

# CHROMATIC QUASISYMMETRIC FUNCTIONS FOR SIGNED GRAPHS

Raquel Melgar  
(joint work with Jean-Christophe Aval)

February 2-4, 2026



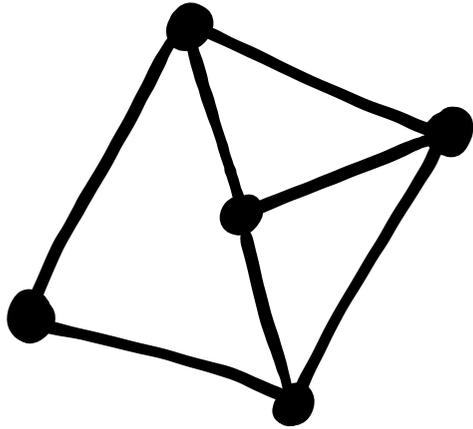
arXiv: 2508.20200

JOURNÉES DE COMBINATOIRE DE BORDEAUX

FIRST PART :

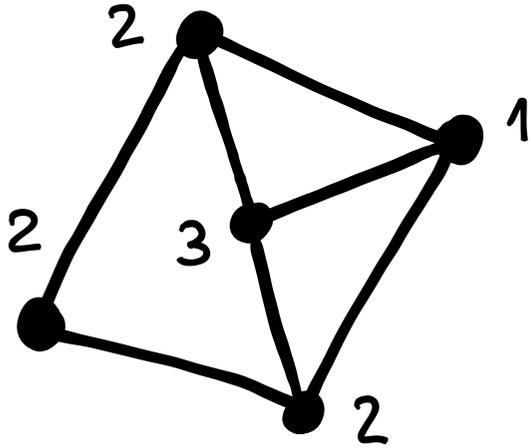
Introduction & preliminaries

Given a graph  $G = (V, E)$



A **proper coloring** of  $G$  is a map  $k: V \rightarrow \mathbb{N}$  verifying  $k(i) \neq k(j)$  for all  $\{i, j\} \in E$

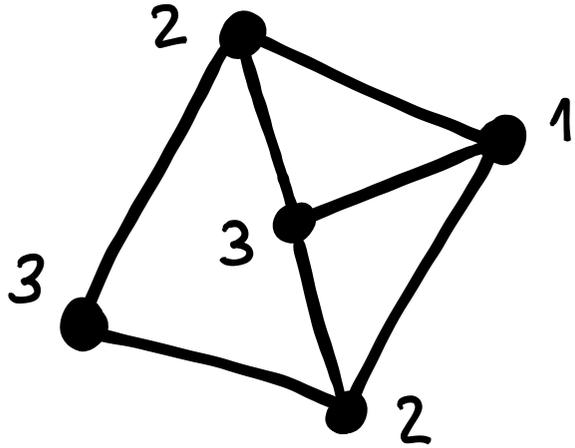
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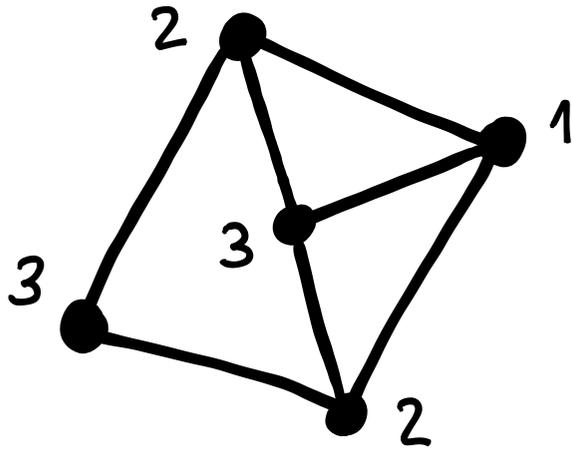
THIS IS NOT A PROPER COLORING

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The function  $\chi_G$  defined as:

$$\chi_G(k) := \# \left\{ \begin{array}{l} \text{maps } k: V \rightarrow \{1, \dots, k\} \text{ that are} \\ \text{proper colorings} \end{array} \right\}$$

is the **chromatic polynomial** of  $G$ .

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\*  $f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots) \quad \forall \sigma \in \mathfrak{S}$

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$$\text{Sym} = \bigoplus_{d=0}^{\infty} \text{Sym}^d$$

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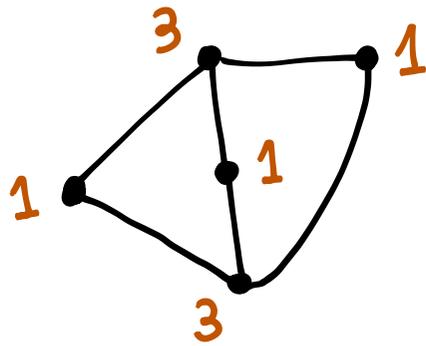
Example:

$$m_{32} = \sum_{i \neq j} x_i^3 x_j^2 \in \text{Sym}^5$$

$$\text{Sym} = \bigoplus_{d=0}^{\infty} \text{Sym}^d$$

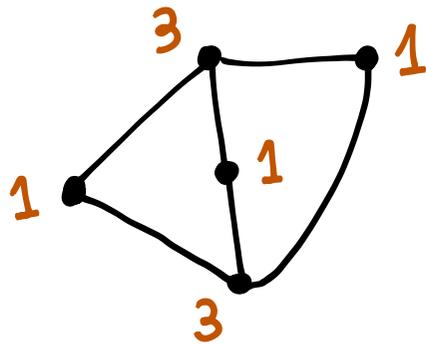
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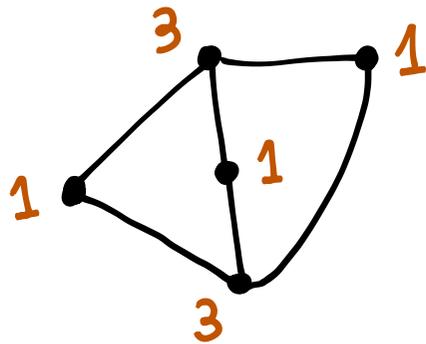
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# THE CHROMATIC SYMMETRIC FUNCTION



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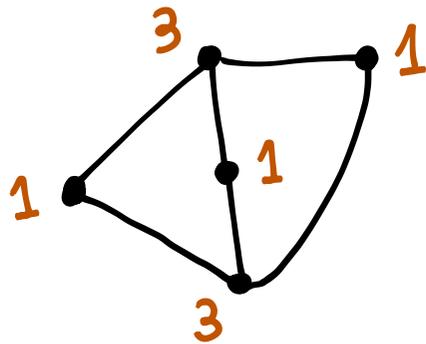
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The chromatic symmetric function of a graph  $G = (V, E)$  is

$$\chi_G(\chi_1, \chi_2, \dots) := \sum_{k \text{ proper coloring}} \chi^k \in \text{Sym}^{|V|}$$

(Stanley 1995)

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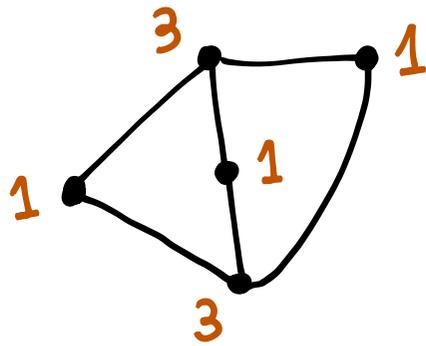
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It specializes  $\chi_G$ :

$$\chi_G(\underbrace{1, \dots, 1}_k, 0, \dots) = \chi_G(k)$$

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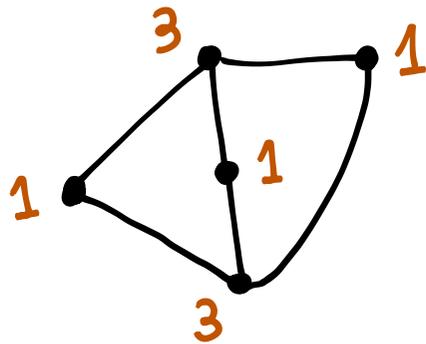
The chromatic symmetric function of a graph  $G = (V, E)$  is

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Does  $X_G$  distinguish non isomorphic graphs?

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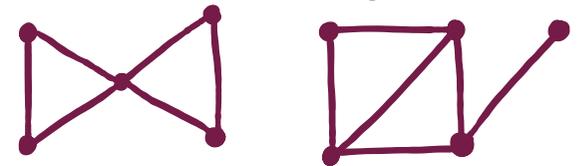
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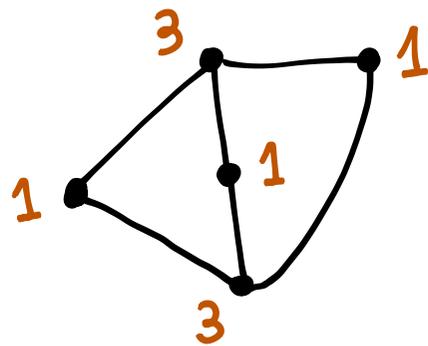
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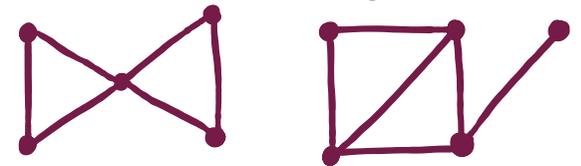
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conjectured YES for trees

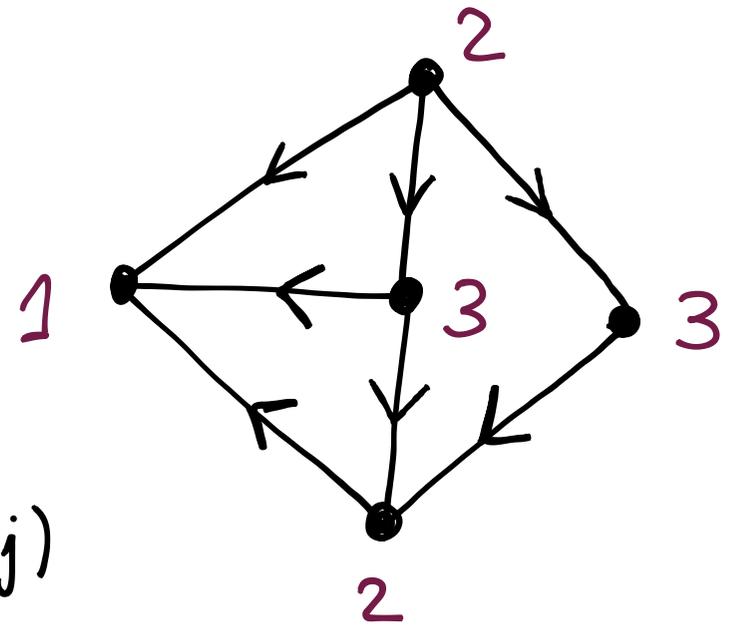
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# SECOND PART:

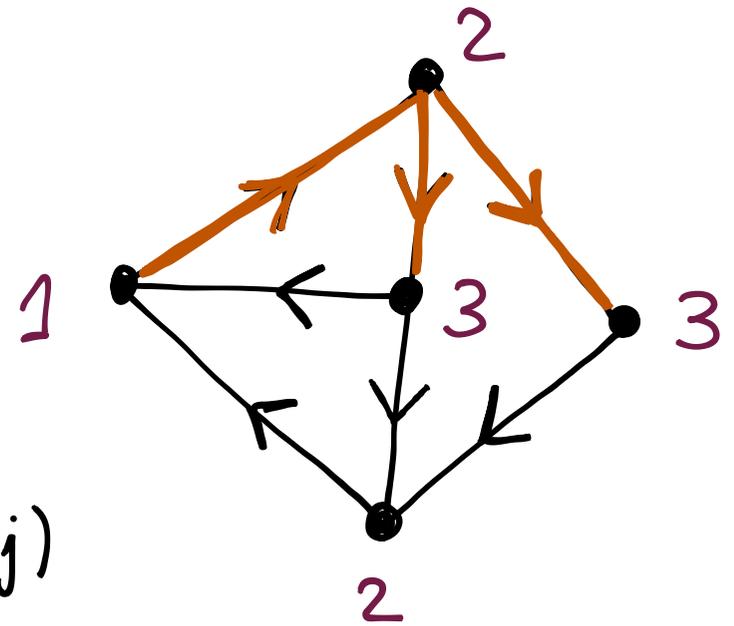
Shreshian, Wachs & Ellzey's  
invariant for directed  
graphs

Given a directed graph  $\vec{G}$   
and a coloring  $\kappa$  of  $G$ , we define  
an **ascent** of  $\kappa$  to be a directed  
edge  $(i, j) \in E$ , verifying  $\kappa(i) < \kappa(j)$

