulated Packing Problem

algebraic numbers represented by polynomials and intervals

excludes hexagonal packing

Given k disk radii $\overbrace{r_1, \dots, r_k}$, is there a triangulated packing of density $> \frac{\pi}{2\sqrt{3}}$

$\forall r_1, \dots, r_k$ with triangulated packings, one is periodic
(Wang algorithm: search for a period)

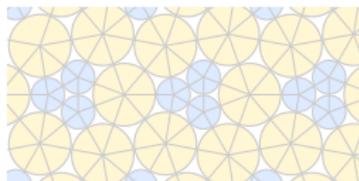
 \Rightarrow **decidable?**

$\exists r_1, \dots, r_k$ whose triangulated packings are all aperiodic

 \Rightarrow **undecidable?**

(Packings and tilings)

triangulated packings



~

tilings by triangles
with local rules

density = weighted proportion of tiles

Dense Packing Problem

algebraic numbers represented by polynomials and intervals

Given k disk radii $\overbrace{r_1, \dots, r_k}$, is there a

excludes hexagonal packing

packing of density $> \frac{\pi}{2\sqrt{3}}$

$\forall r_1, \dots, r_k$ with dense packings, one is periodic
(interval arithmetic and subdivision until needed precision)

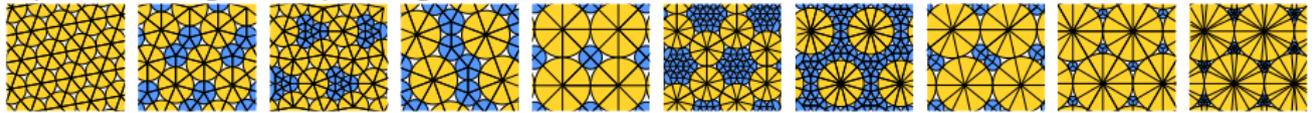
 \Rightarrow **decidable?**

$\exists r_1, \dots, r_k$ whose dense packings are all aperiodic

 \Rightarrow **not possible!**

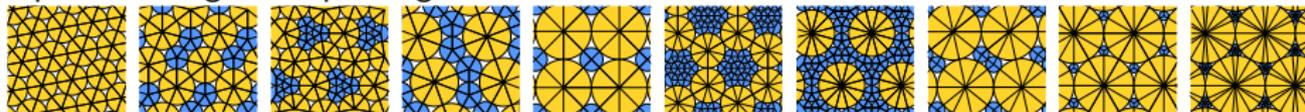
Triangulated = optimal?

Optimal triangulated packings:



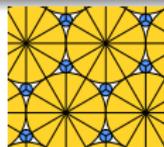
Triangulated = optimal?

Optimal triangulated packings:



Conjecture (Connelly 2018)

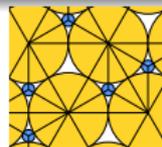
If a finite set of disks allows **saturated** triangulated packings then one of them is optimal.



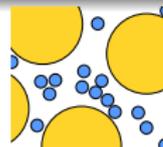
triangulated
saturated



non triangulated
saturated



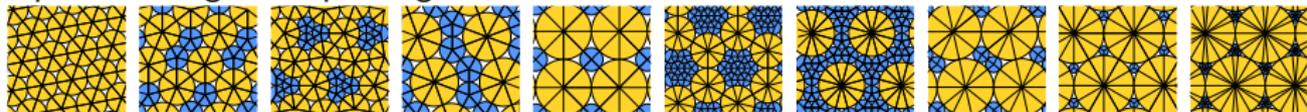
triangulated
non saturated



non triangulated
non saturated

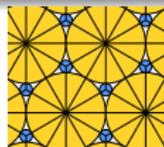
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Conjecture (Connelly 2018)

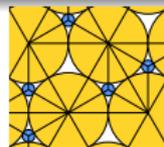
If a finite set of disks allows **saturated** triangulated packings then one of them is optimal.



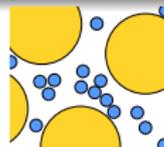
triangulated
saturated



non triangulated
saturated



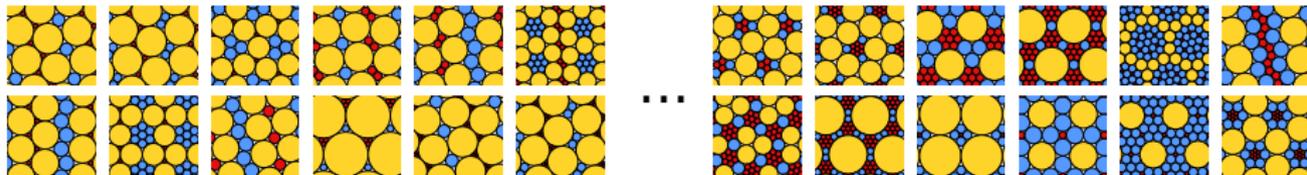
triangulated
non saturated



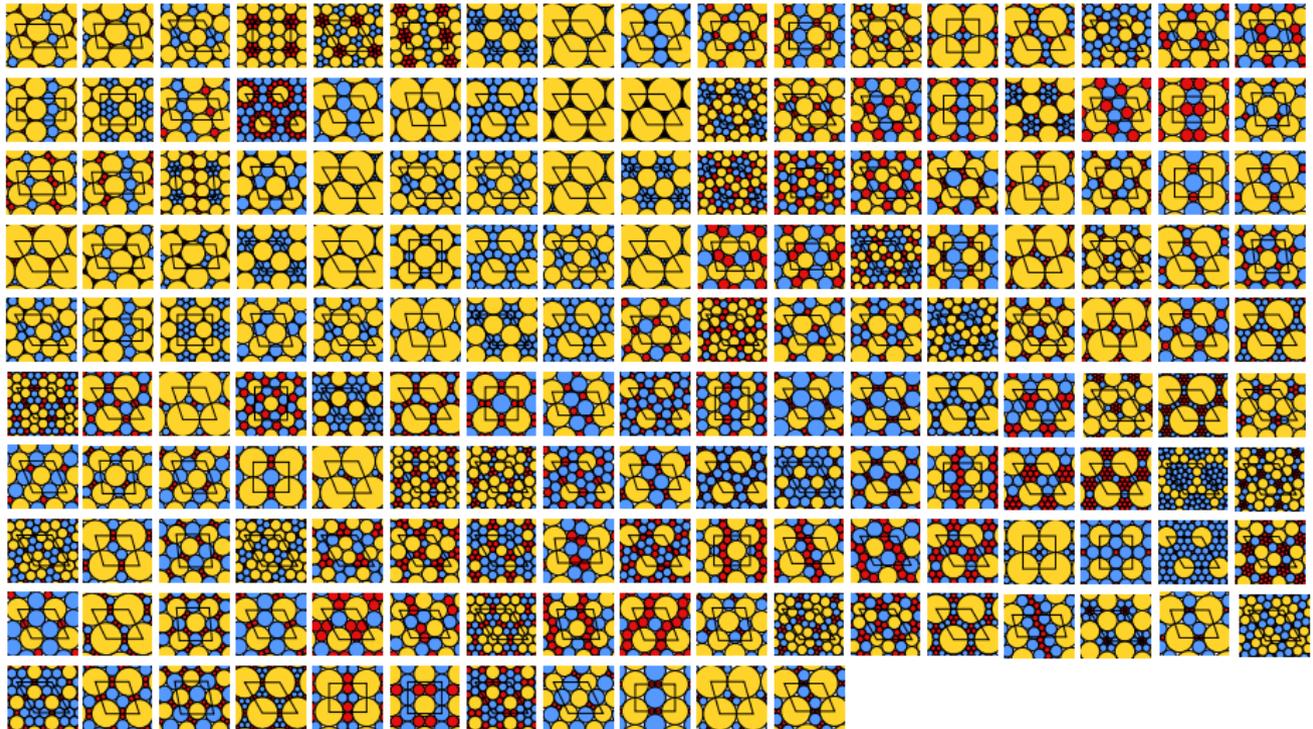
non triangulated
non saturated

Theorem (Fernique, Hashemi, Sizova 2019)

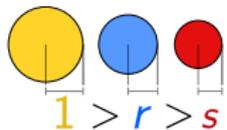
: there are 164 pairs (r, s) allowing triangulated packings.
 $1 > r > s$



Disk packings

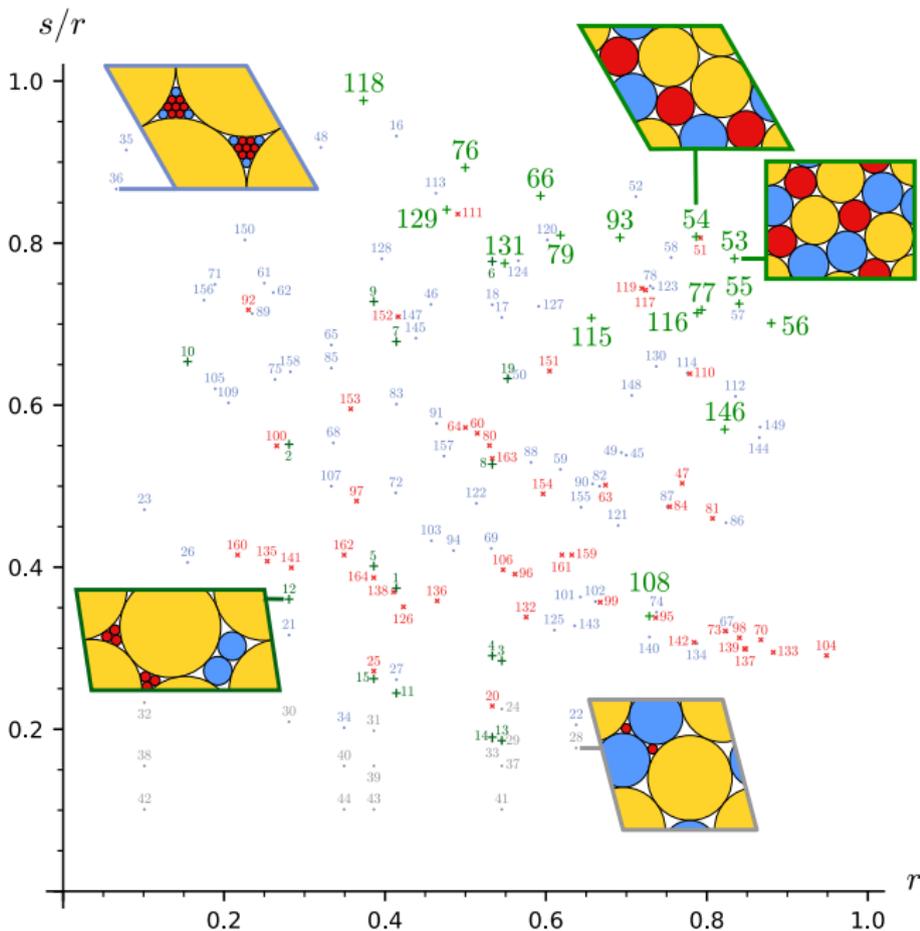


Disk packings



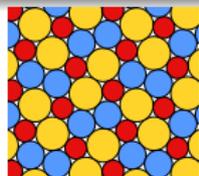
164 (r, s) allowing
triangulated packings:

- 15 cases: non saturated
- 16+16 cases:
a **ternary** or **binary**
triangulated packing
is densest
- 45 cases: a binary
non triangulated
packing is denser

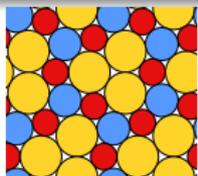


Theorem (Fernique, P 2023)

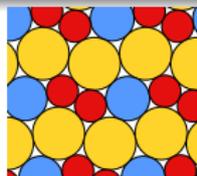
Each of the following packings is optimal for disks of radii 1, r and s :



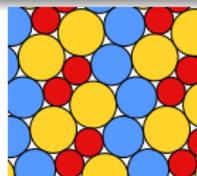
$r \approx 0.8$ $s \approx 0.7$ $\delta^* \approx 90.9\%$



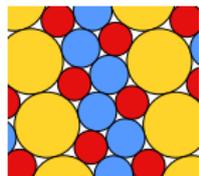
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 90.9\%$



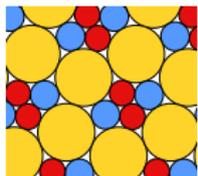
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 91\%$



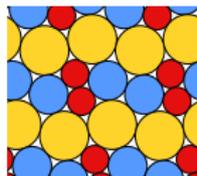
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 90.9\%$



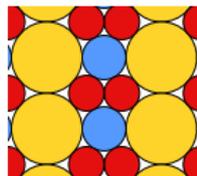
$r \approx 0.6$ $s \approx 0.5$ $\delta^* \approx 91.2\%$



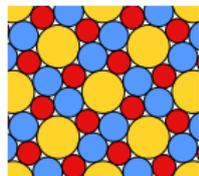
$r \approx 0.5$ $s \approx 0.4$ $\delta^* \approx 91.5\%$



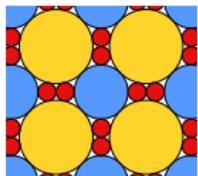
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 91\%$



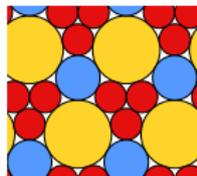
$r \approx 0.6$ $s \approx 0.5$ $\delta^* \approx 91.3\%$



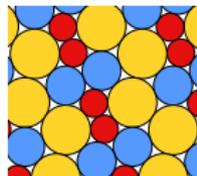
$r \approx 0.7$ $s \approx 0.5$ $\delta^* \approx 91.1\%$



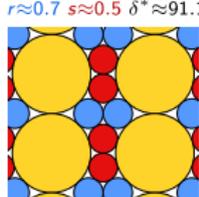
$r \approx 0.7$ $s \approx 0.2$ $\delta^* \approx 93.2\%$



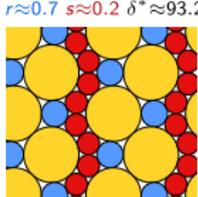
$r \approx 0.7$ $s \approx 0.5$ $\delta^* \approx 91.5\%$



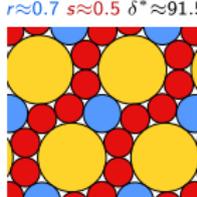
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 91\%$



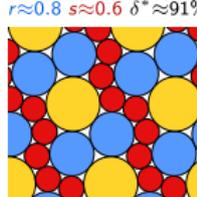
$r \approx 0.4$ $s \approx 0.4$ $\delta^* \approx 92.2\%$



$r \approx 0.5$ $s \approx 0.4$ $\delta^* \approx 91.8\%$



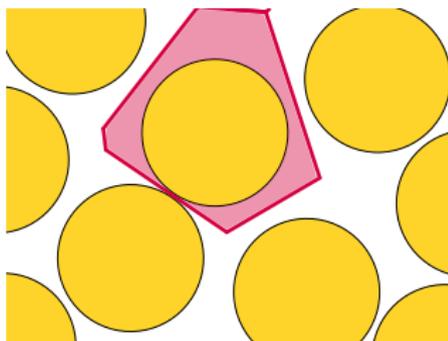
$r \approx 0.5$ $s \approx 0.4$ $\delta^* \approx 91.7\%$



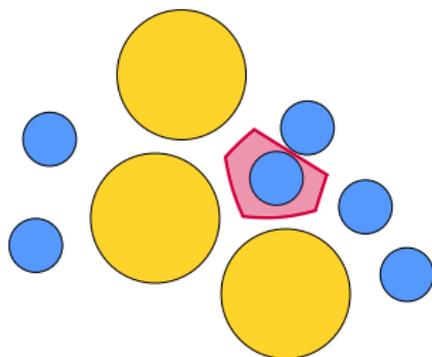
$r \approx 0.8$ $s \approx 0.5$ $\delta^* \approx 91.2\%$

FM-triangulation

1-disk packing



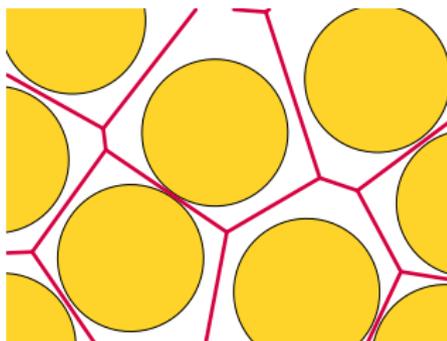
multi-size disk packing



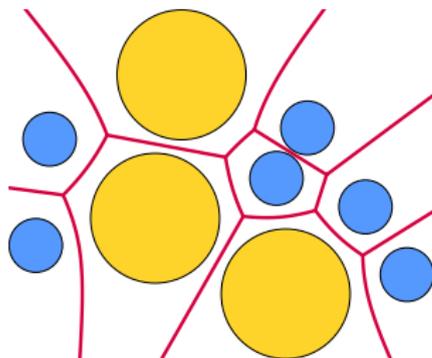
Voronoi cell of a disk in a packing: set of points closer to this disk than to any other

FM-triangulation

1-disk packing



multi-size disk packing

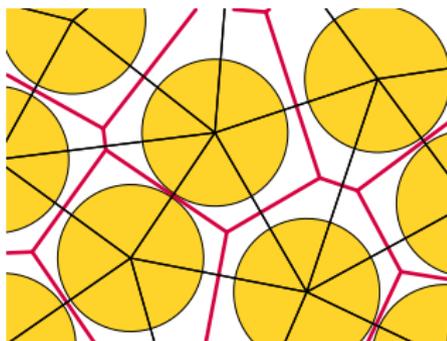


Voronoi cell of a disk in a packing: set of points closer to this disk than to any other

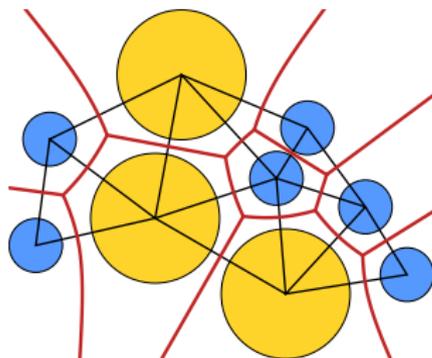
Voronoi diagram of a packing: partition of the plane into Voronoi cells

FM-triangulation

1-disk packing



multi-size disk packing



Voronoi cell of a disk in a packing: set of points closer to this disk than to any other

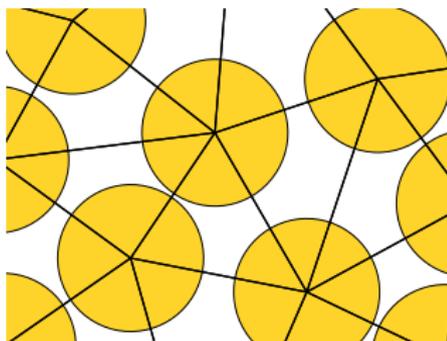
Voronoi diagram of a packing: partition of the plane into Voronoi cells

FM-triangulation of a packing: dual graph of the Voronoi diagram

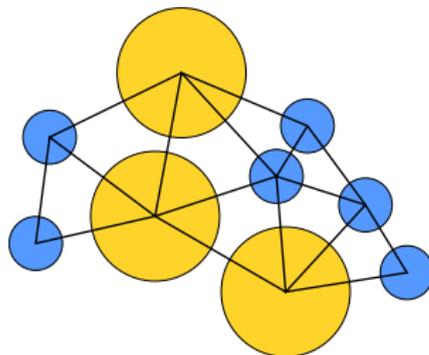
Fejes Tóth, Mólnar

FM-triangulation

1-disk packing



multi-size disk packing



Voronoi cell of a disk in a packing: set of points closer to this disk than to any other

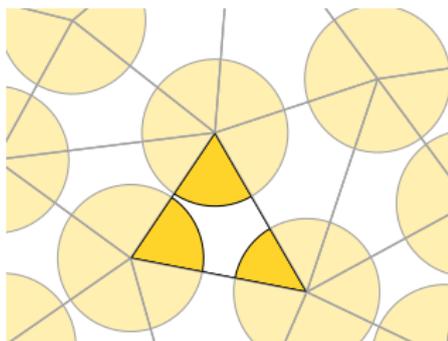
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FM-triangulation of a packing: dual graph of the Voronoi diagram

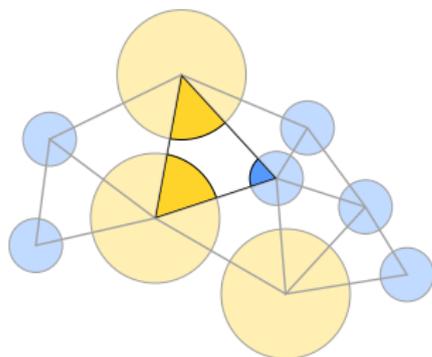
Fejes Tóth, Mólnar

FM-triangulation

1-disk packing



multi-size disk packing



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Voronoi diagram of a packing: partition of the plane into Voronoi cells

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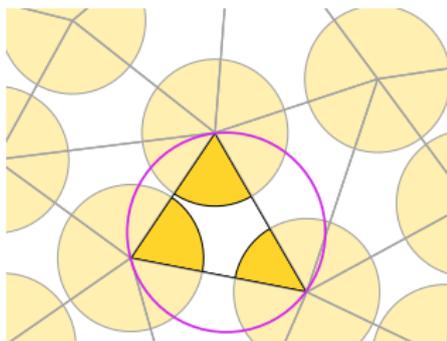
Fejes Tóth, Mólнар

Density of a triangle Δ in a packing = its proportion covered by disks

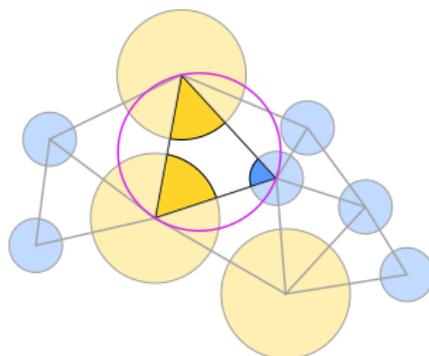
$$\delta_{\Delta} = \frac{\text{area}(\Delta \cap P)}{\text{area}(\Delta)}$$

FM-triangulation

1-disk packing



multi-size disk packing



Voronoi cell of a disk in a packing: set of points closer to this disk than to any other

Voronoi diagram of a packing: partition of the plane into Voronoi cells

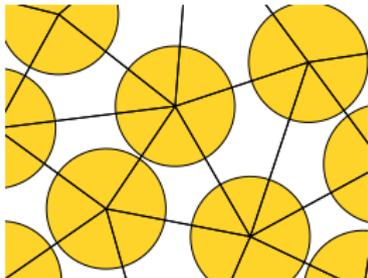
FM-triangulation of a packing: dual graph of the Voronoi diagram

Fejes Tóth, Mólнар

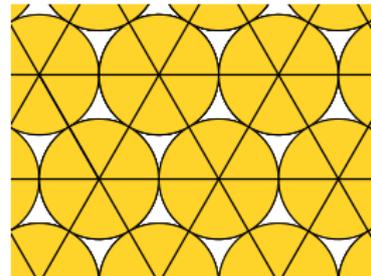
Density of a triangle Δ in a packing = its proportion covered by disks

$$\delta_{\Delta} = \frac{\text{area}(\Delta \cap P)}{\text{area}(\Delta)}$$

Local density redistribution

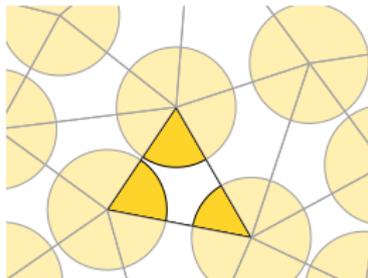


P of density $\delta(P)$



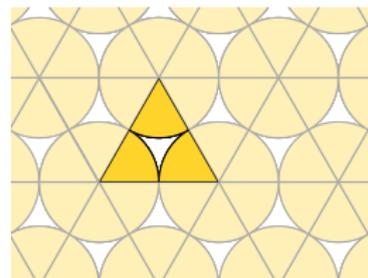
P^* of density δ^*

Local density redistribution



P of density $\delta(P)$

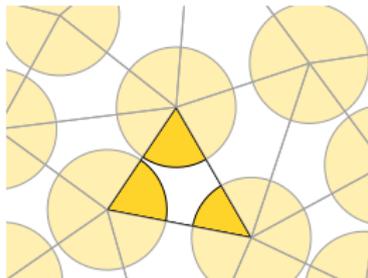
$$\forall \Delta, \delta(\Delta) \leq \delta(\triangle) = \delta^*$$