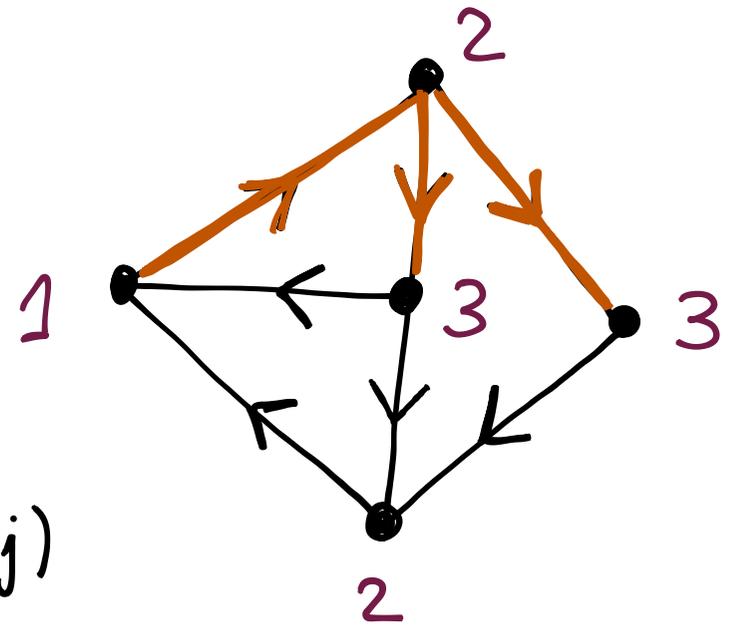
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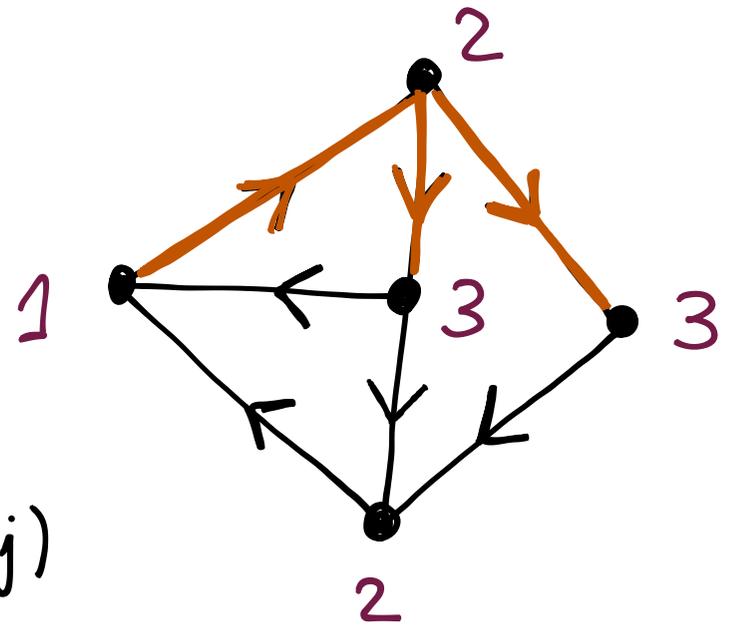


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Then the **chromatic quasisymmetric function** of  $\vec{G}$  is

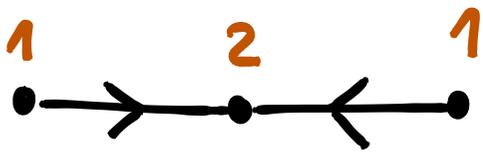
$$X_{\vec{G}}(\alpha_1, \alpha_2, \dots; t) := \sum_{\kappa \text{ proper}} \alpha^\kappa \cdot t^{\text{asc}(\kappa)}$$

AN EXAMPLE:  $\vec{G} := \bullet \rightarrow \bullet \leftarrow \bullet$

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$$\begin{aligned} \chi_{\vec{G}}(\underline{x}; t) &= \sum t^{\text{asc}(k)} \cdot x^k \\ &= t^2 \left( x_1^2 x_2 + x_1^2 x_3 + \dots \right) \end{aligned}$$

AN EXAMPLE:  $\vec{G} := \overset{1}{\bullet} \rightarrow \overset{5}{\bullet} \leftarrow \overset{3}{\bullet}$

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$$= t^2 \left( x_1^2 x_2 + x_1^2 x_3 + \dots + 2 x_1 x_3 x_5 + \dots \right)$$

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Easy to check that  $X_{\vec{G}}(\underline{x}; 1) = X_G(\underline{x})$

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In general, for a composition  $c = (c_1, c_2, \dots, c_s)$ ,

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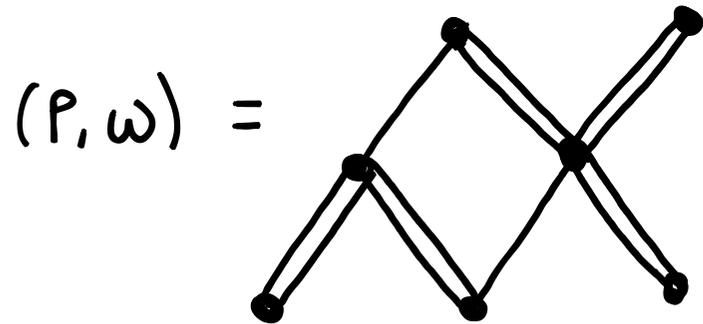
- Proposition: Let  $\vec{G}$  be a directed graph  $\chi_{\vec{G}}(\underline{x}; t) \in QSym[t]$   
[SW16]

## EXAMPLE AND FUNDAMENTAL BASIS

The enumerators of  $(P, \omega)$ -partitions are in  $\mathcal{QSym}$

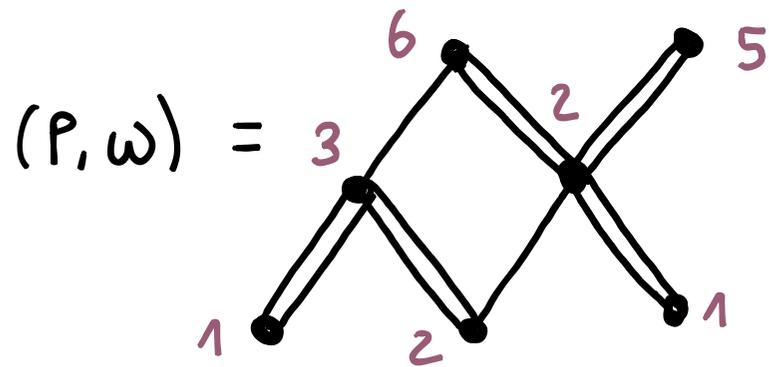
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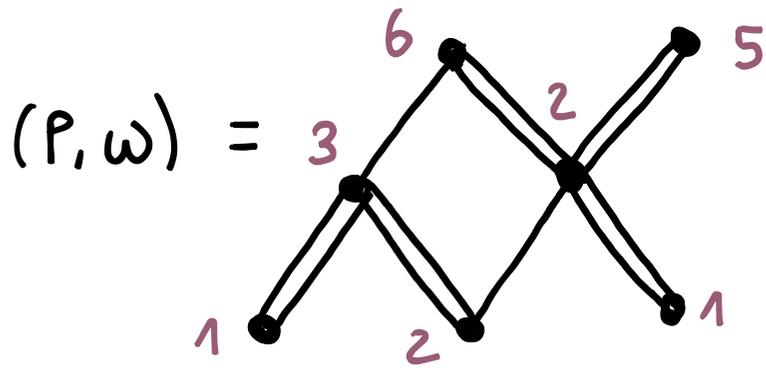
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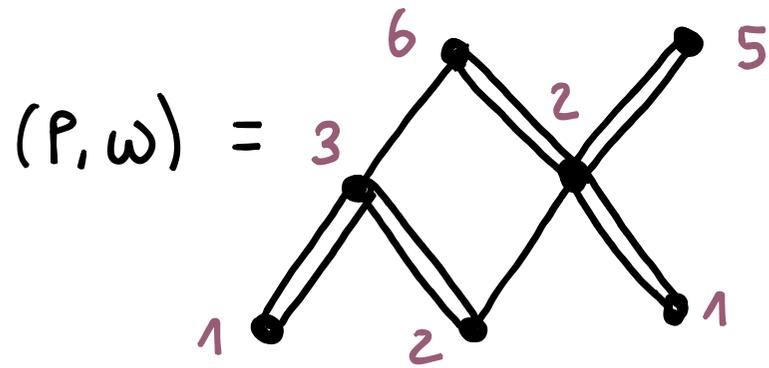
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$$I(P, \omega) := \sum_f x_{f(1)} \cdots x_{f(d)} \in \mathcal{QSym}$$

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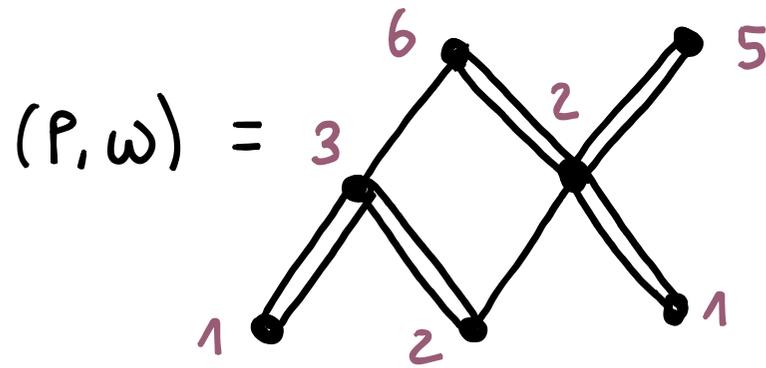
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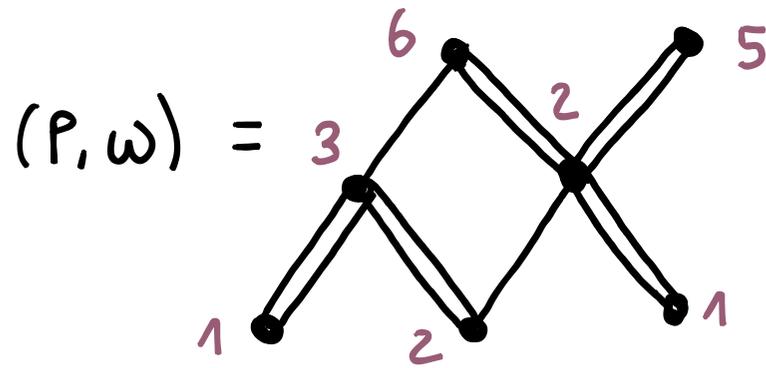
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$$I(\downarrow) = \sum_{0 \leq i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3}$$

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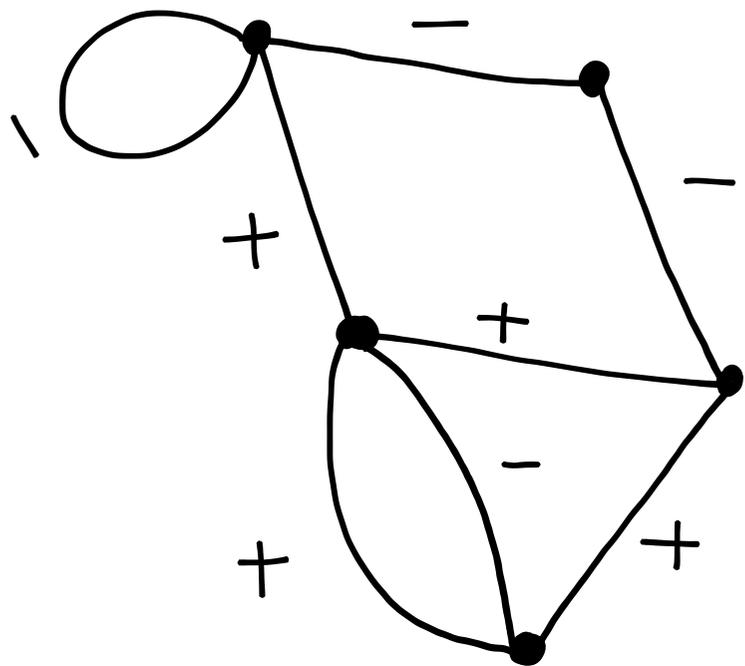
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THIRD PART:

*Signed graphs*



### III. SIGNED GRAPHS

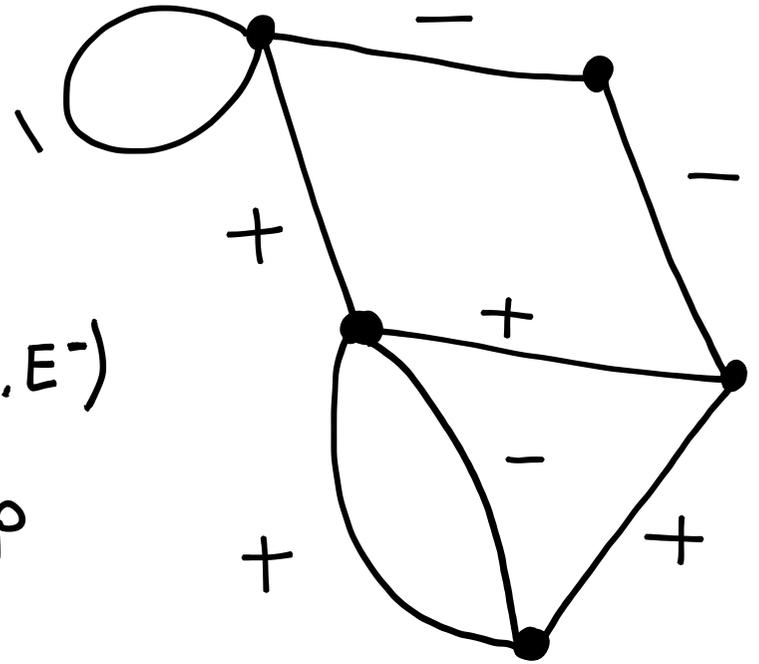
# PROPER COLORING OF SIGNED GRAPHS

Given a signed graph  $\Sigma = (V, E^+, E^-)$   
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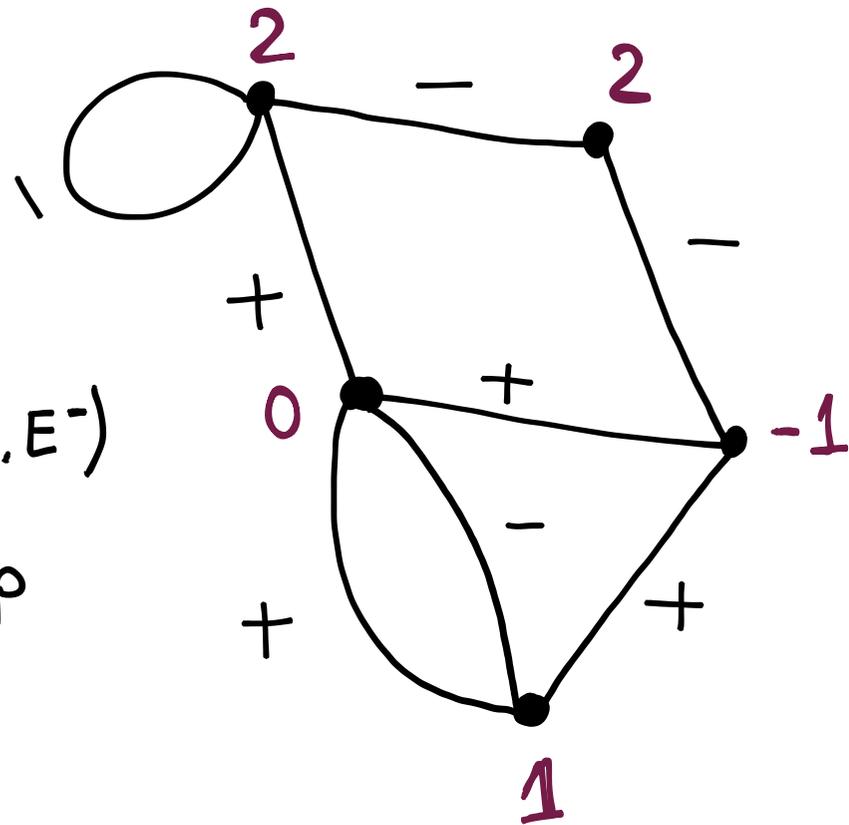
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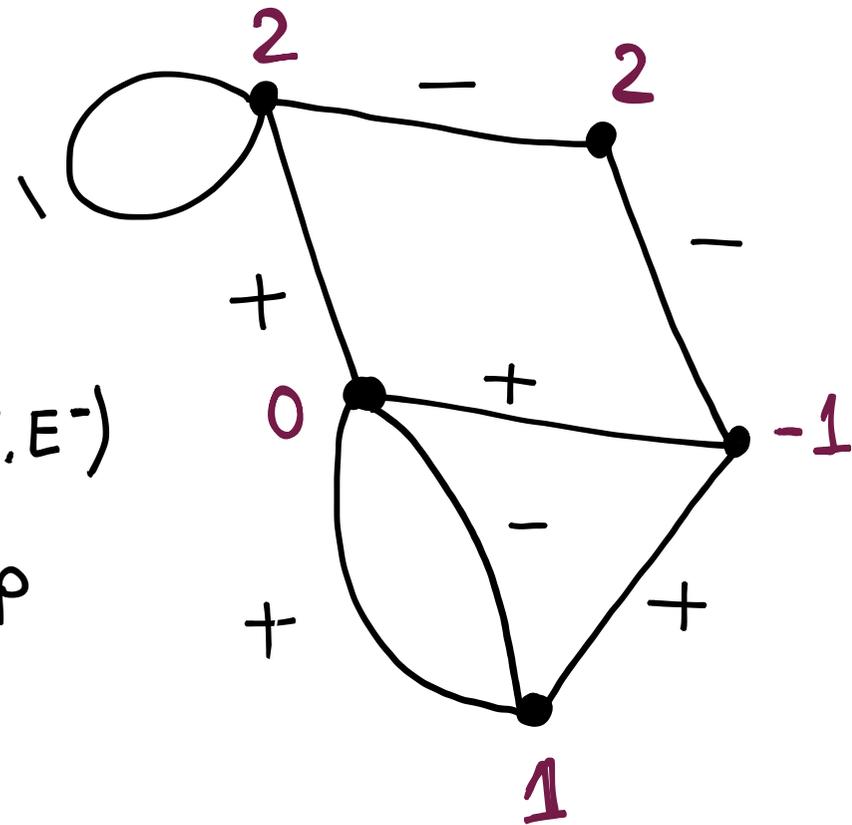
# PROPER COLORING OF SIGNED GRAPHS

Given a signed graph  $\Sigma = (V, E^+, E^-)$   
 a **proper coloring** of  $\Sigma$  is a map

$\kappa$ :  $V \longrightarrow \mathbb{Z}$  verifying:

$$* \kappa(i) \neq \kappa(j) \quad \forall \{i, j\} \in E^+$$

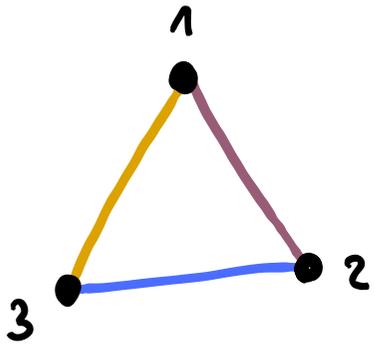
$$* \kappa(i) \neq -\kappa(j) \quad \forall \{i, j\} \in E^-$$



• Definition (Zaslavsky 80's)

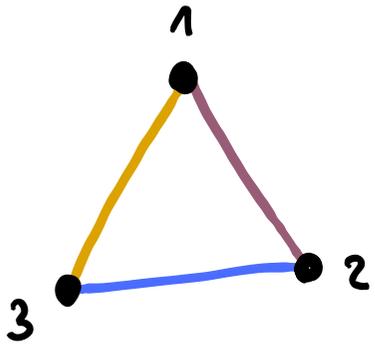
$$\chi_{\Sigma}(k) := \# \left\{ \begin{array}{l} \text{proper colorings} \\ \text{with colors} \\ \{0, \pm 1, \dots, \pm k\} \end{array} \right\}$$

# THE GEOMETRIC P.O.V. OF GRAPH COLORING

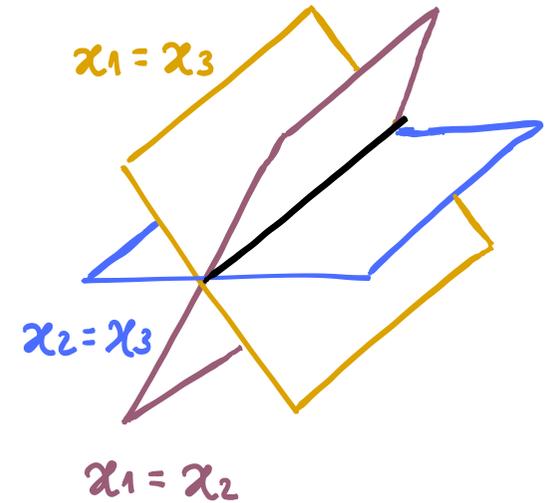


→ Consider in  $\mathbb{R}^{|V|}$   
 $H_G := \{ \{x_i = x_j\} : \{i, j\} \in E \}$

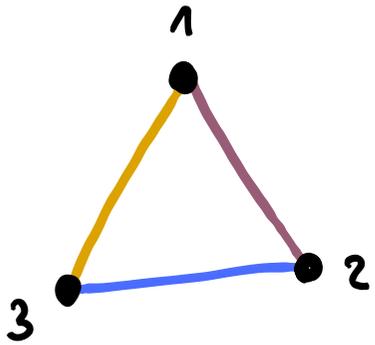
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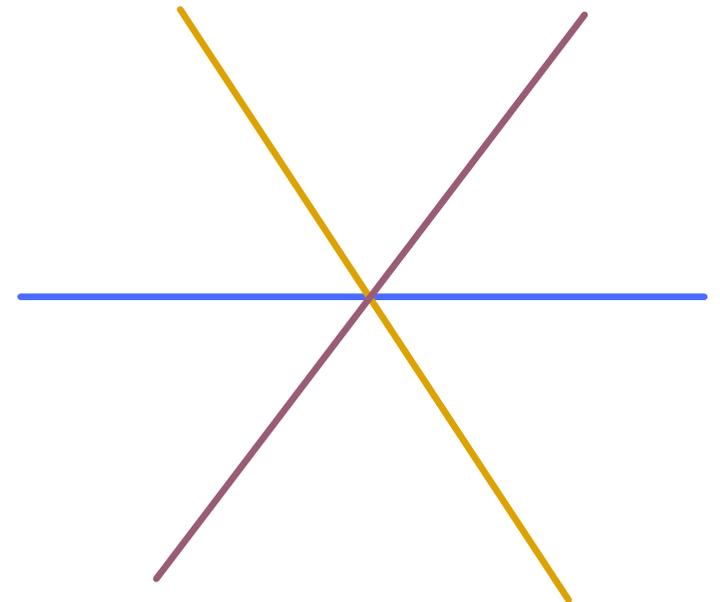
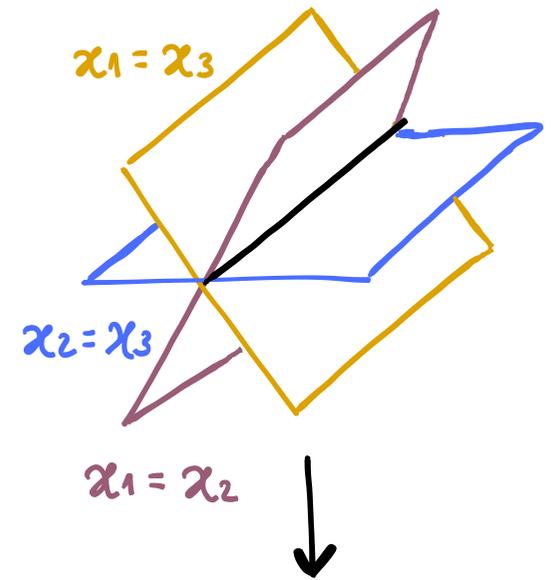
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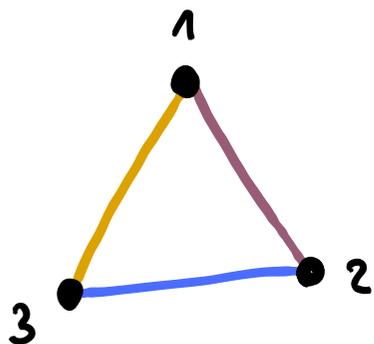
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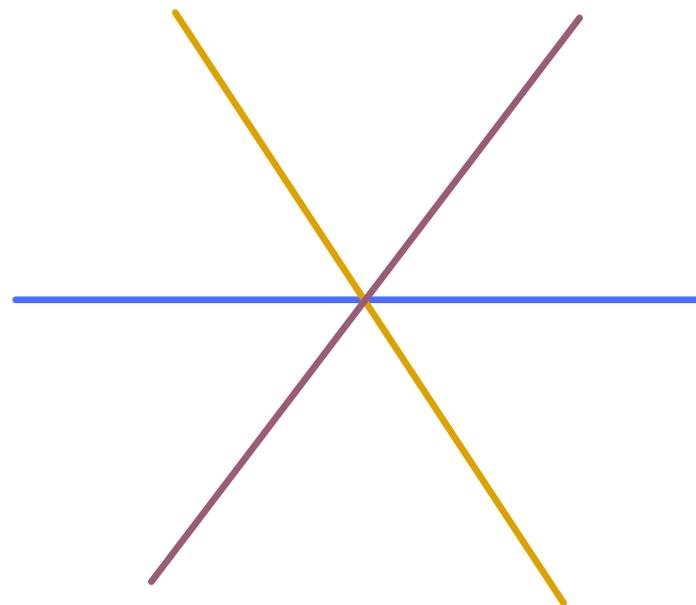
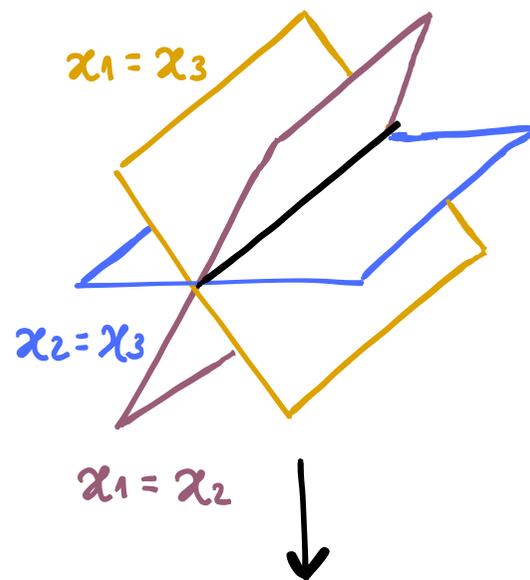
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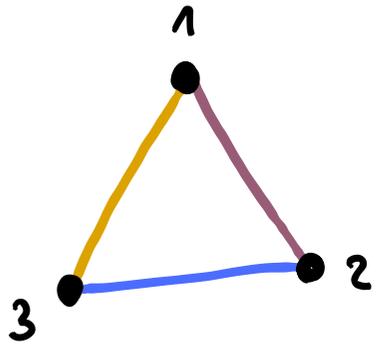
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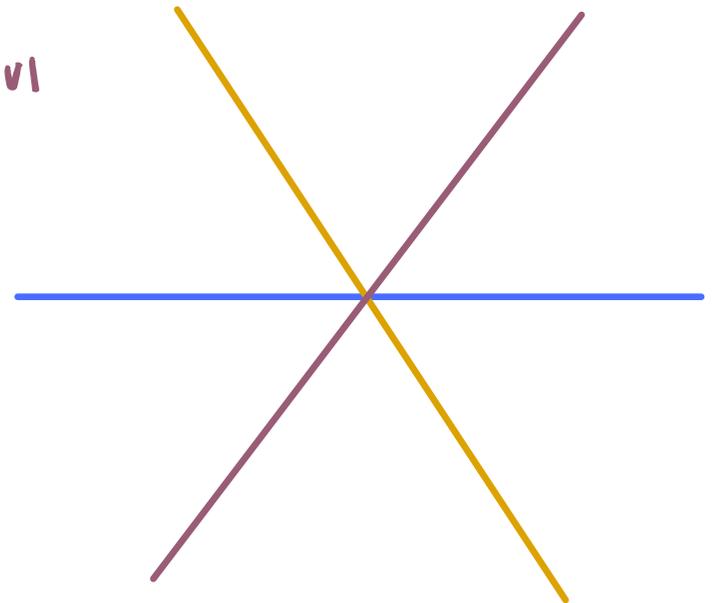
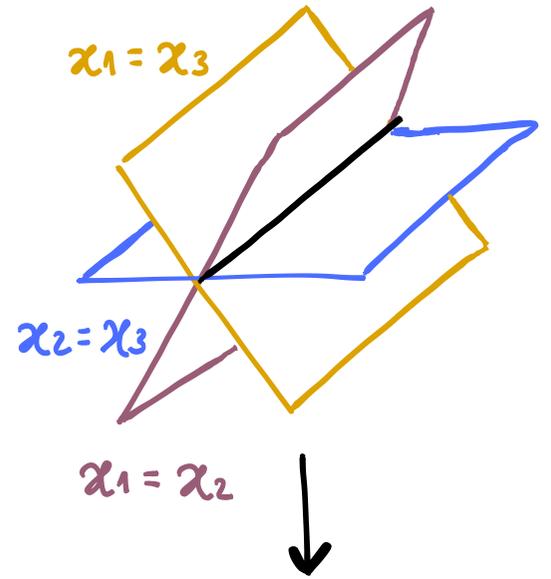