P^* of density δ^*

$$\delta(\triangle) = \delta^*$$

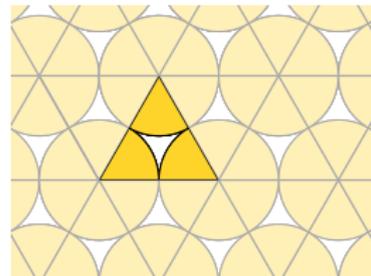
Local density redistribution



P of density $\delta(P)$

$$\forall \Delta, \delta(\Delta) \leq \delta(\triangle) = \delta^*$$

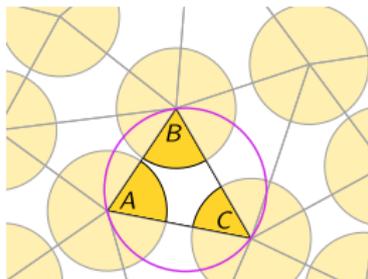
$$\delta(P) \leq \delta^*$$



P^* of density δ^*

$$\delta(\triangle) = \delta^*$$

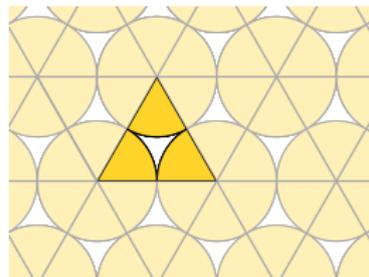
Local density redistribution



P of density $\delta(P)$

$$\forall \Delta, \delta(\Delta) \leq \delta(\triangle) = \delta^*$$

$$\delta(P) \leq \delta^*$$



P^* of density δ^*

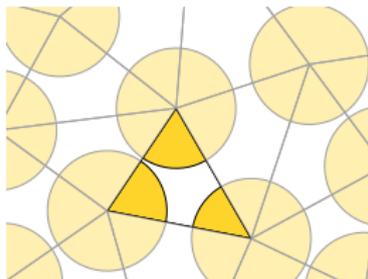
$$\delta(\triangle) = \delta^*$$

Proof:

- the smallest angle of any Δ is at least $\frac{\pi}{6}$
- thus the largest angle is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$
- density of a triangle Δ : $\delta(\Delta) = \frac{\pi/2}{\text{area}(\Delta)}$
- the area of a triangle ABC with the largest angle \hat{A} : $\frac{|AB| \cdot |AC| \cdot \sin \hat{A}}{2} \geq \frac{2 \cdot 2 \cdot \frac{\sqrt{3}}{2}}{2} = \sqrt{3}$
- thus the density of ABC is less or equal to $\frac{\pi/2}{\sqrt{3}} = \delta^*$

$$2 > R = \frac{|AB|}{2 \sin \hat{C}} \geq \frac{1}{\sin \hat{C}} \implies \hat{C} > \frac{\pi}{6}$$

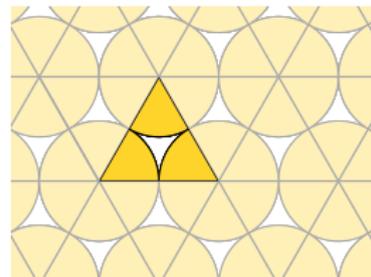
Local density redistribution



P of density $\delta(P)$

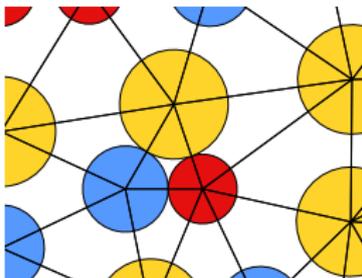
$$\forall \Delta, \delta(\Delta) \leq \delta(\triangle) = \delta^*$$

$$\delta(P) \leq \delta^*$$

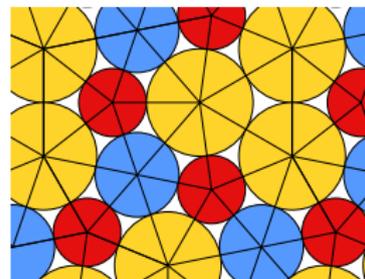


P^* of density δ^*

$$\delta(\triangle) = \delta^*$$

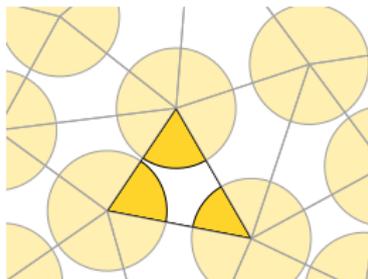


P of density $\delta(P)$



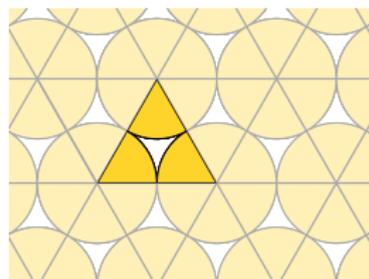
P^* of density δ^*

Local density redistribution



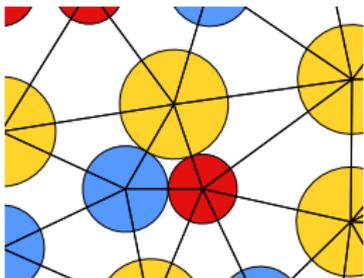
P of density $\delta(P)$

$$\forall \Delta, \delta(\Delta) \leq \delta(\triangle) = \delta^*$$



P^* of density δ^*

$$\delta(\triangle) = \delta^*$$

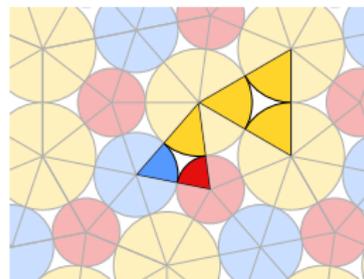


P of density $\delta(P)$

Triangles in P^* have different densities:

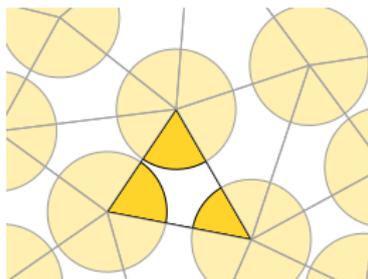
$$\delta(\triangle) < \delta^* < \delta(\triangle)$$

Hopeless to bound the density by δ^* in each triangle...



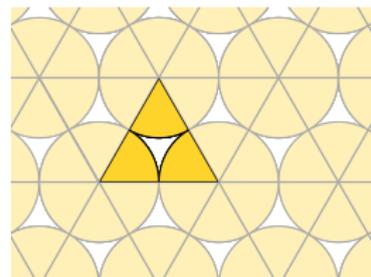
P^* of density δ^*

Local density redistribution



P of density $\delta(P)$

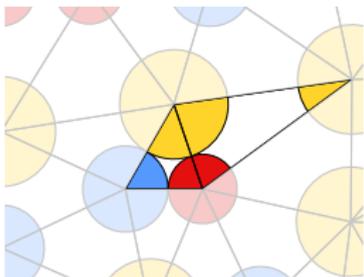
$$\forall \Delta, \delta(\Delta) \leq \delta(\triangle) = \delta^*$$



P^* of density δ^*

$$\delta(\triangle) = \delta^*$$

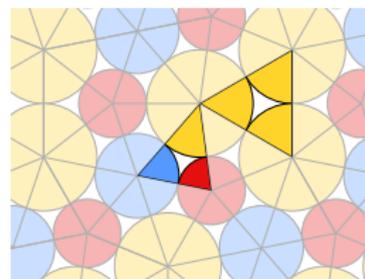
$$\delta(P) \leq \delta^*$$



P of density $\delta(P) \leq \delta'(P)$

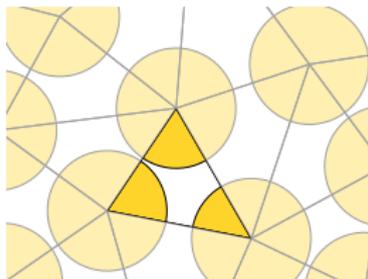
redistributed density δ' :

dense triangles
share their density
with neighbors



P^* of density δ^*

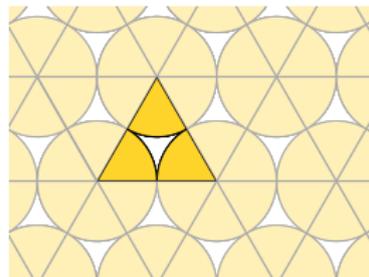
Local density redistribution



P of density $\delta(P)$

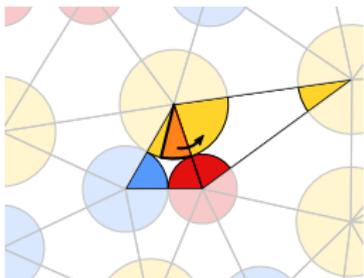
$$\forall \Delta, \delta(\Delta) \leq \delta(\triangle) = \delta^*$$

$$\delta(P) \leq \delta^*$$



P^* of density δ^*

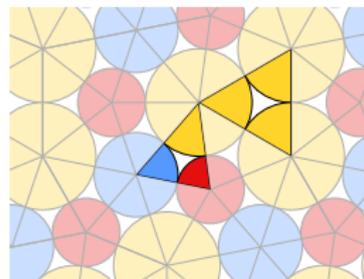
$$\delta(\triangle) = \delta^*$$



P of density $\delta(P) \leq \delta'(P)$

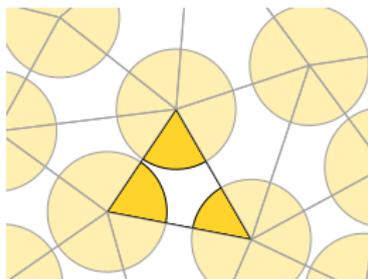
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P^* of density δ^*

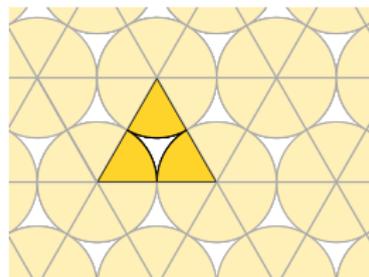
Local density redistribution



P of density $\delta(P)$

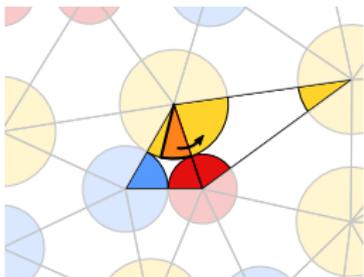
$$\forall \Delta, \delta(\Delta) \leq \delta(\triangle) = \delta^*$$

$$\delta(P) \leq \delta^*$$



P^* of density δ^*

$$\delta(\triangle) = \delta^*$$



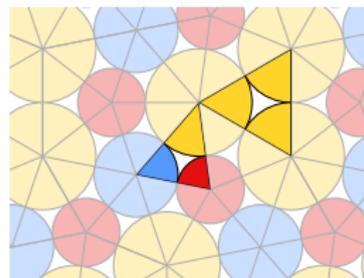
P of density $\delta(P) \leq \delta'(P)$

$$\forall \Delta, \delta'(\Delta) \leq \delta^*$$

$$\delta(P) \leq \delta'(P) \leq \delta^*$$

redistributed density δ' :

dense triangles
share their density
with neighbors



P^* of density δ^*

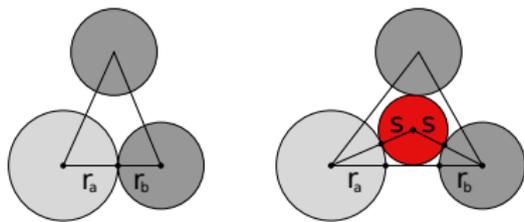
Verifying inequalities on compact sets

How to check $\delta'(\Delta) \leq \delta^*$ on each possible triangle Δ ? (there is a continuum of them)

Verifying inequalities on compact sets

How to check $\delta'(\Delta) \leq \delta^*$ on each possible triangle Δ ? (there is a continuum of them)

FM-triangulation properties + saturation \Rightarrow uniform bound on edge length

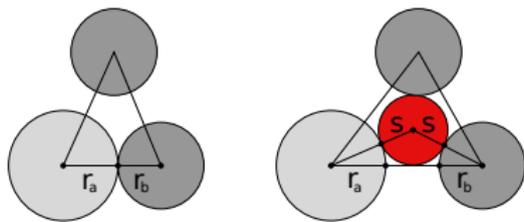


$$r_a + r_b \leq c \leq r_a + r_b + 2s$$

Verifying inequalities on compact sets

How to check $\delta'(\Delta) \leq \delta^*$ on each possible triangle Δ ? (there is a continuum of them)

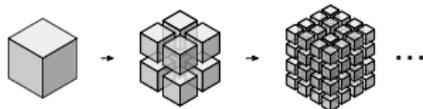
FM-triangulation properties + saturation \Rightarrow uniform bound on edge length



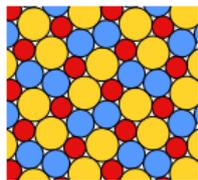
$$r_a + r_b \leq c \leq r_a + r_b + 2s$$

- Interval arithmetic: to verify $\delta'(\Delta_{a,b,c}) \leq \delta^*$ for all $(a, b, c) \in [\underline{a}, \bar{a}] \times [\underline{b}, \bar{b}] \times [\underline{c}, \bar{c}]$, we verify $[\underline{\delta}, \bar{\delta}] \leq \delta^*$ where $[\underline{\delta}, \bar{\delta}] = \delta'(\Delta_{[\underline{a}, \bar{a}], [\underline{b}, \bar{b}], [\underline{c}, \bar{c}]})$

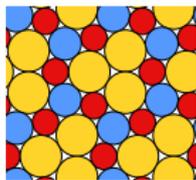
- If $\delta^* \in [\underline{\delta}, \bar{\delta}]$, recursive subdivision:



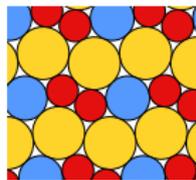
Our proof worked for these cases:



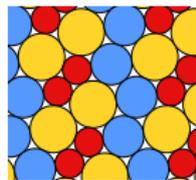
$r \approx 0.8$ $s \approx 0.7$ $\delta^* \approx 90.9\%$



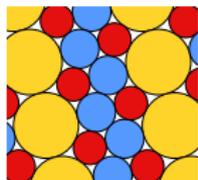
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 90.9\%$



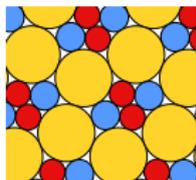
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 91\%$



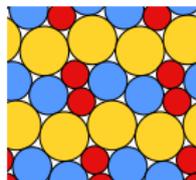
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 90.9\%$



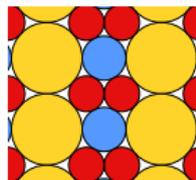
$r \approx 0.6$ $s \approx 0.5$ $\delta^* \approx 91.2\%$



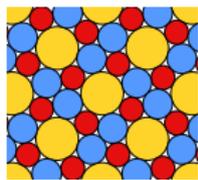
$r \approx 0.5$ $s \approx 0.4$ $\delta^* \approx 91.5\%$



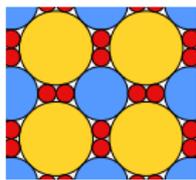
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 91\%$



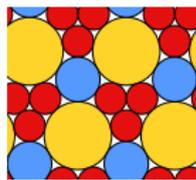
$r \approx 0.6$ $s \approx 0.5$ $\delta^* \approx 91.3\%$



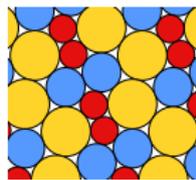
$r \approx 0.7$ $s \approx 0.5$ $\delta^* \approx 91.1\%$



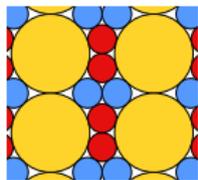
$r \approx 0.7$ $s \approx 0.2$ $\delta^* \approx 93.2\%$



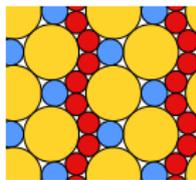
$r \approx 0.7$ $s \approx 0.5$ $\delta^* \approx 91.5\%$



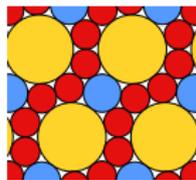
$r \approx 0.8$ $s \approx 0.6$ $\delta^* \approx 91\%$



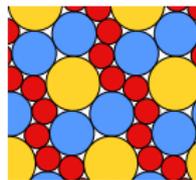
$r \approx 0.4$ $s \approx 0.4$ $\delta^* \approx 92.2\%$



$r \approx 0.5$ $s \approx 0.4$ $\delta^* \approx 91.8\%$



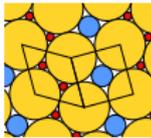
$r \approx 0.5$ $s \approx 0.4$ $\delta^* \approx 91.7\%$



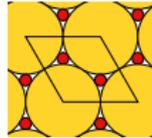
$r \approx 0.8$ $s \approx 0.5$ $\delta^* \approx 91.2\%$

And these:

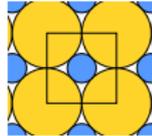
$$\delta^* \approx 93\%$$



$$\delta^* \approx 95\%$$

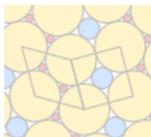


$$\delta^* \approx 92\%$$

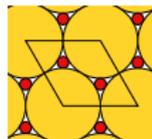


And these:

$$\delta^* \approx 93\%$$



$$\delta^* \approx 95\%$$

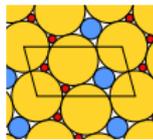


$$\delta^* \approx 92\%$$

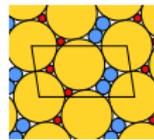


Disk packings

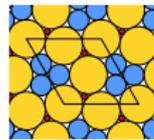
And these:



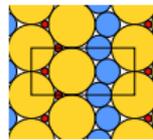
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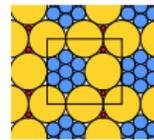
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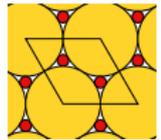
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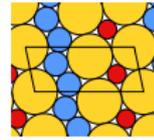
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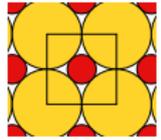
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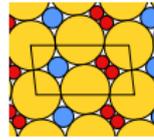
b_8



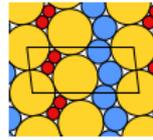
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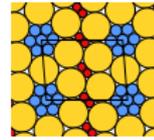
b_4



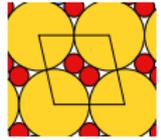
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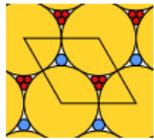
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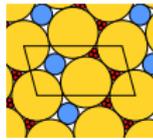
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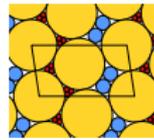
b_7



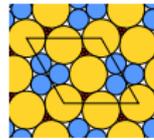
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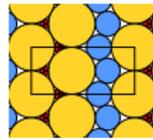
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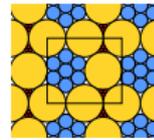
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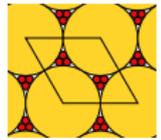
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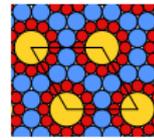
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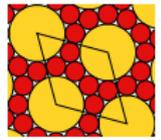
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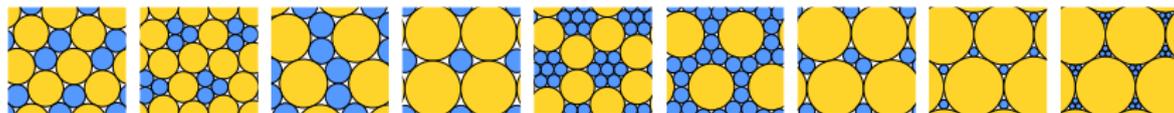
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19

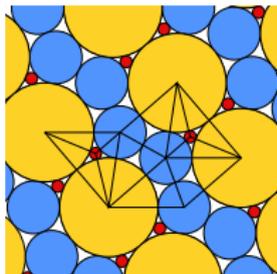


b_6

45 counter examples: *flip-and-flow method*

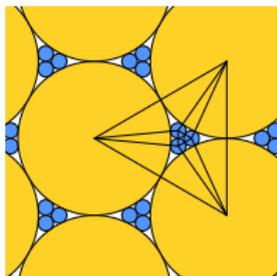
When the ratio of two disks is close enough to the ratio in a **dense binary packing**, we can pack these disks in a **similar (non triangulated) manner** and still get high density

triangulated ternary
packing

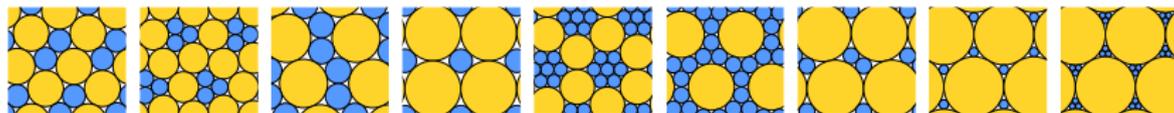


$$\delta \leq 0.931369 \quad s \approx 0.121445$$

dense binary
packing

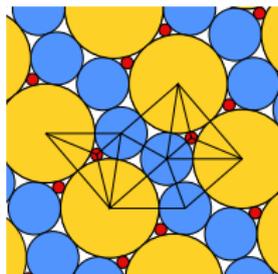


$$\delta \approx 0.962430 \quad r \approx 0.101021$$

45 counter examples: *flip-and-flow method*

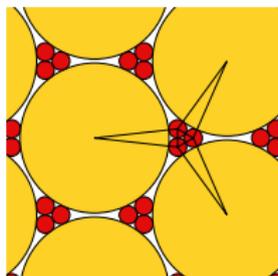
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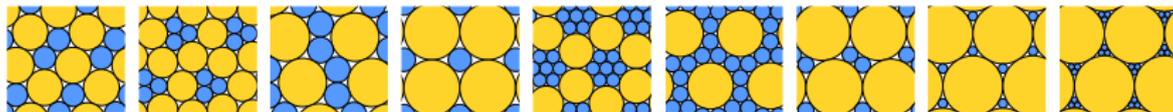
$$\delta \leq 0.931369 \quad s \approx 0.121445$$

dense non-triangulated
packing



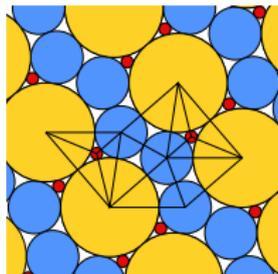
$$\delta \geq 0.937371 \quad s \approx 0.121445$$

45 counter examples: *flip-and-flow* method

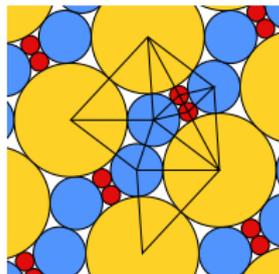


When the ratio of two disks is close enough to the ratio in a **dense binary packing**, we can pack these disks in a **similar (non triangulated) manner** and still get high density