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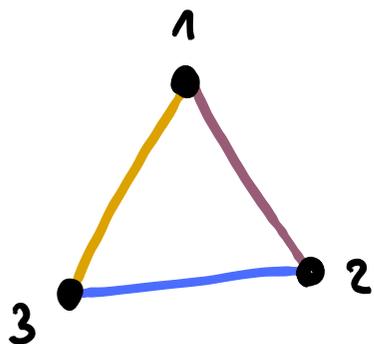
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$$\text{Then } \chi_G(\underline{x}) = \sum_{k \text{ proper}} x^k = \sum_{\alpha \in \mathbb{N}^{|V|} \setminus HG} x_{\alpha_1} \cdots x_{\alpha_{|V|}}$$

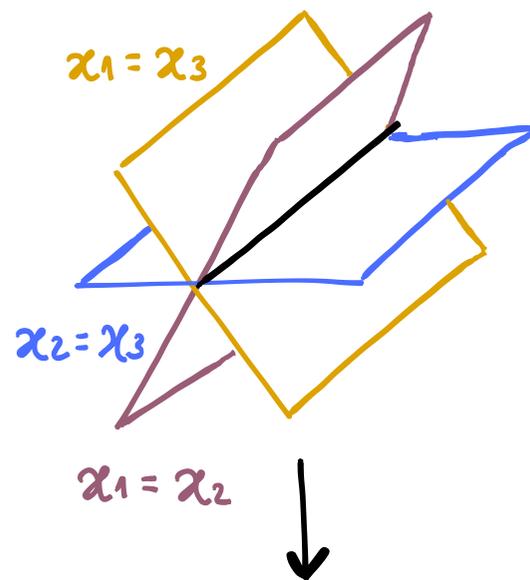


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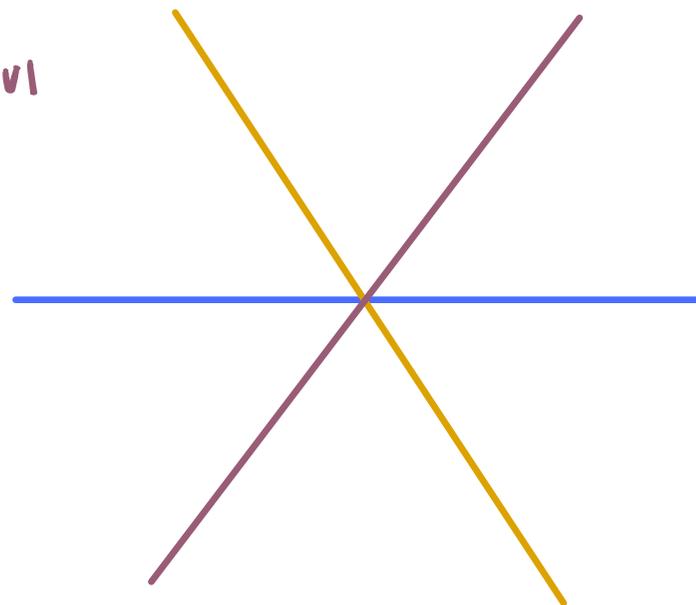
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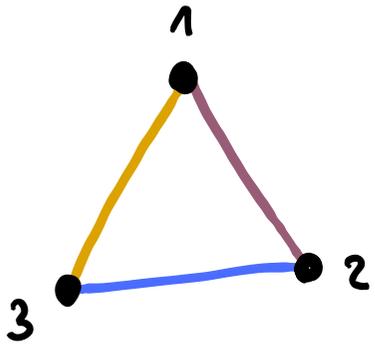
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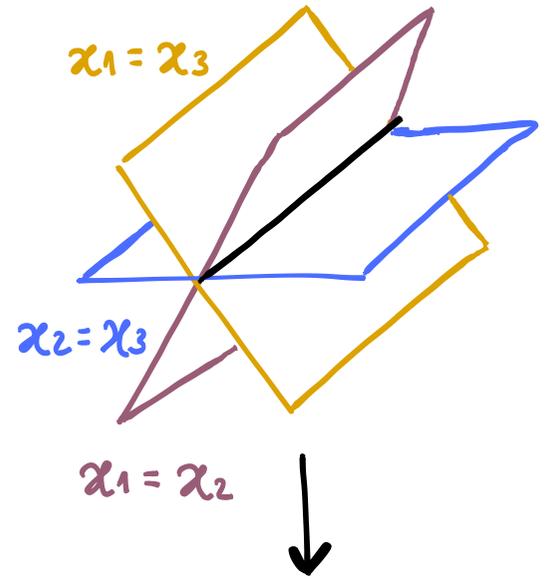


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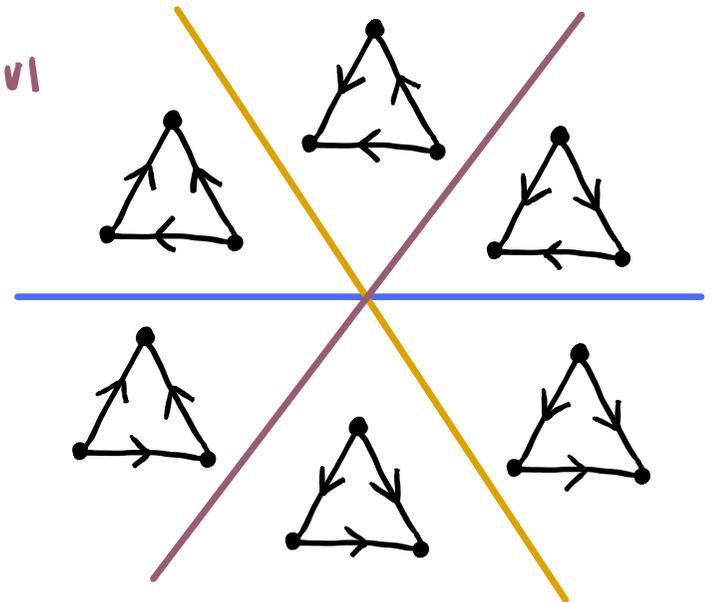
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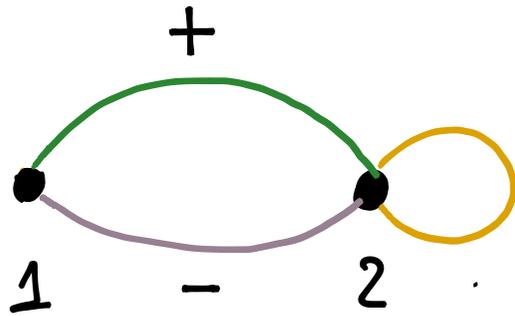
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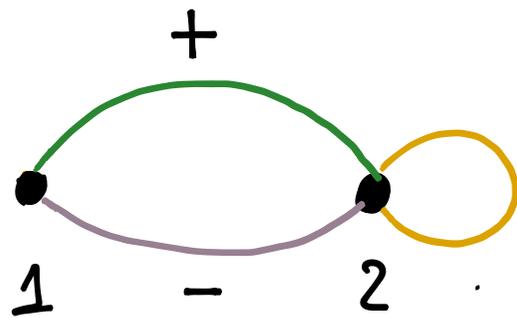
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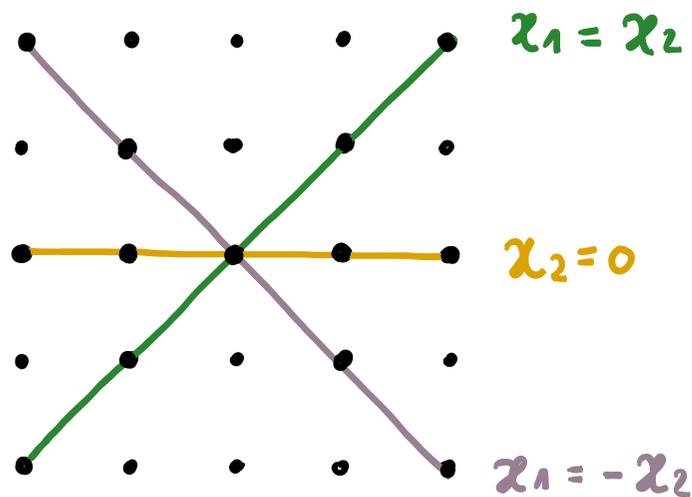
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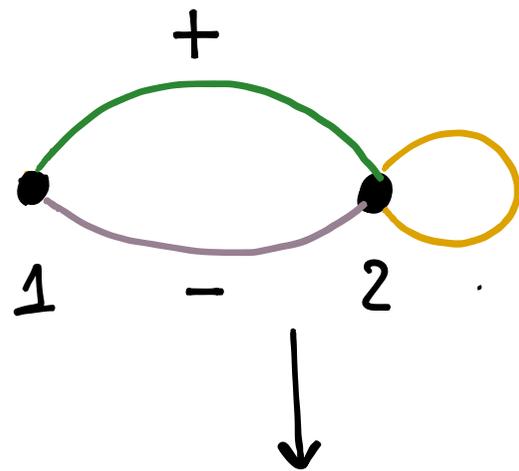
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$$\mathcal{H}_\Sigma \subseteq \mathbb{R}^{|\mathcal{V}|}$$



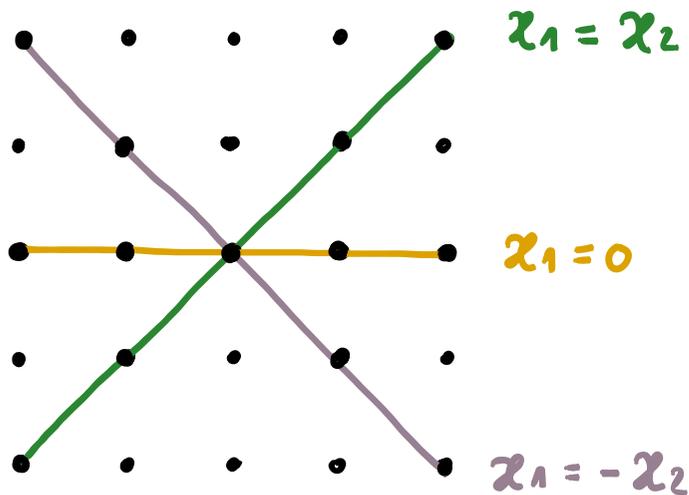
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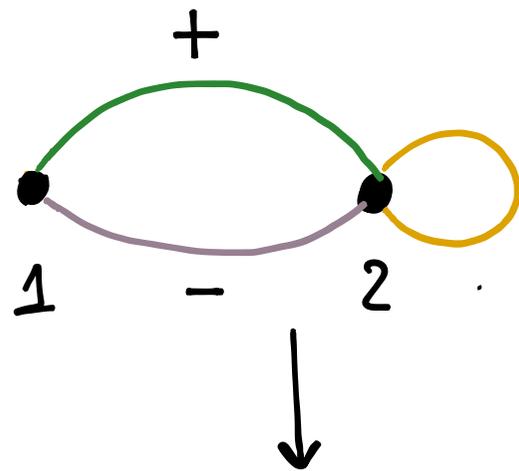
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There is a bijection between the acyclic orientations of  $\Sigma$  and the chambers of  $\mathcal{H}_\Sigma$