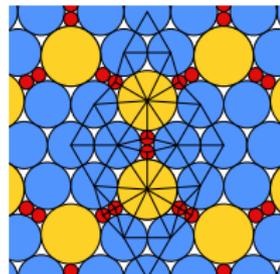
triangulated ternary packing



$$\delta \leq 0.931369 \quad s \approx 0.121445$$

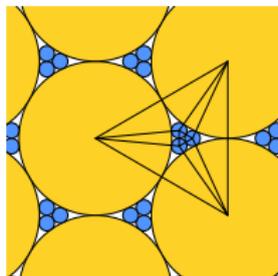


$$\delta \leq 0.924522 \quad s \approx 0.166169$$

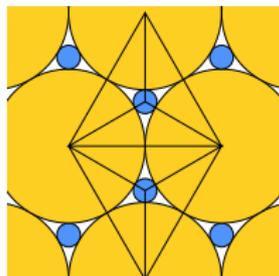


$$\delta \leq 0.917352 \quad s \approx 0.240205$$

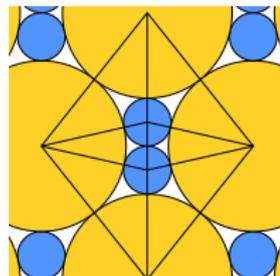
dense binary packing



$$\delta \approx 0.962430 \quad r \approx 0.101021$$

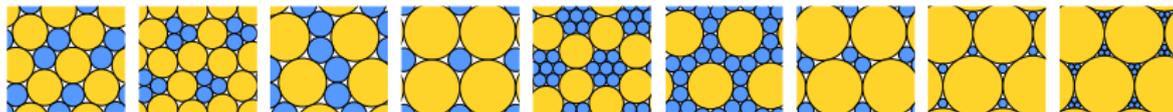


$$\delta \approx 0.950308 \quad r \approx 0.154701$$



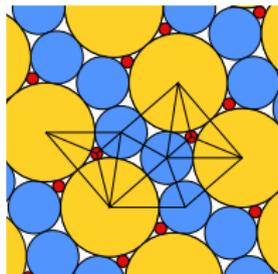
$$\delta \approx 0.931901 \quad r \approx 0.280776$$

45 counter examples: *flip-and-flow* method

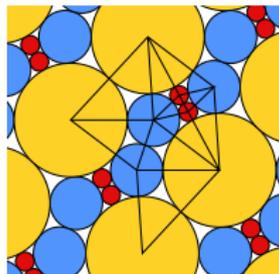


When the ratio of two disks is close enough to the ratio in a **dense binary packing**, we can pack these disks in a **similar (non triangulated) manner** and still get high density

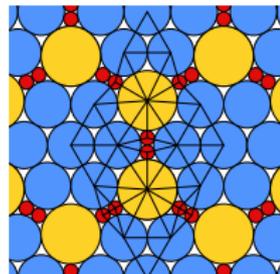
triangulated ternary packing



$$\delta \leq 0.931369 \quad s \approx 0.121445$$

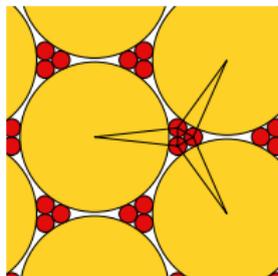


$$\delta \leq 0.924522 \quad s \approx 0.166169$$

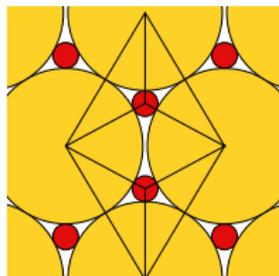


$$\delta \leq 0.917352 \quad s \approx 0.240205$$

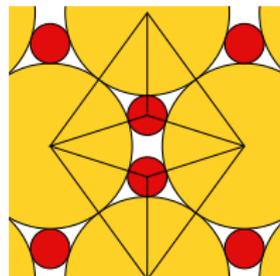
dense non-triangulated packing



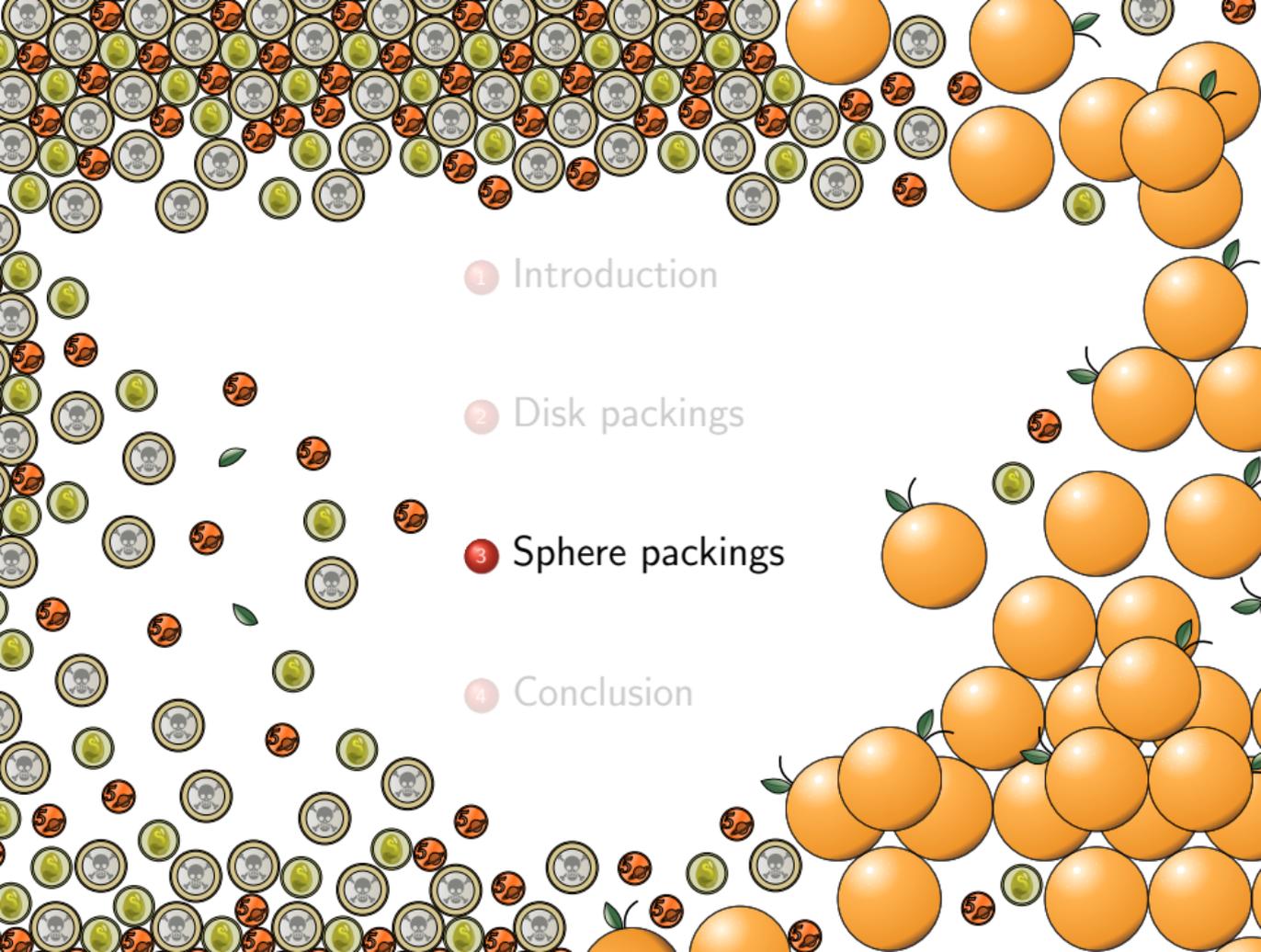
$$\delta \geq 0.937371 \quad s \approx 0.121445$$



$$\delta \geq 0.939305 \quad s \approx 0.166169$$



$$\delta \geq 0.918420 \quad s \approx 0.240205$$



1 Introduction

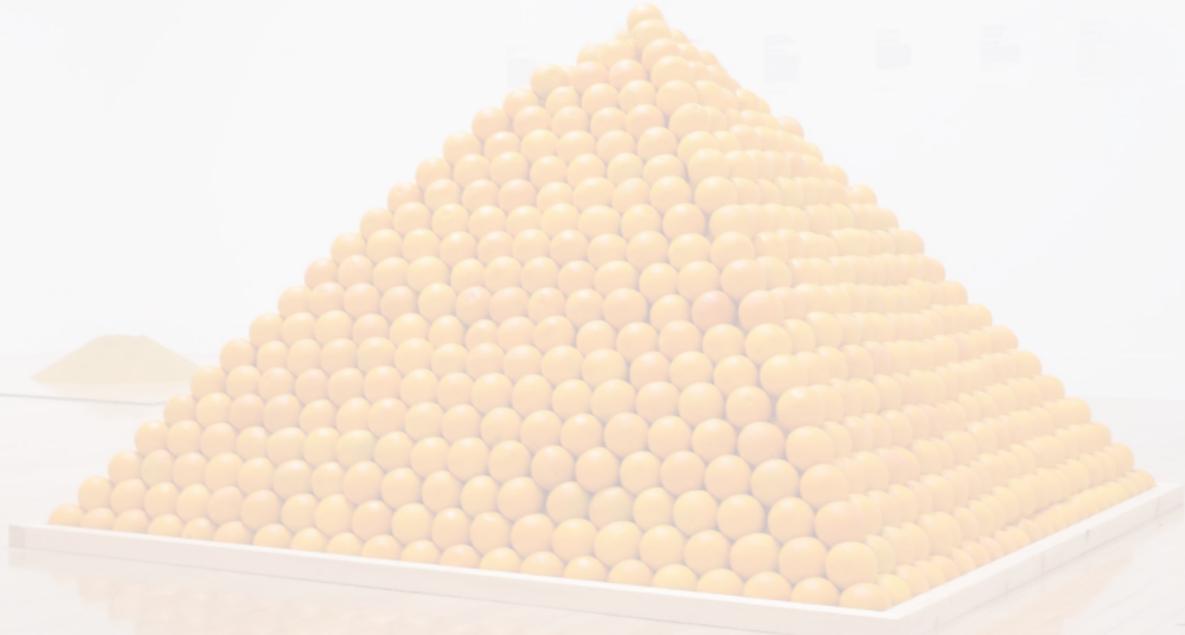
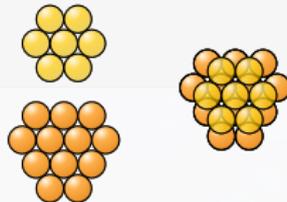
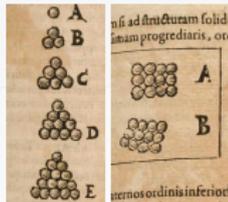
2 Disk packings

3 Sphere packings

Conclusion

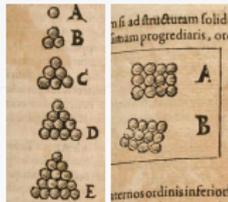
Kepler conjecture: ●-packings

3D close ●-packings



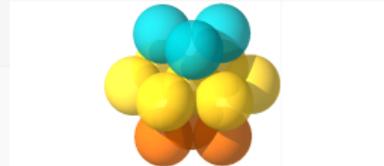
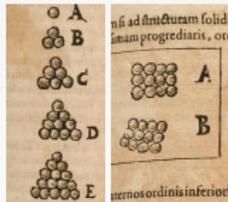
Kepler conjecture: -packings

3D close -packings



Kepler conjecture: -packings

3D close -packings



Kepler conjecture: -packings

3D close -packings

$$\delta^* = \frac{\pi}{3\sqrt{2}} \approx 74\%$$



HCP



FCC

Hales, Ferguson, 1998–2014

(Conjectured by Kepler, 1611)

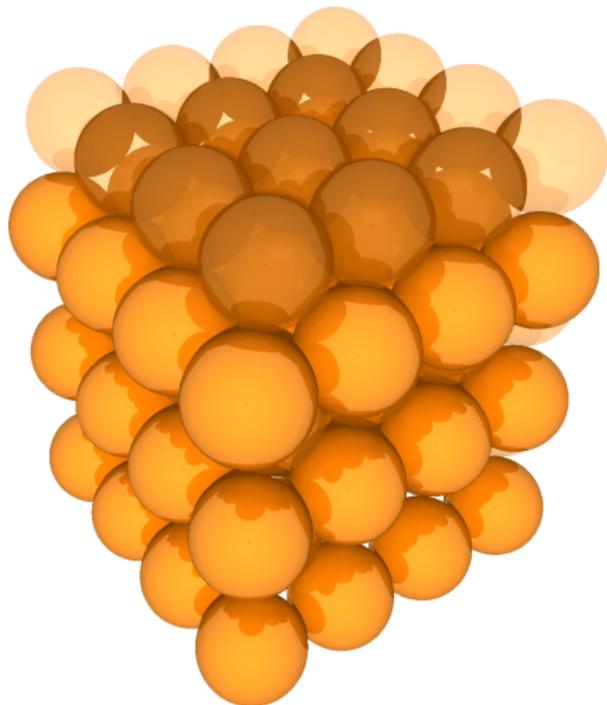
Close packings are optimal.

- close packings are optimal among lattice packings Gauss, 1831
- 18th problem of the Hilbert's list 1900
- 6 preprints by Hales and Ferguson ArXiv 1998
250 pages and > 180000 lines of code
- reviewing: 13 reviewers, 4 years... "99% certain" 1999–2003
- published proof: 300 pages, 3 computer programs DCG 2006
- Flyspeck project: formal proof (HOL Light proof assistant) 2003–2014
Forum of Mathematics, Pi 2017

sphere



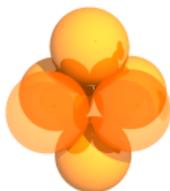
close packing



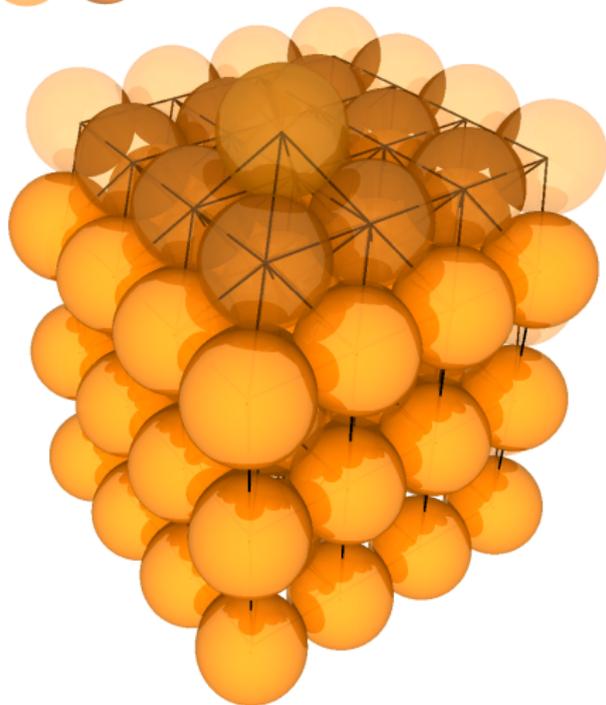
Sphere packings

Rock salt -packings

sphere

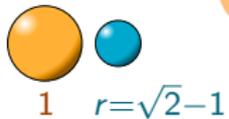


close packing

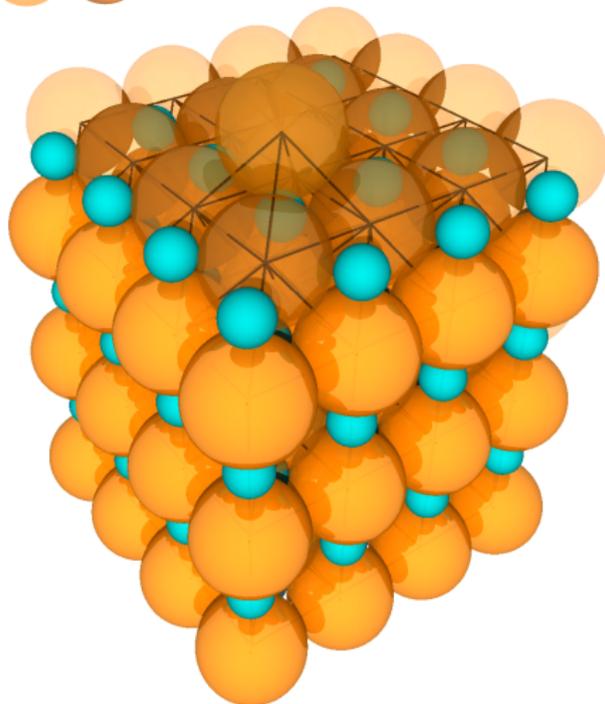


Rock salt -packings

rock salt spheres

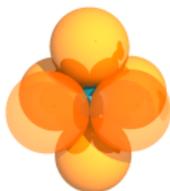
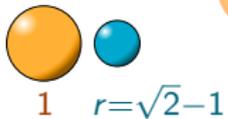


rock salt packing

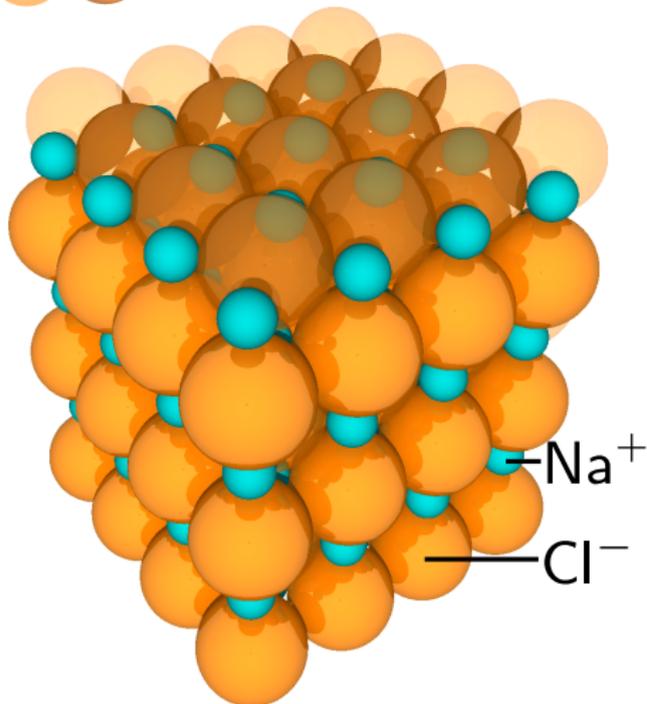


Rock salt -packings

rock salt spheres

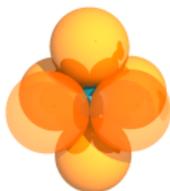


rock salt packing



Rock salt -packings

rock salt spheres

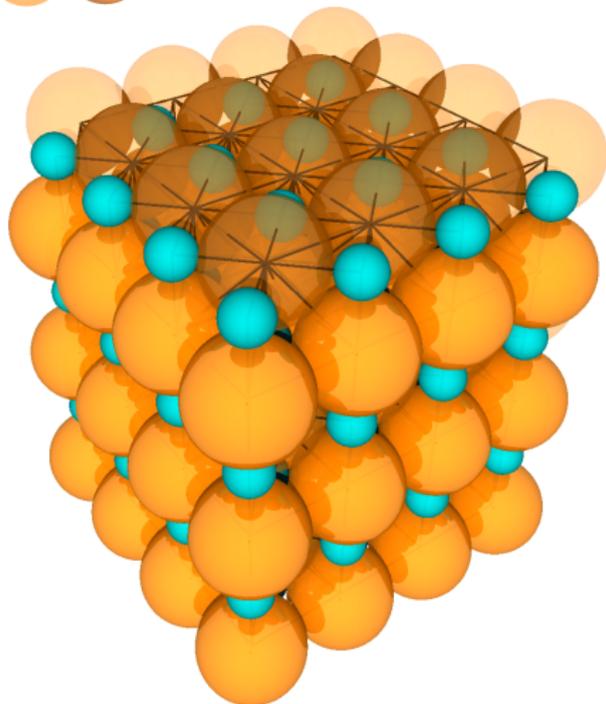


rock salt packing

triangulated \rightarrow simplicial
(contact graph is a “tetrahedration”)

Fernique, 2019

The only simplicial 2-sphere packings in 3D are rock salt packings.



Rock salt -packings

rock salt spheres



rock salt packing

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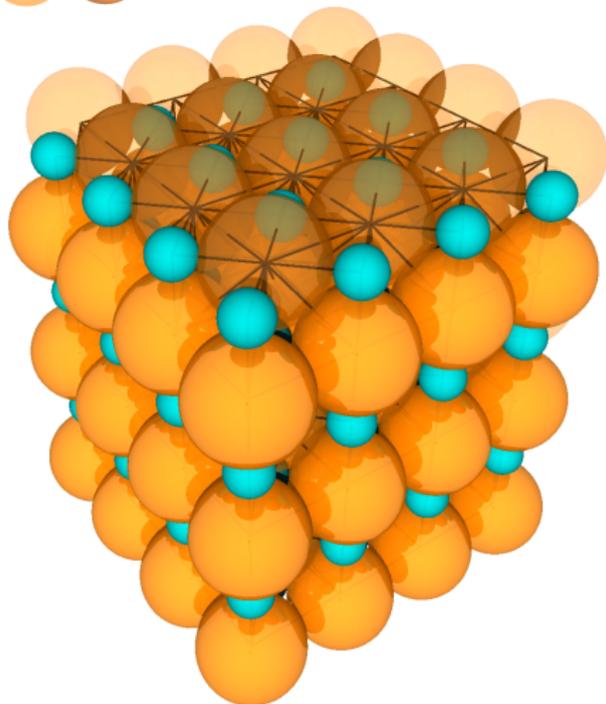
The only simplicial 2-sphere packings in 3D are rock salt packings.

Salt conjecture

open problem

Rock salt packings are optimal

$$\delta^* = \left(\frac{5}{3} - \sqrt{2}\right)\pi \approx 79\%.$$



Upper density bound for $\odot\bullet$ -packings in 2D

Florian, 1960

The density of a packing never exceeds the density in the following triangle:



Upper density bound for yellow-blue -packings in 2D

Florian, 1960

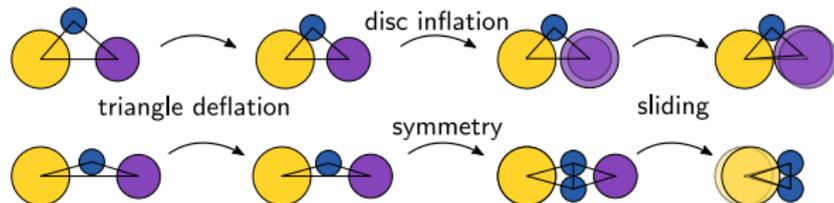
The density of a packing never exceeds the density in the following triangle:

**Proof:**

- Reduce the dimension of the set of triangles ($3 \rightarrow 1$)

Fejes Tóth, Mólnar, 1958

For any triangle, there is a denser triangle with at least two contacts between disks.