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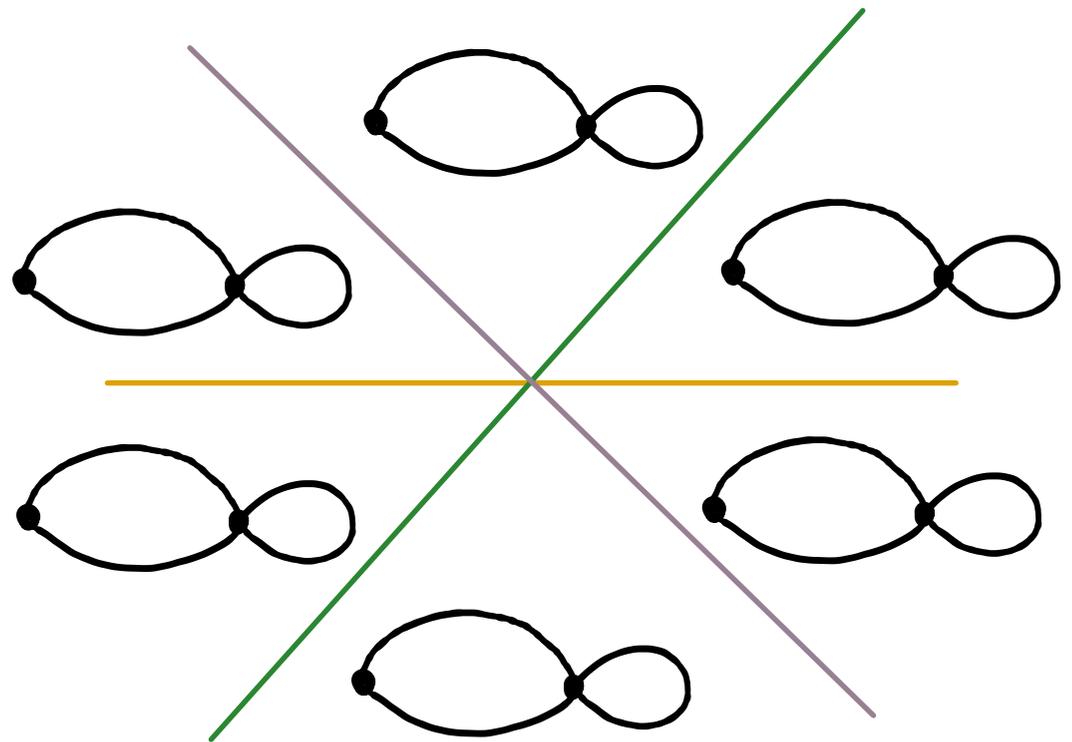
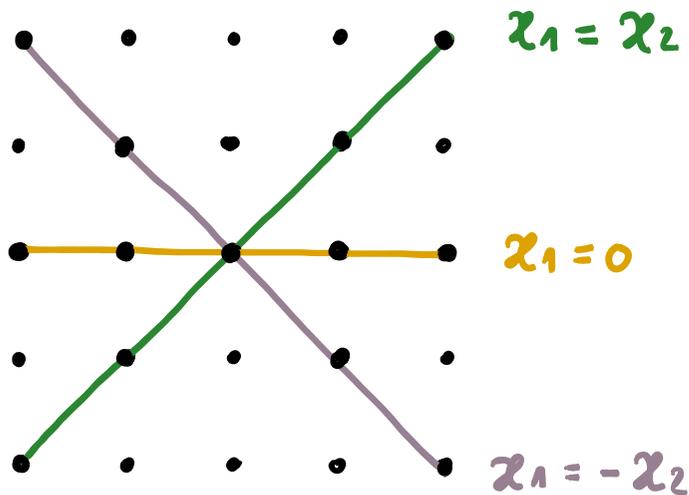
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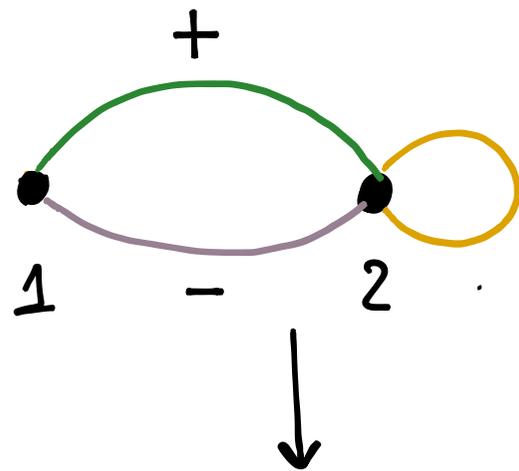
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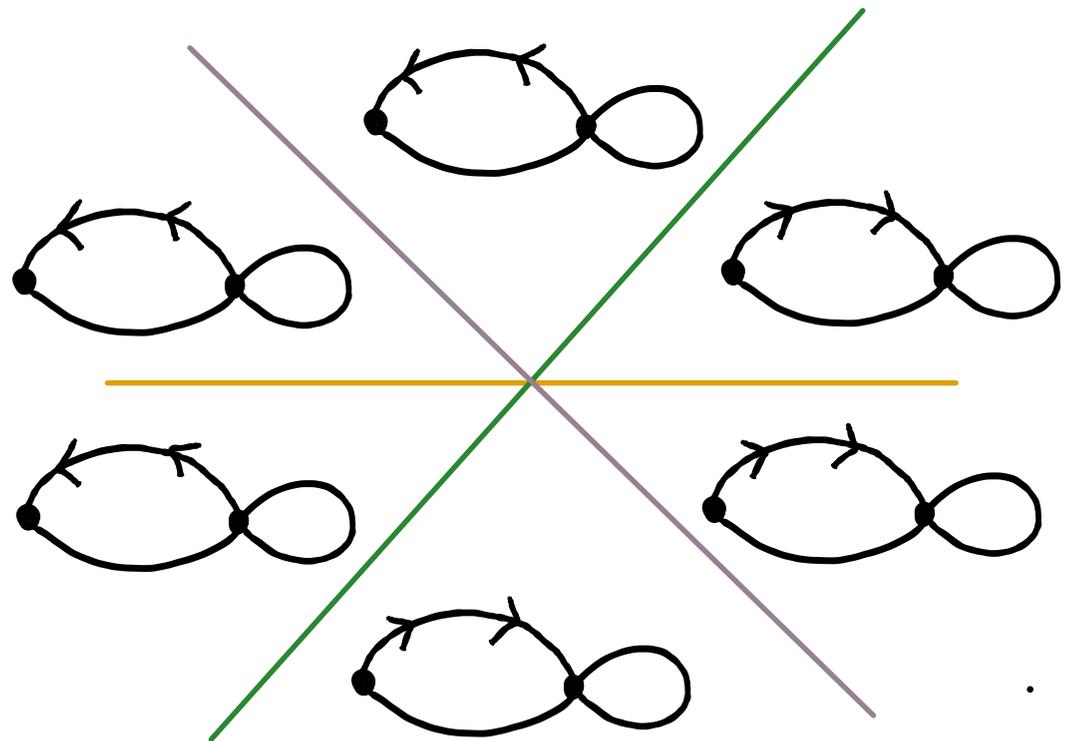
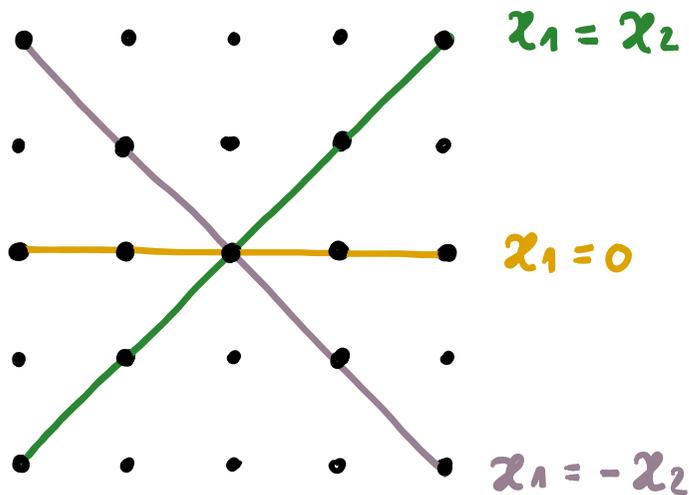
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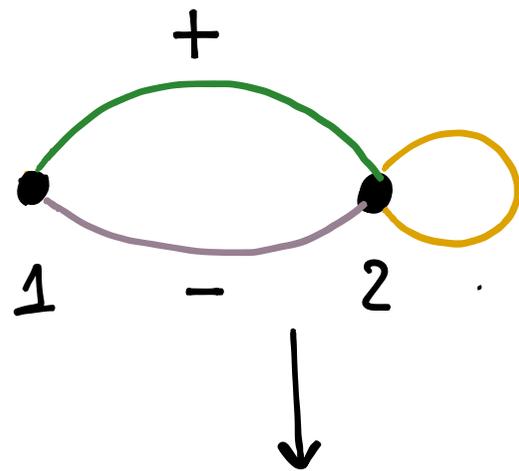
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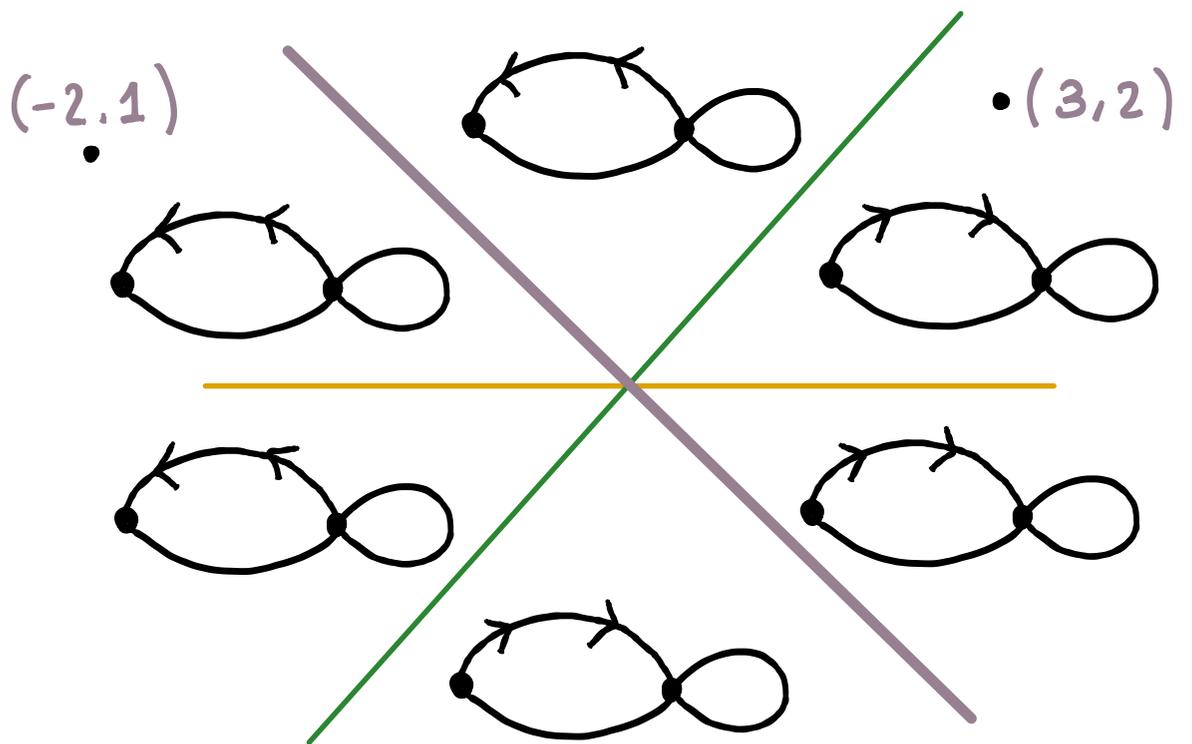
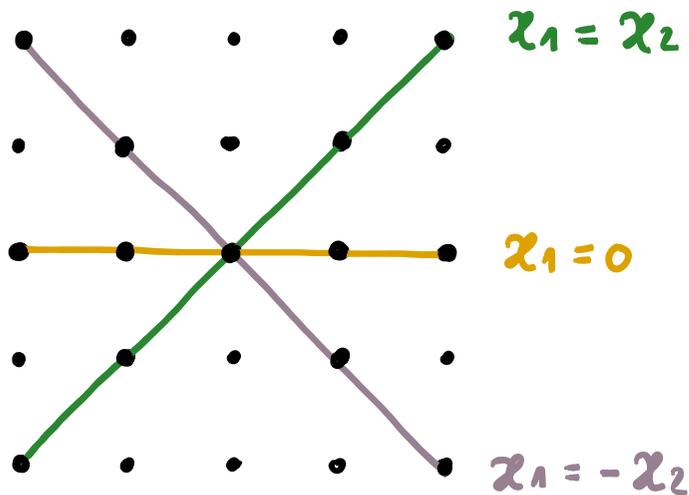
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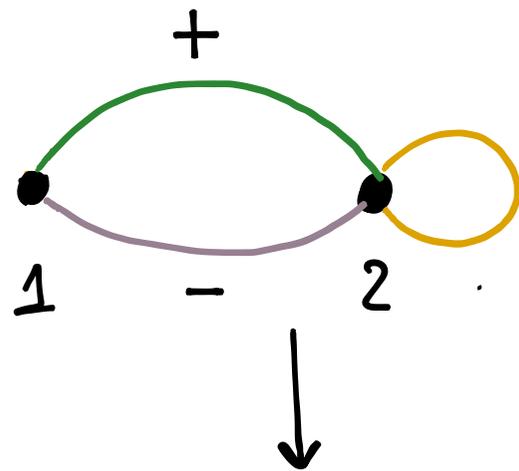
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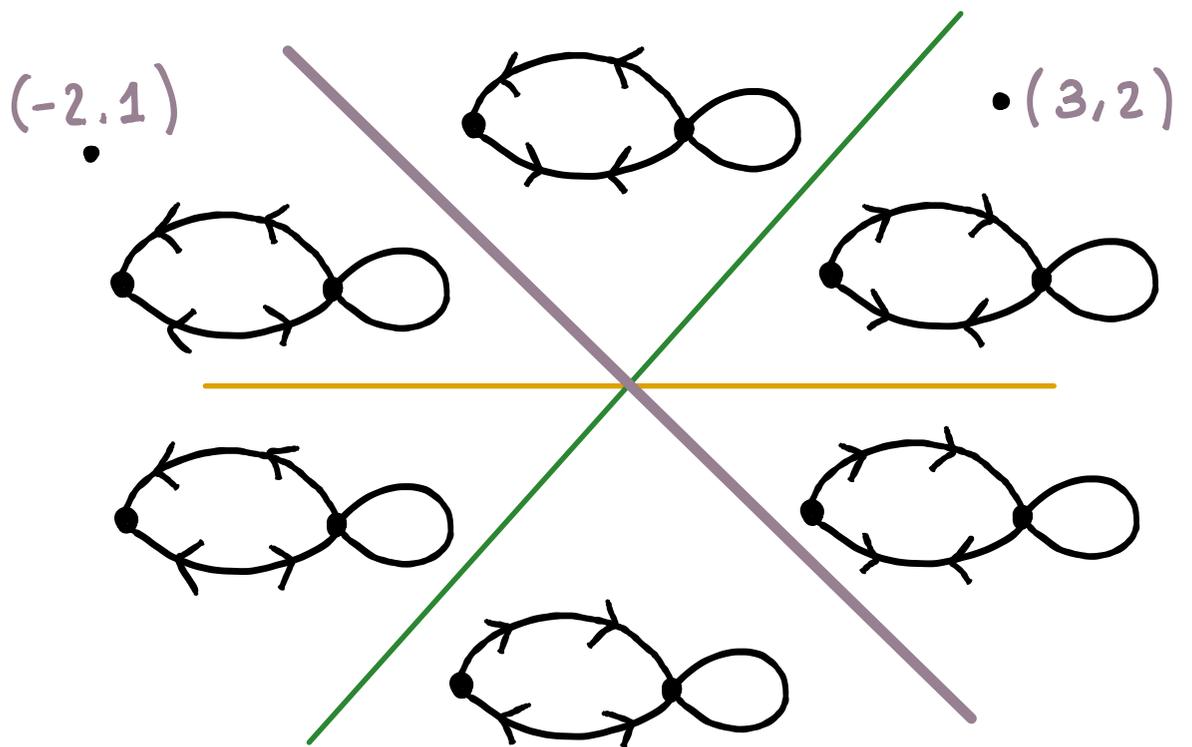
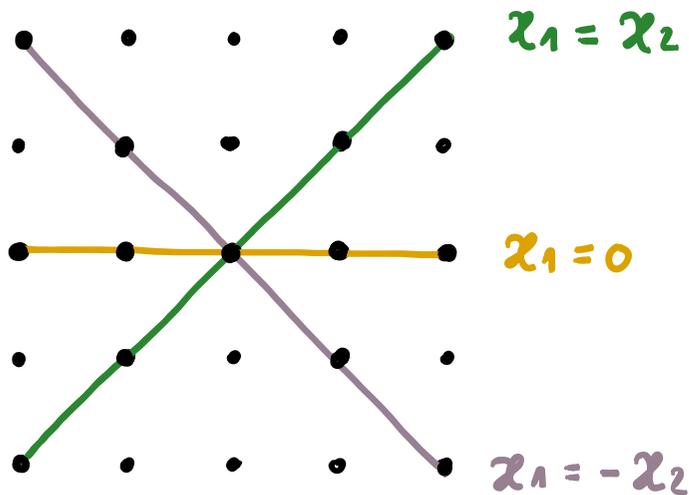
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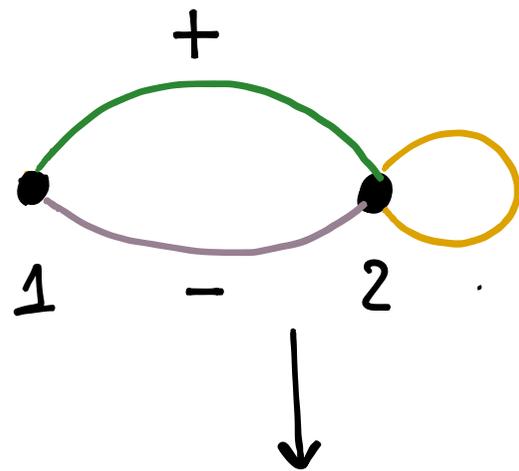
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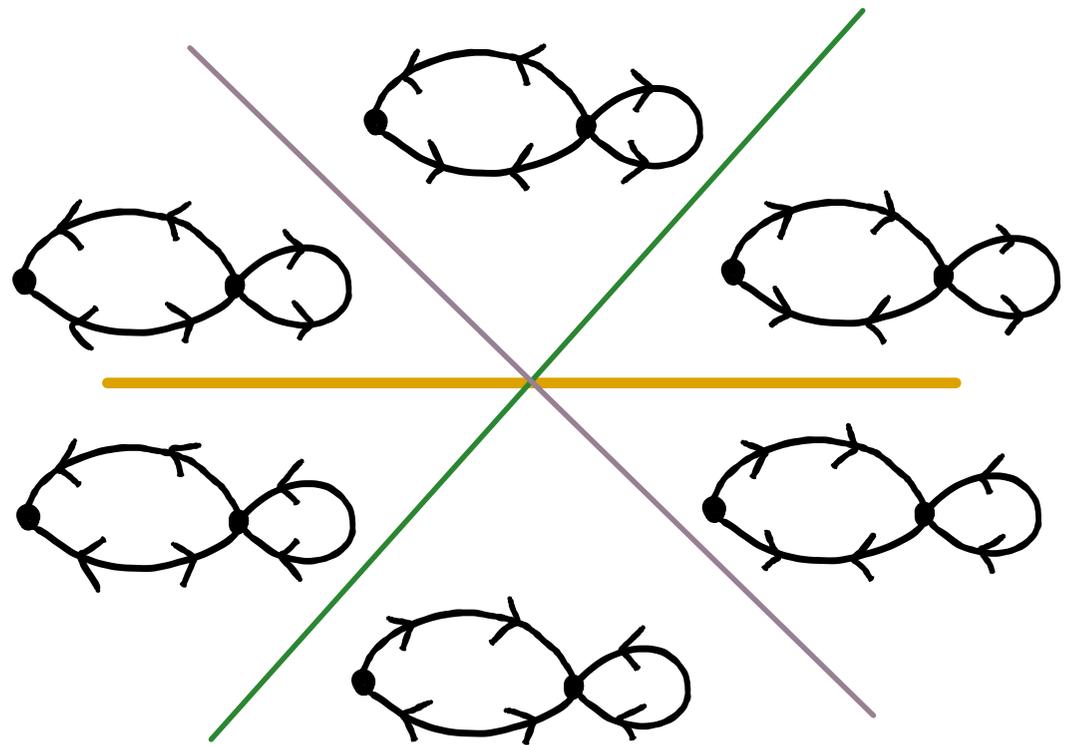
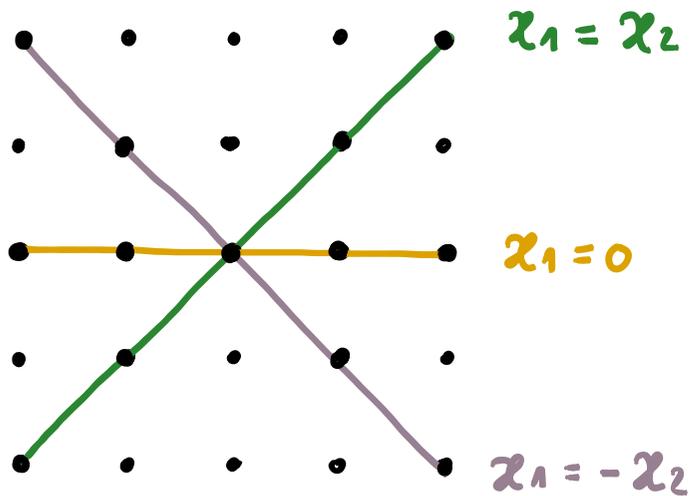
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# THE SIGNED-SYMMETRIC INVARIANT