Upper density bound for $\odot\bullet$ -packings in 2D

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The density of a packing never exceeds the density in the following triangle:

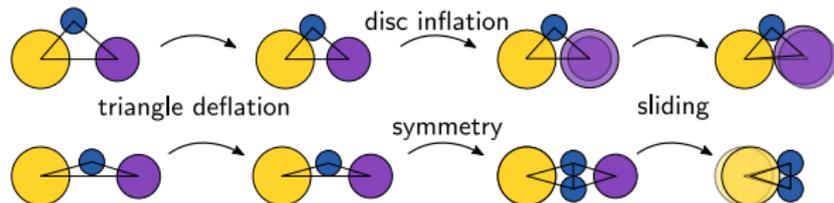


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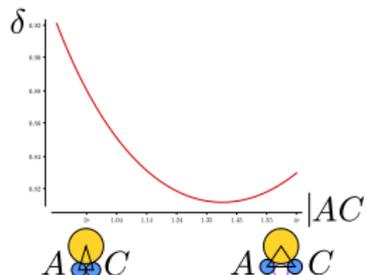
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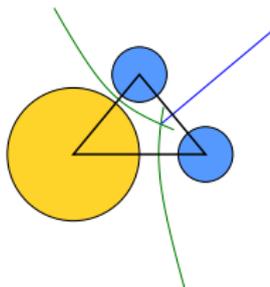


- Among all triangles with 2 contacts,  is the densest

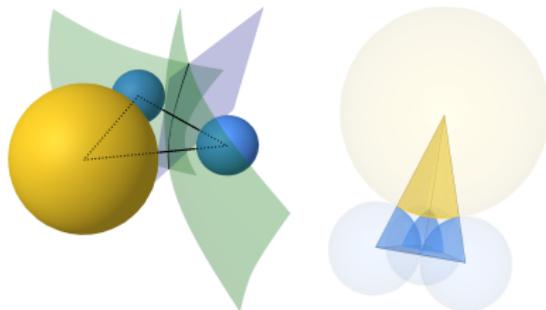


Upper density bound for $\text{orange}\text{-blue}$ -packings

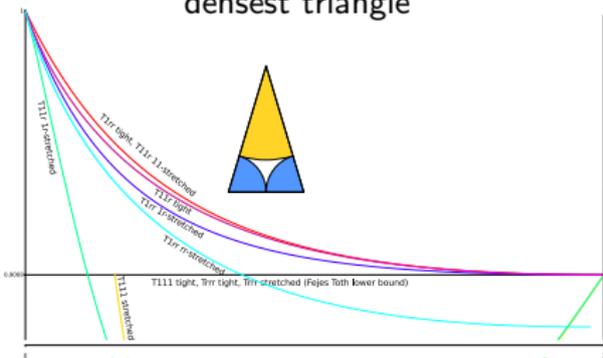
FM-triangulation (triangles)



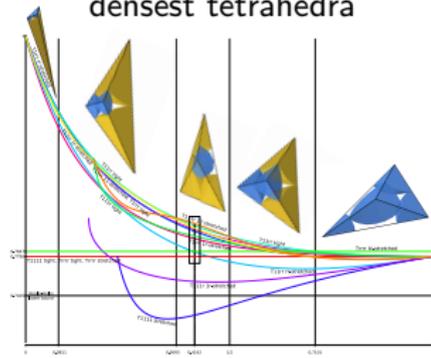
FM-simplicial partition (tetrahedra)



densest triangle



densest tetrahedra



Upper density bound for $\circlearrowleft\circlearrowright$ -packings

Theorem, $r = \sqrt{2} - 1$

in progress

Each of the following tetrahedra is densest among the tetrahedra with the same spheres:



$$\delta_{1111} \approx 0.7209$$



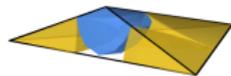
$$\delta_{11rr} \approx 0.8105$$



$$\delta_{1rrr} \approx 0.8065$$



$$\delta_{rrrr} \approx 0.7847$$



$$\delta_{111r} \approx 0.8125$$

Upper density bound for $\textcircled{O}\textcircled{B}$ -packings

Theorem, $r = \sqrt{2} - 1$

in progress

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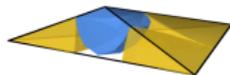
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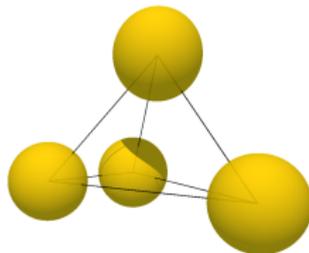
$$\delta_{rrrr} \approx 0.7847$$



$$\delta_{111r} \approx 0.8125$$

Proof:

- Reduce the dimension of the set ($6 \rightarrow 5$):
move spheres towards support sphere center



Upper density bound for $\textcircled{\small r} \textcircled{\small r} \textcircled{\small r} \textcircled{\small r}$ -packingsTheorem, $r = \sqrt{2} - 1$

in progress

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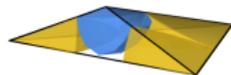
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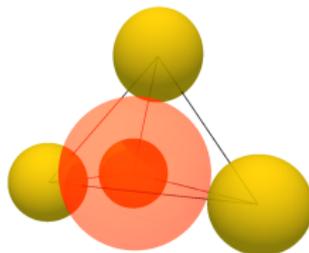
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Upper density bound for $\textcircled{\text{orange}}\textcircled{\text{blue}}$ -packings

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in progress

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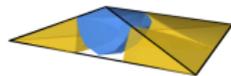
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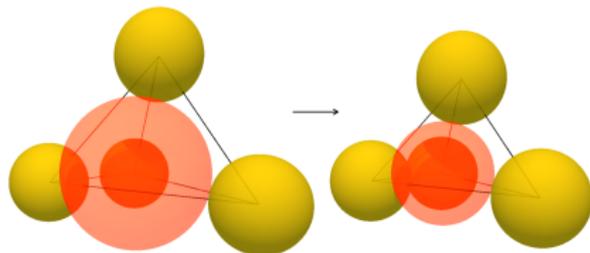
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in progress

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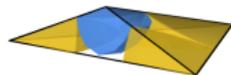
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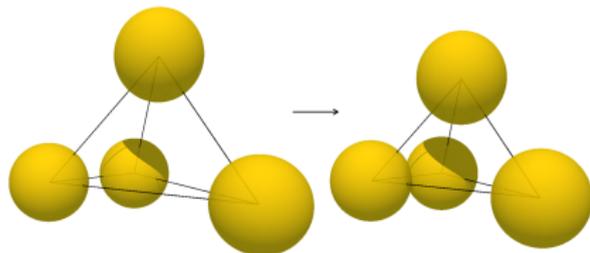
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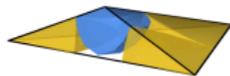
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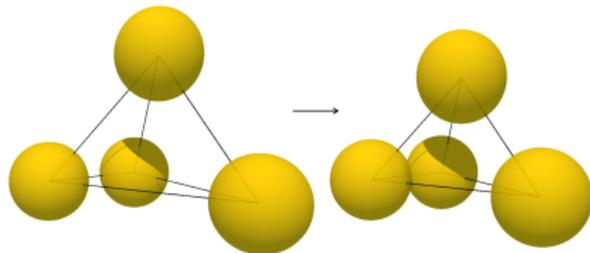
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Proof:

- Reduce the dimension of the set ($6 \rightarrow 5$):
move spheres towards support sphere center



- Computer-assisted proof for tetrahedra with one contact:
recursive subdivision + interval arithmetic
 ≈ 1000 lines of code

75 hours of CPU time

Why the computations are so slow

interval arithmetic + huge formulas \rightarrow loss of precision

Why the computations are so slow

interval arithmetic + huge formulas \rightarrow loss of precision

Example: to compute the support sphere radius, we need to solve $Ar^2 + Br + C = 0$

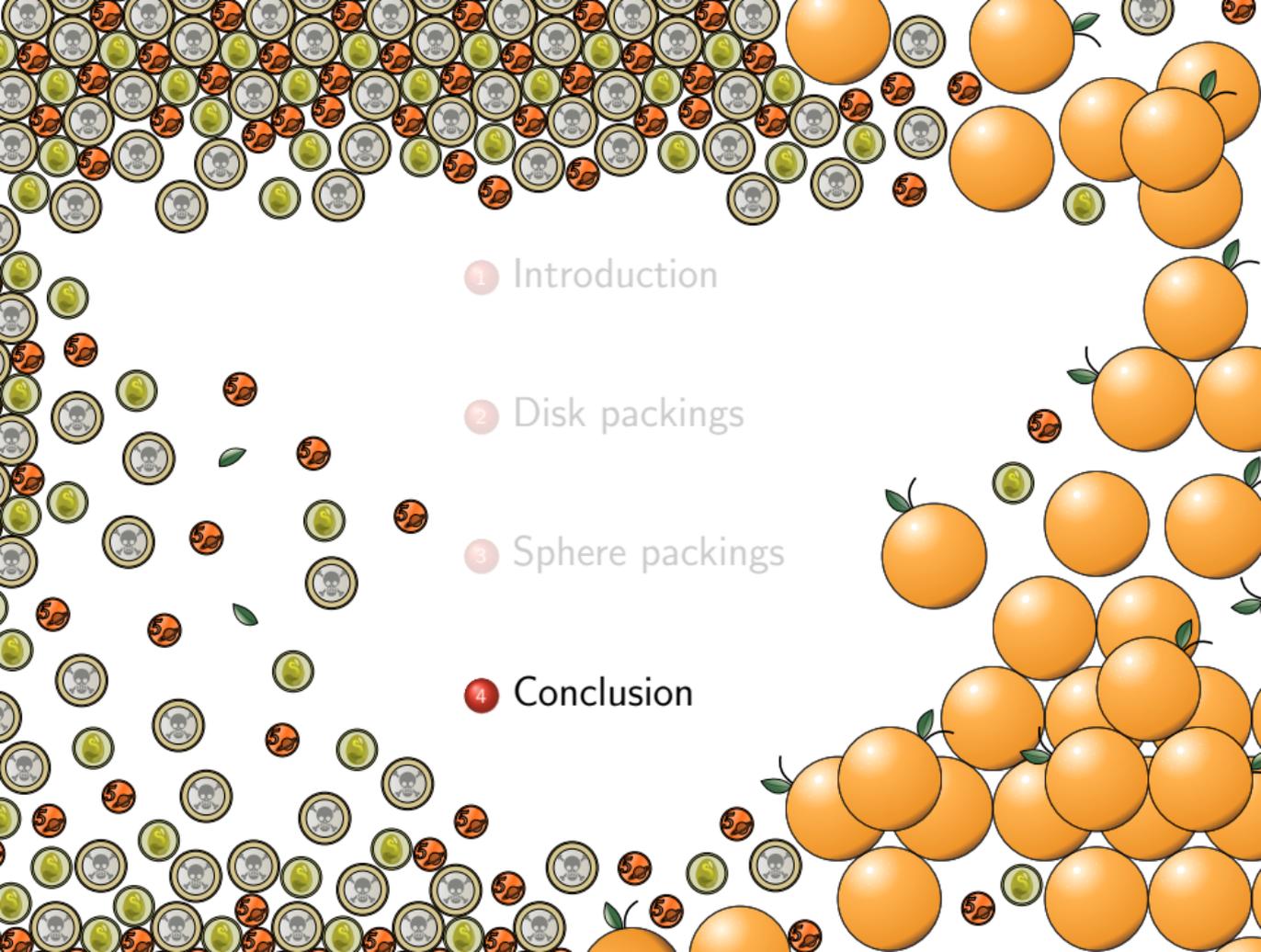
Thanks to dimension reduction:
compute with fixed radii and edge lengths, then "simplify"

$$r_x = r_y = r_z = r_w = 1, \quad a = 2;$$

$$A = 4 \left(b \left((c-4)(d-f) - (b-c-d+e-f-4)e \right) - \left((c+d-e-f-4)d + 4(e-f) \right) c - f \left(d(e-4) - 4e + 4f + 16 \right) \right)$$

$$B = 8 \left(b \left((c-4)(d-f) - (b-c-d+e-f-4)e \right) - \left((c+d-e-f-4)d + 4(e-f) \right) c - f \left(d(e-4) - 4e + 4f + 16 \right) \right)$$

$$C = b \left(4(c-4)(d-f) - e(2c(d-2) - b(e-4) - 4(d+e+f-4)) \right) + (d-4) \left(c(d(c-4) + 4(e-f)) - 4f(e-4) \right)$$



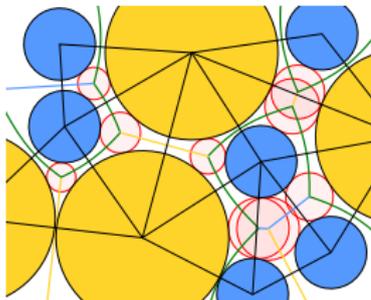
1 Introduction

2 Disk packings

3 Sphere packings

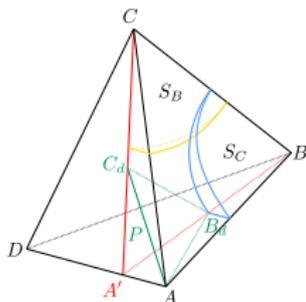
4 Conclusion

Techniques

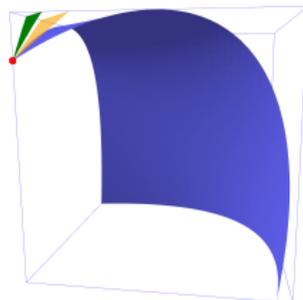


properties of triangulations

Geometry:

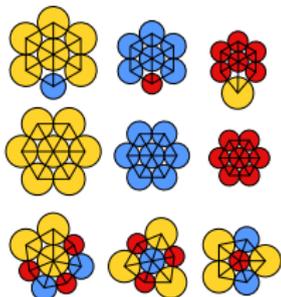


... and "tetrahedrizations"



differential geometry

Computer assistance:



case enumeration
Python, C++

$$A_{1111} = 4(d^2 - e^2)^2 + 4f^4 + ((d^2 - 8)e^2 - 8d^2)f^2$$

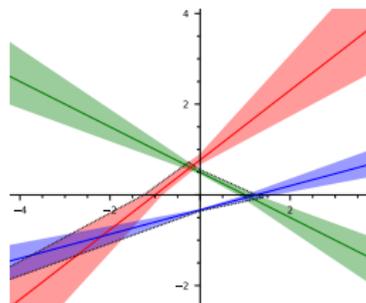
$$B_{1111} = 8(d^2 - e^2)^2 + 8f^4 + 2((d^2 - 8)e^2 - 8d^2)f^2$$

$$C_{1111} = d^2e^2f^2$$

$$8 \left(\begin{array}{l} \arctan \left(\frac{\sqrt{-4(d^2 - e^2)^2 - 4f^4 - ((d^2 - 8)e^2 - 8d^2)f^2}}{(d^2 + 2)e^2 + 2d^2 + (d^2 + 4)e^2 - 2f^2} \right) \\ + \arctan \left(\frac{\sqrt{-4(d^2 - e^2)^2 - 4f^4 - ((d^2 - 8)e^2 - 8d^2)f^2}}{(d^2 + 2)f^2 + 2d^2 - 2e^2 + (d^2 + 4)d^2} \right) \\ + \arctan \left(\frac{\sqrt{-4(d^2 - e^2)^2 - 4f^4 - ((d^2 - 8)e^2 - 8d^2)f^2}}{(d^2 + 2)f^2 - 2d^2 + 2e^2 + (d^2 + 4)f^2} \right) \\ - \arctan \left(\frac{\sqrt{-4(d^2 - e^2)^2 - 4f^4 - ((d^2 - 8)e^2 - 8d^2)f^2}}{2(d^2 + e^2 + f^2 - 3d^2)} \right) \end{array} \right)$$

$$\delta_{1111} = \frac{\quad}{\sqrt{-4(d^2 - e^2)^2 - 4f^4 - ((d^2 - 8)e^2 - 8d^2)f^2}}$$

symbolic calculus
SageMath



interval arithmetic
MPFI (RIF SageMath)
Boost (C++)



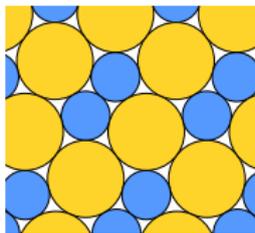
Thank you for your attention!

5 How to find triangulated packings

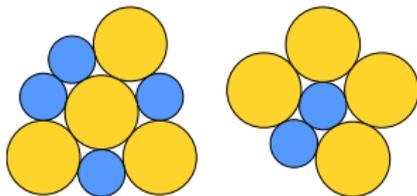
6 Other “spherical” questions: from pollen grains to kissing problem

How to find triangulated packings

packing is triangulated



each disk has a "corona"

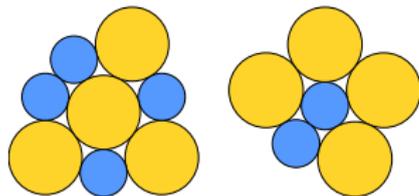
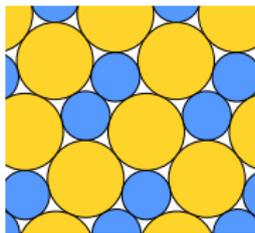


How to find triangulated packings

packing is triangulated



each disk has a "corona"



To find disk sizes with triangulated packings, we run through all possible combinations of symbolic coronas of two disks (finite number):

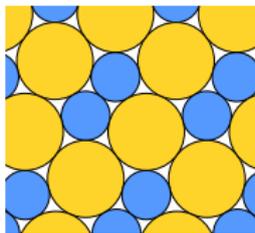
symbolic corona

$$\begin{array}{ccc} r & 1 & \\ r & 1 & r \\ 1 & r & 1 \end{array}$$

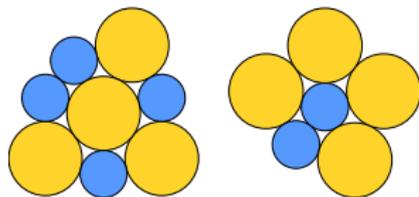
(Fernique, Hashemi, Sizova 2019)

How to find triangulated packings

packing is triangulated

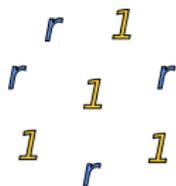


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To find disk sizes with triangulated packings, we run through all possible combinations of symbolic coronas of two disks (finite number):

symbolic corona



value of r

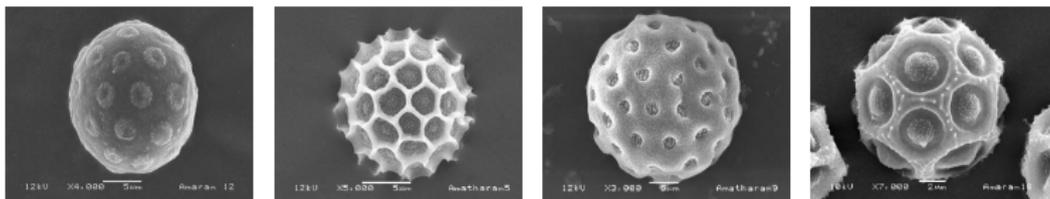
$$6 \times \widehat{11r} + 1 \times \widehat{r1r} = 2\pi$$

$$r \approx 0.63$$

(Fernique, Hashemi, Sizova 2019)

Tammes 1930: configuration of pores on a pollen grain

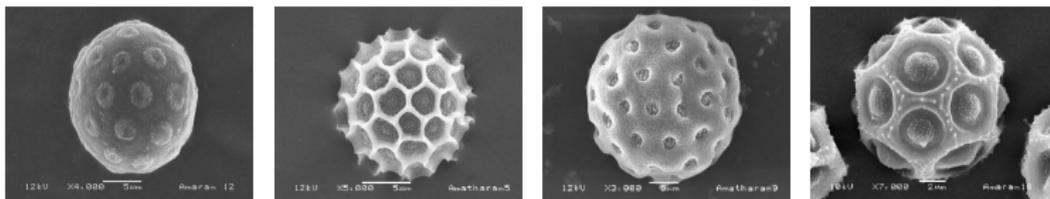
spherical codes



maximize the number of spherical caps of a given radius on a sphere

Tammes 1930: configuration of pores on a pollen grain

spherical codes

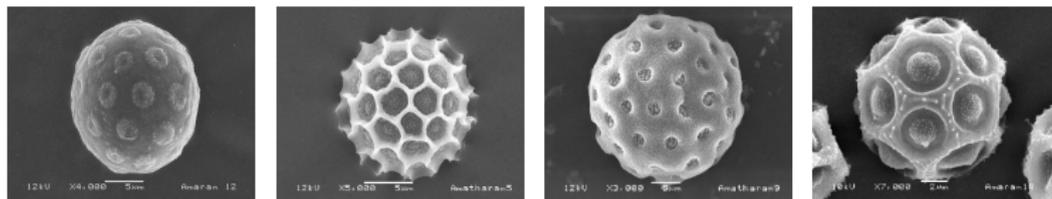


maximize the number of spherical caps of a given radius on a sphere

place n points on a sphere to maximize the distance between two nearest points

Tammes 1930: configuration of pores on a pollen grain

spherical codes



maximize the number of spherical caps of a given radius on a sphere

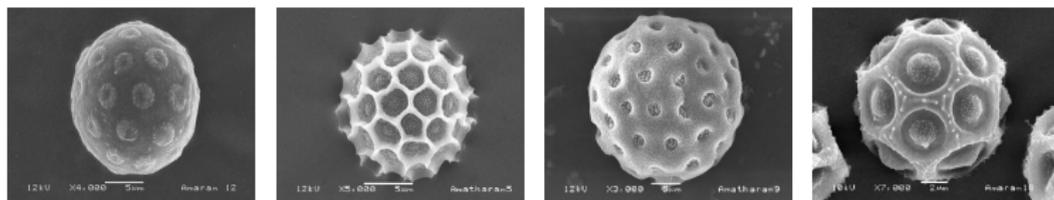
place n points on a sphere to maximize the distance between two nearest points

find the smallest possible radius of a central sphere tangent to n unit spheres

solved for $n = 3, \dots, 14$, and 24 (1943 – 2015)

Tammes 1930: configuration of pores on a pollen grain

spherical codes



maximize the number of spherical caps of a given radius on a sphere

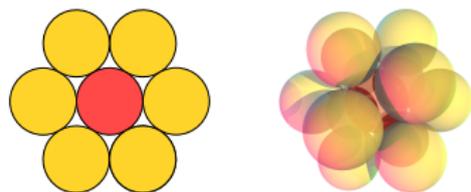
place n points on a sphere to maximize the distance between two nearest points

find the smallest possible radius of a central sphere tangent to n unit spheres

solved for $n = 3, \dots, 14$, and 24 (1943 – 2015)

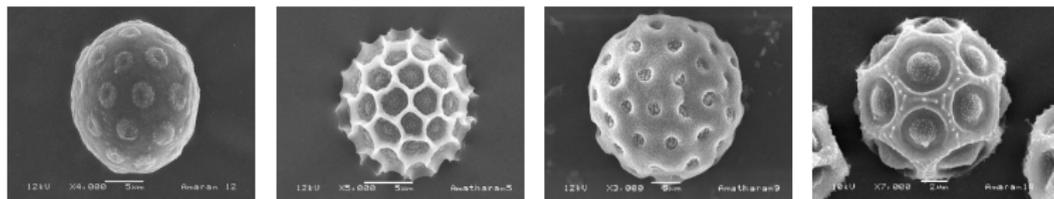
kissing number: how many unit spheres can touch the same unit central sphere in \mathbb{R}^d

solved for $d = 2 : 6, 3 :$



Tammes 1930: configuration of pores on a pollen grain

spherical codes



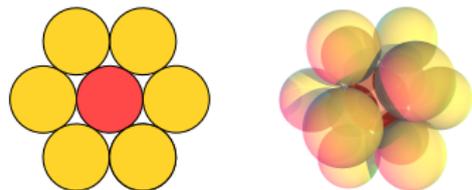
maximize the number of spherical caps of a given radius on a sphere

place n points on a sphere to maximize the distance between two nearest points

find the smallest possible radius of a central sphere tangent to n unit spheres

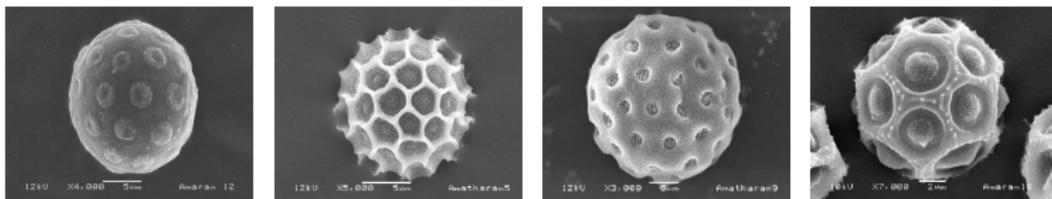
solved for $n = 3, \dots, 14$, and 24 (1943 – 2015)

kissing number: how many unit spheres can touch the same unit central sphere in \mathbb{R}^d
solved for $d = 2 : 6, 3 : 12$ (1953), $4 : 24$ (2003), $8 : 240, 24 : 196560$ (1979)



Tammes 1930: configuration of pores on a pollen grain

spherical codes



maximize the number of spherical caps of a given radius on a sphere

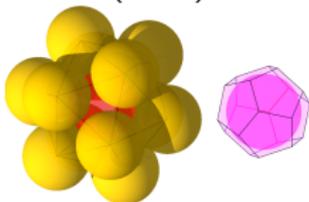
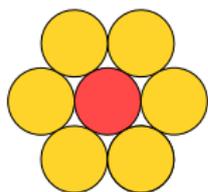
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dodecahedral conjecture

smallest Voronoi cell in sphere packing
(proved in 2010)