Recall 
$$\chi_G(x_1, x_2, \dots) = \sum_{\alpha \in \mathbb{N}^{|V|} \setminus HG} x_{\alpha_1} \cdots x_{\alpha_{|V|}} \in \text{Sym}$$

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Definition [Kuroda & Tsujie] 2021

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Where does this function live?

# THE ALGEBRA OF SIGNED-SYMMETRIC FUNCTIONS

\* Definition: The group of signed permutations  $\mathfrak{S}\mathfrak{T}$  is the set of permutations  $\sigma$  of  $\mathbb{Z}$  verifying

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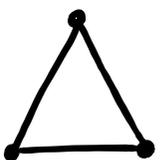
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• Proposition [KT21]:  $\chi_{\Sigma}(\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{SSym}$

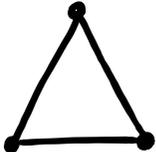
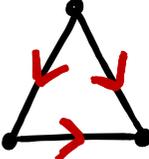
FOURTH PART:

*our research*

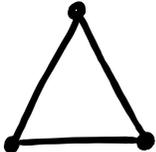
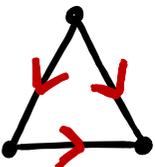
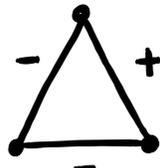
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SIGNED GRAPH		

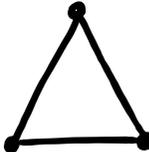
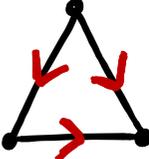
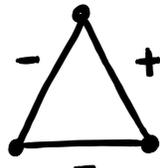
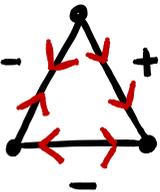
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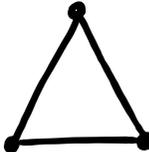
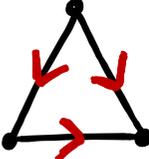
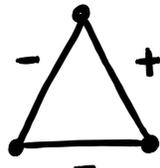
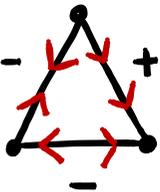
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SIMPLE GRAPH	$G$ :  Stanley '95 $\chi_G(x_1, x_2, \dots) \in \text{Sym}$	$\vec{G}$ :  Shreshian Wachs & Ellzey $\chi_{\vec{G}}(x_1, x_2, \dots; t) \in \text{QSym}[t]$
SIGNED GRAPH	$\Sigma$ :  Wolfgang Kuroda & Tsujie $\chi_{\Sigma}(\dots x_{-1}, x_0, x_1, \dots) \in \text{SSym}$	$\vec{\Sigma}$ : 

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# THE DEFINITION OF THE INVARIANT

$$X_{\vec{\Sigma}}(\dots x_{-1}, x_0, x_1, \dots; t) := \sum_{\kappa \text{ proper}} x^{\kappa} \cdot t^{\underline{\text{asc}(\kappa)}}$$

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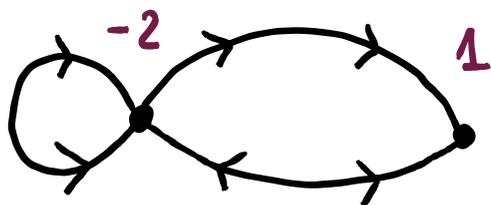
$$\left\{ \begin{array}{l} \text{positive edges} \\ \begin{array}{c} i \rightarrow j \text{ verifying} \\ \kappa(i) < \kappa(j) \end{array} \end{array} \right\} \cup \left\{ \begin{array}{l} \text{introverted edges} \\ \begin{array}{c} i \rightarrow j \text{ verifying} \\ \kappa(i) + \kappa(j) < 0 \end{array} \end{array} \right\} \cup \left\{ \begin{array}{l} \text{extroverted edges} \\ \begin{array}{c} i \leftarrow j \text{ verifying} \\ \kappa(i) + \kappa(j) > 0 \end{array} \end{array} \right\}$$

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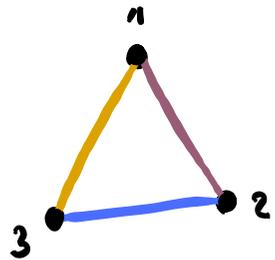
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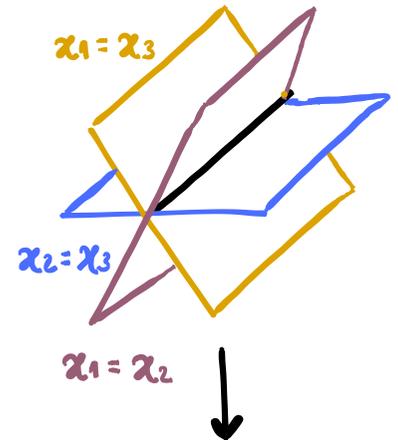
$$\longrightarrow x^{\kappa} \cdot t^{\text{asc}(\kappa)} = x_{-2} x_1 \cdot t$$

# WHY THIS DEFINITION FOR $\text{asc}(k)$ ?



→ Consider in  $\mathbb{R}^{|V|}$

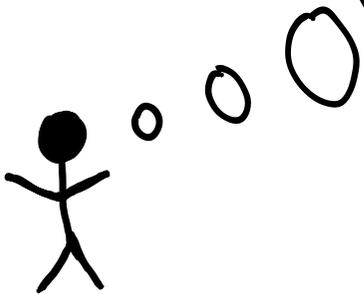
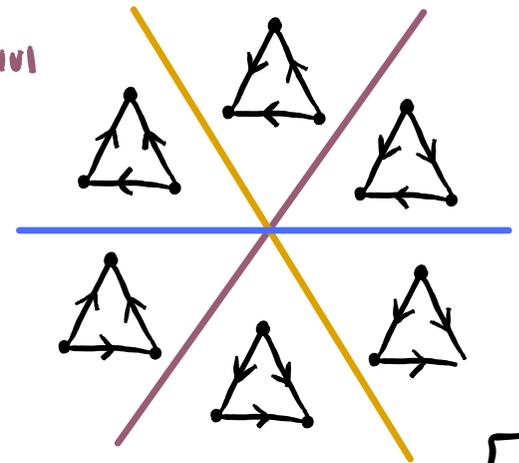
$$H_G := \{ \{x_i = x_j\} : \{i, j\} \in E \}$$



$$* \left\{ \begin{array}{l} \text{proper} \\ \text{colorings of} \\ G \end{array} \right\} \longleftrightarrow \left\{ \mathbb{N}^{|V|} \setminus H_G \right\}$$

$$\text{Then } \chi_G(\underline{x}) = \sum_{k \text{ proper}} x^k = \sum_{\alpha \in \mathbb{N}^{|V|} \setminus H_G} x_{\alpha_1} \cdots x_{\alpha_{|V|}}$$

$$* \left\{ \begin{array}{l} \text{acyclic} \\ \text{orientations} \\ \text{of } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{"chambers"} \\ \text{of } H_G \end{array} \right\}$$



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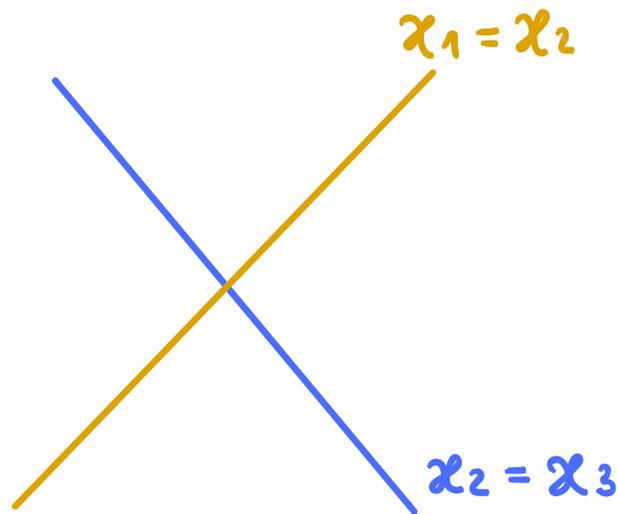
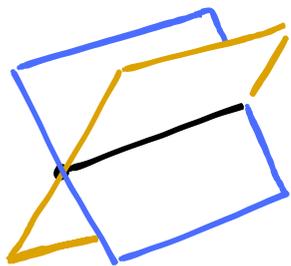
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$\mathcal{HG} \subseteq \mathbb{R}^3$

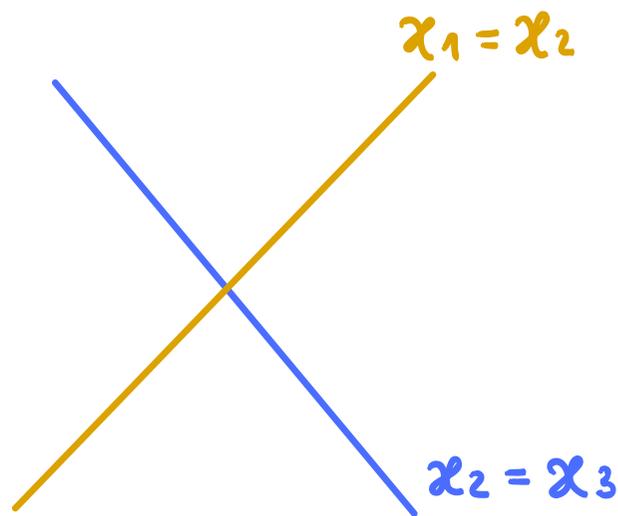
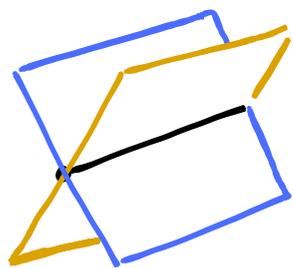


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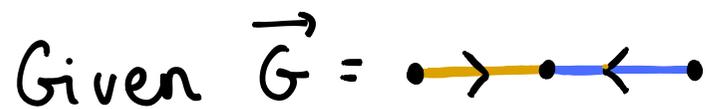
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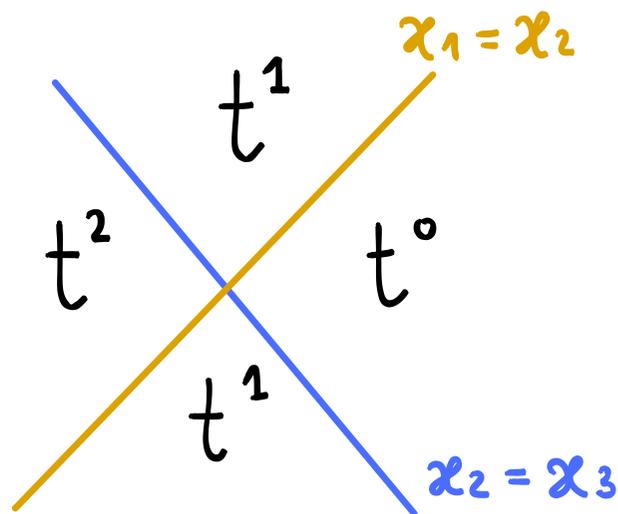
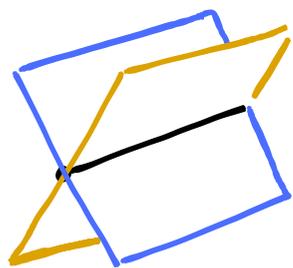
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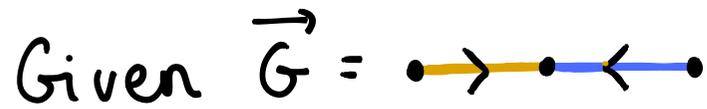
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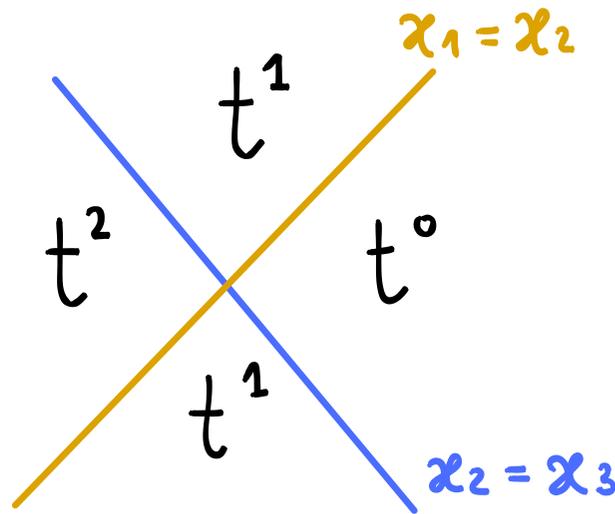
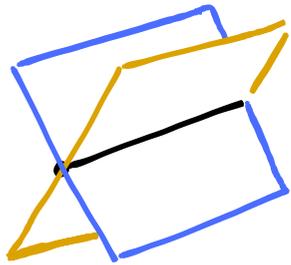
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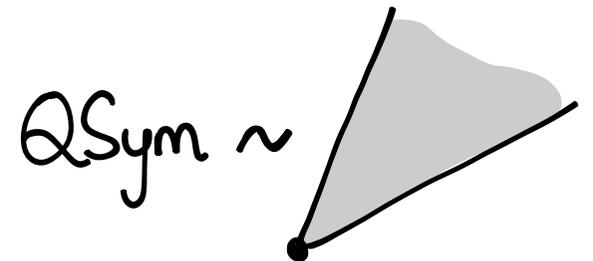


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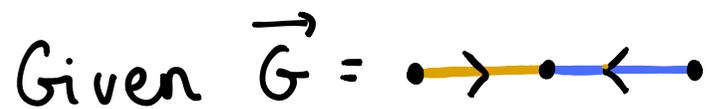
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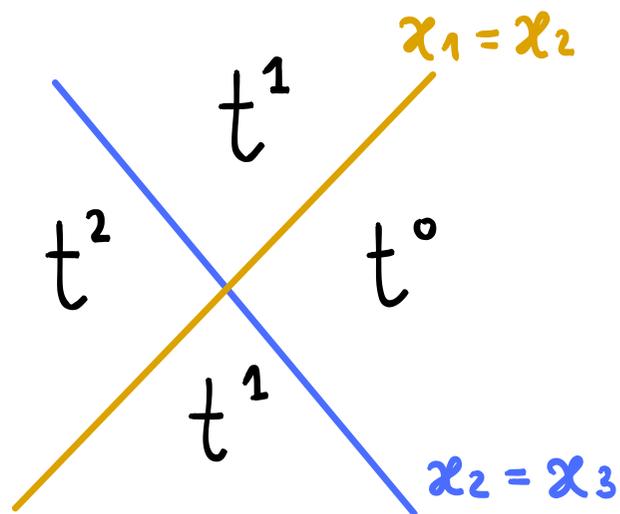
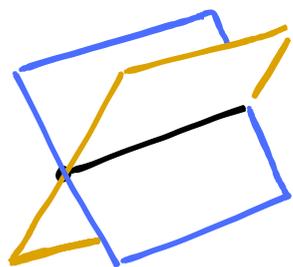


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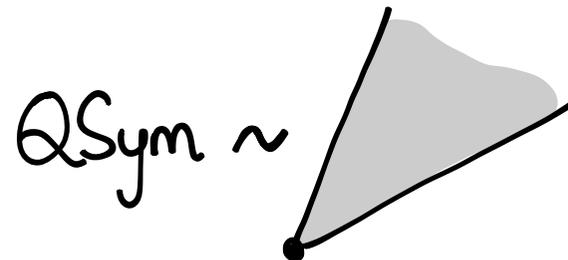
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Our definition for  $\text{asc}(k)$  has the same property!

Intuition:



THE ALGEBRA WHERE  $X_{\Sigma}^{\rightarrow}$  LIVES

## THE ALGEBRA WHERE $X_{\Sigma}^{\vec{}}$ LIVES

The bases of  $SQSym$  are indexed by pairs

$$(k, \lambda) = \left( k, \begin{pmatrix} a_1 \cdots a_r \\ b_1 \cdots b_r \end{pmatrix} \right) \text{ where } k \in \mathbb{N}_0 \text{ and } (a_i, b_i) \neq (0, 0)$$

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The monomial basis  $\{M_{(\kappa, \lambda)}\}_{(\kappa, \lambda)}$  is given by

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$$\begin{aligned} M_{(2, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix})} &= x_0^2 x_1 x_{-1} x_2^2 + x_0^2 x_1 x_{-1} x_3^2 + \dots \\ &= \sum_{i < j} x_0^2 x_i x_{-i} x_j^2 \end{aligned}$$

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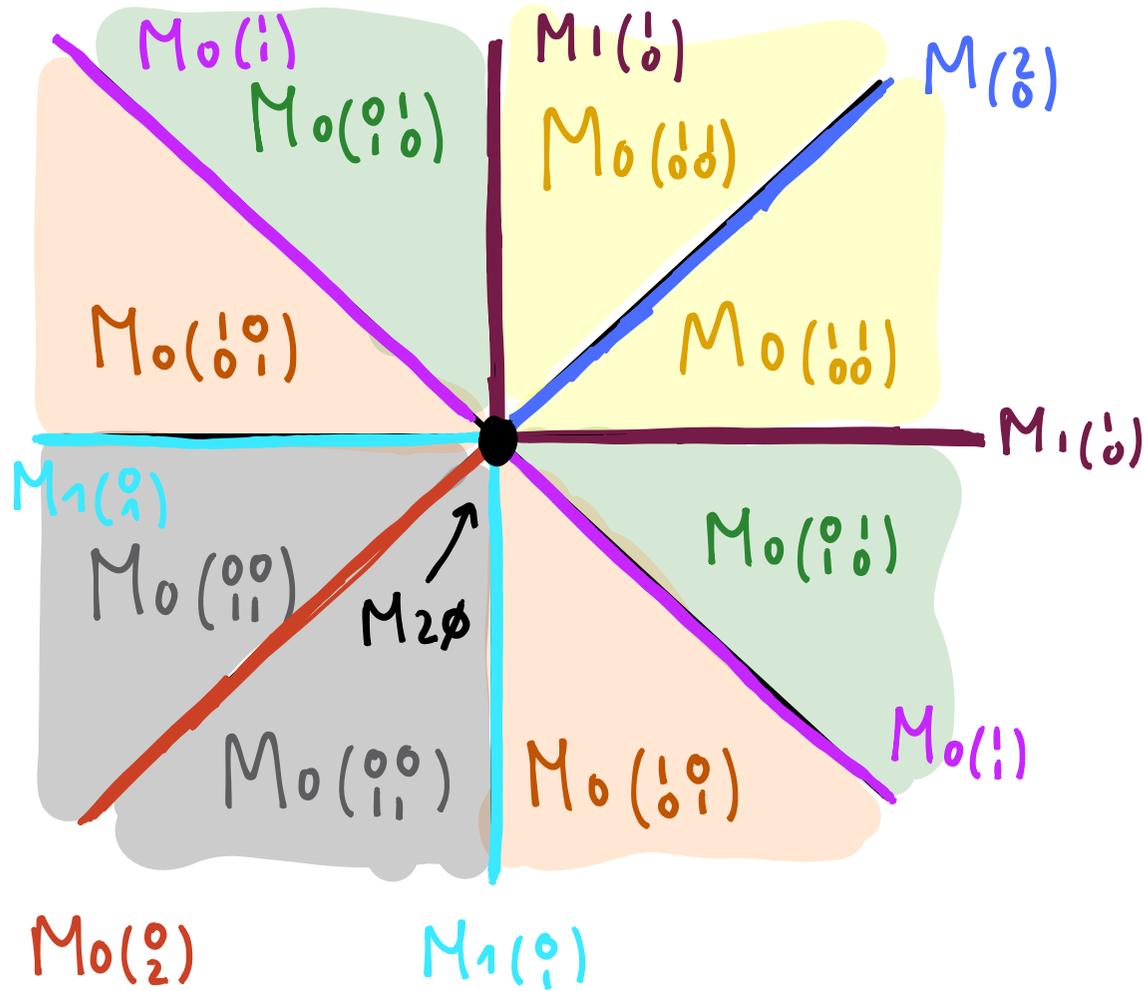
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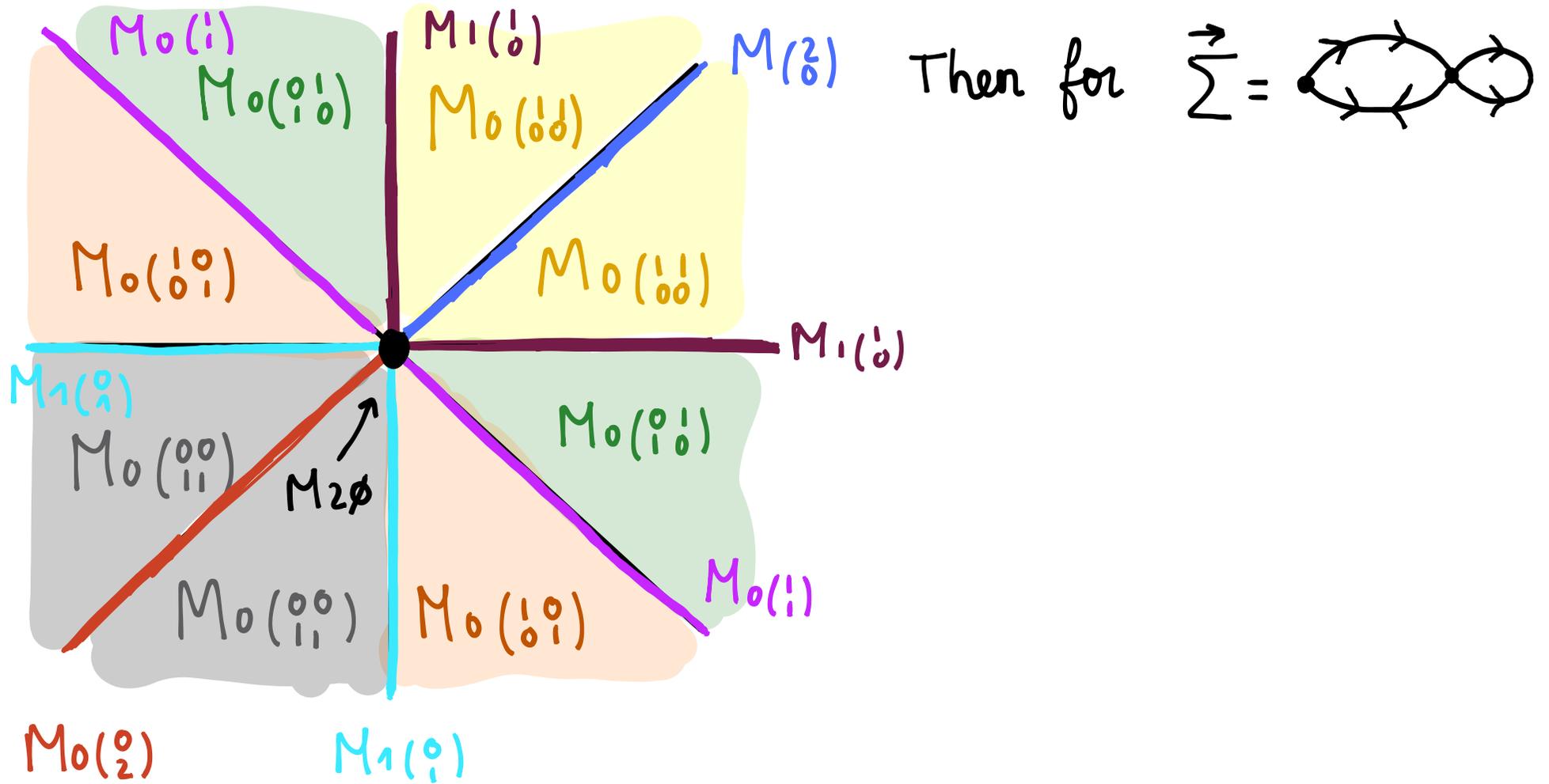
We can define:

$$\text{SQSym} = \text{Vect} \left\{ M_{(k, \lambda)} \right\}_{(k, \lambda)}$$

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## A CHARACTERIZATION OF SQSym

• Definition: Let  $f \in \mathbb{Q}[[\dots x_{-1}, x_0, x_1, \dots]]$ , for  $i \in \mathbb{N}$   
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Let  $f \in \mathbb{Q}[[\dots, x_{-1}, x_0, x_1, \dots]]$  of bounded degree

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$$R_i(f \cdot g) = R_i(f) \cdot R_i(g) = R_{i+1}(f) \cdot R_{i+1}(g) = R_{i+1}(f \cdot g)$$

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And the generating function is  $\sum_{d \geq 0} \dim(SQS_{\text{Sym}}^d) t^d = \frac{1-t}{1+4t+2t^2}$

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We know how to extract a basis!

$$\bar{\mathcal{F}} \subseteq \mathcal{F}$$

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## A FORMULA FOR $\chi_{\vec{\Sigma}}$

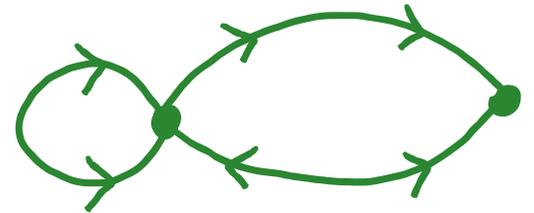
Theorem: Given a directed signed graph  $\Sigma$  on  $d$  vertices

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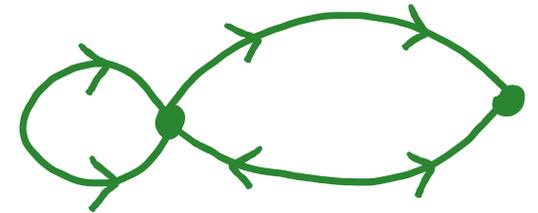
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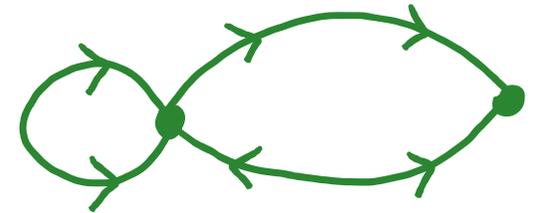


$\pi$	$\text{inv}_{\vec{\Sigma}}(\pi)$	$\text{sgn}(\pi)$	$(\Sigma, \pi) - \text{rank}$	$DES_{\Sigma}(\pi)$
$\bar{2}\bar{1}12$	3	++	1234	{0,1}
$\bar{1}\bar{2}21$	2	++	1223	{0,1}
$2\bar{1}1\bar{2}$	1	+−	1234	{0,1}
$2\bar{1}\bar{1}\bar{2}$	2	−+	1223	{1}
$\bar{2}1\bar{1}\bar{2}$	2	−+	1234	{0,1}
$1\bar{2}\bar{2}\bar{1}$	1	+−	1223	{0,1}
$21\bar{1}\bar{2}$	0	−−	1234	{0,1}
$12\bar{2}\bar{1}$	1	−−	1223	{0}

# A FORMULA FOR $\chi_{\vec{\Sigma}}$

Theorem: Given a directed signed graph  $\Sigma$  on  $d$  vertices

$$[\text{Aval}, \mathbf{M}] \quad \chi_{\vec{\Sigma}}(x; t) = \sum_{\sigma \in S\overline{d}} t^{\text{inv}_{\vec{\Sigma}}(\sigma)} \cdot \Gamma_{\text{DES}_{\Sigma}(\sigma)}^{\text{sgn}(\sigma)}$$



$\pi$	$\text{inv}_{\vec{\Sigma}(\pi)}$	$\text{sgn}(\pi)$	$(\Sigma, \pi) - \text{rank}$	$\text{DES}_{\Sigma}(\pi)$
$\overline{21}12$	3	++	1234	{0,1}
$\overline{12}21$	2	++	1223	{0,1}
$2\overline{1}1\overline{2}$	1	+-	1234	{0,1}
$2\overline{1}1\overline{2}$	2	-+	1223	{1}
$\overline{2}1\overline{1}2$	2	-+	1234	{0,1}
$1\overline{2}2\overline{1}$	1	+-	1223	{0,1}
$21\overline{1}2$	0	--	1234	{0,1}
$122\overline{1}$	1	--	1223	{0}

$$\begin{aligned} \chi_{\vec{\Sigma}}(x; t) &= t^3 \Gamma_{\{0,1\}}^{++} + \\ &t^2 \left[ \Gamma_{\{0,1\}}^{++} + \Gamma_{\{0,1\}}^{-+} + \Gamma_{\{1\}}^{++} \right] + \\ &t \left[ 2 \Gamma_{\{0,1\}}^{+-} + \Gamma_{\{1\}}^{--} \right] + \Gamma_{\{0,1\}}^{--} \end{aligned}$$

## A SYMMETRY RESULT

When does  $X_{\Sigma}^{\rightarrow}(\underline{x}; t)$  satisfy stronger symmetry conditions?

## A SYMMETRY RESULT

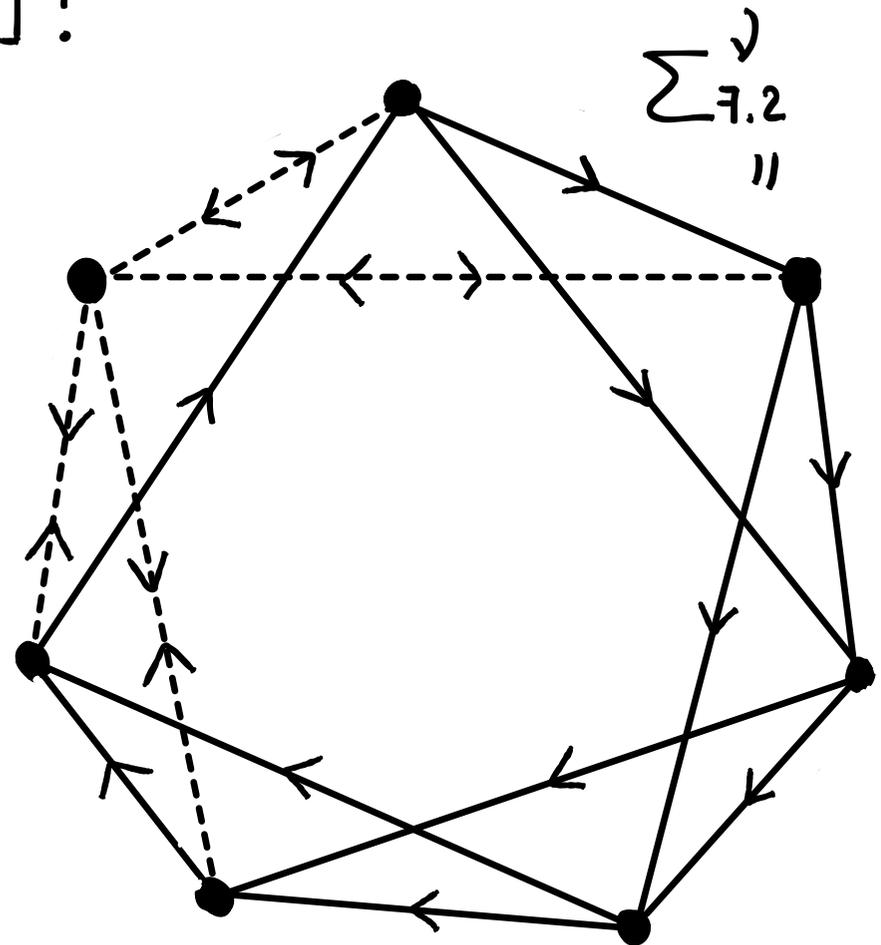
When does  $X_{\Sigma}^{\vec{\alpha}}(\underline{x}; t)$  satisfy stronger symmetry conditions?

When does  $X_{\Sigma}^{\vec{\alpha}}(\underline{x}; t) \in \text{SSym}[t]$ ?

## A SYMMETRY RESULT

When does  $X_{\Sigma}^{\vec{v}}(\underline{x}; t)$  satisfy stronger symmetry conditions?

When does  $X_{\Sigma}^{\vec{v}}(\underline{x}; t) \in \text{SSym}[t]$ ?



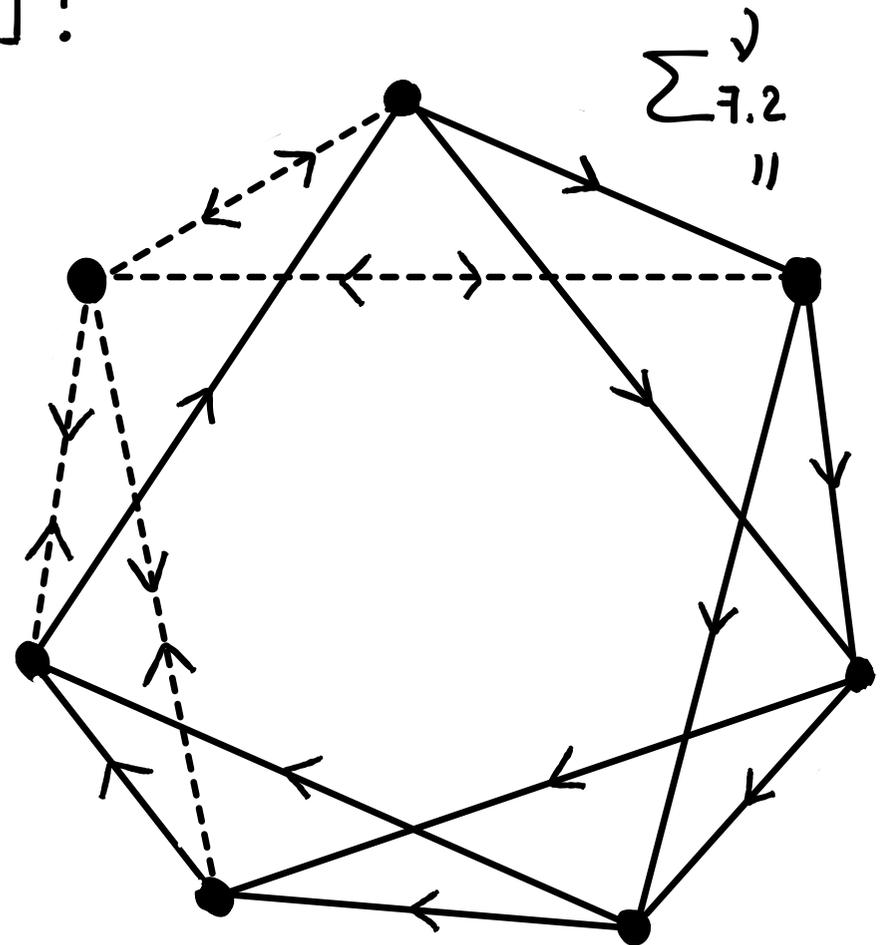
## A SYMMETRY RESULT

When does  $X_{\Sigma}^{\vec{\nu}}(\underline{x}; t)$  satisfy stronger symmetry conditions?

When does  $X_{\Sigma}^{\vec{\nu}}(\underline{x}; t) \in \text{SSym}[t]$ ?

• Theorem [Aval, M]

$$X_{\sum_{d,k}^{\vec{\nu}}}(\underline{x}; t) \in \text{SSym}[t]$$



## SOME BIBLIOGRAPHY

- **Ellzey**(2017). *Chromatic Quasisymmetric Functions of Directed Graphs* Proceedings of the 29th FPSAC
- **Kuroda & Tsujie** (2021). *Chromatic Signed-Symmetric Functions of Signed Graphs*. arXiv preprint
- **Nadeau, Spink & Tewari** (2024). *Quasisymmetric divided differences* arXiv preprint
- **Shareshian & Wachs** (2016). *Chromatic quasisymmetric functions*. Adv. Math.
- **Stanley** (1995). *A symmetric function generalization of the chromatic polynomial of a graph*. Adv. Math.
- **Zaslavsky** (1991). *Orientation of signed graphs*. Eur. J. Comb.

MERCI BEAUCOUP!

questions?

