

# Combinatorics of one variable catalytic equations: a geometric wiring

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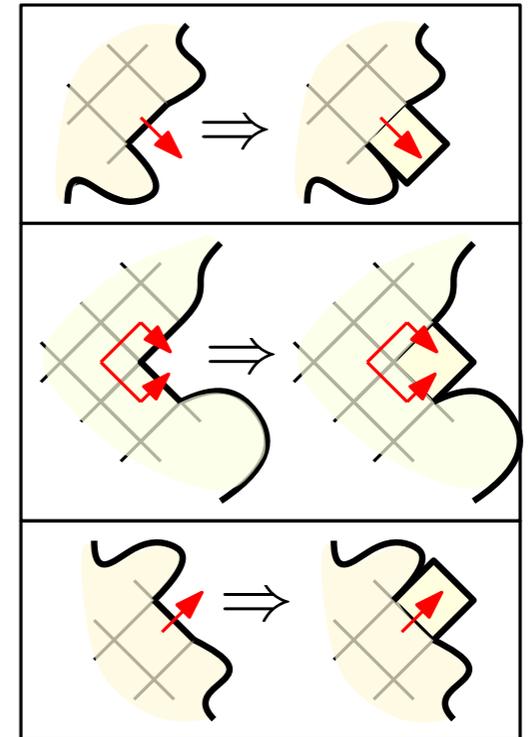
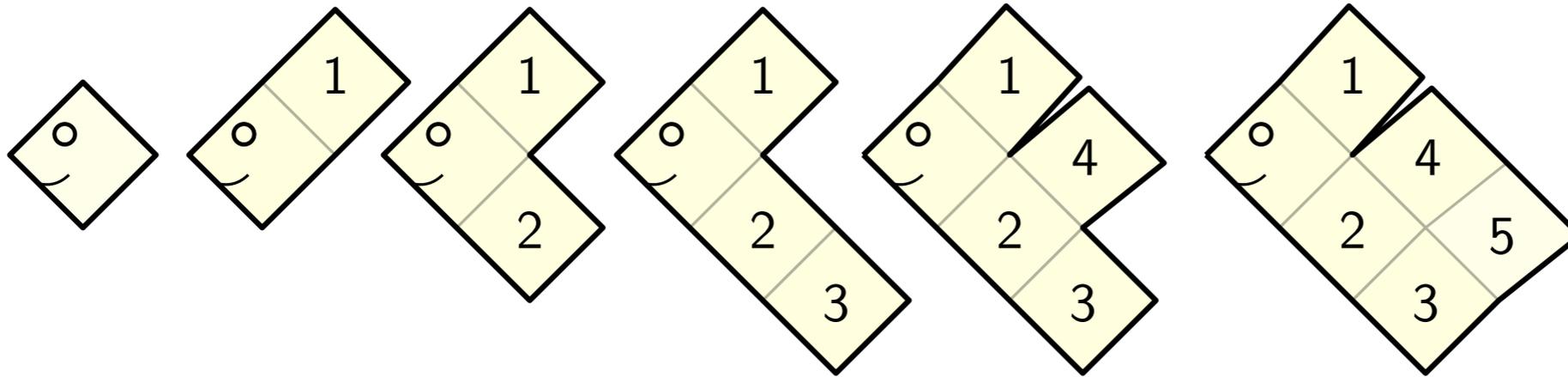
Based on works with Gilles Schaeffer,  
and with Jean-Luc Baril, Khaydar Nurligareev, Medhi Naima et Sergei Kirgizov.

Fighting fish

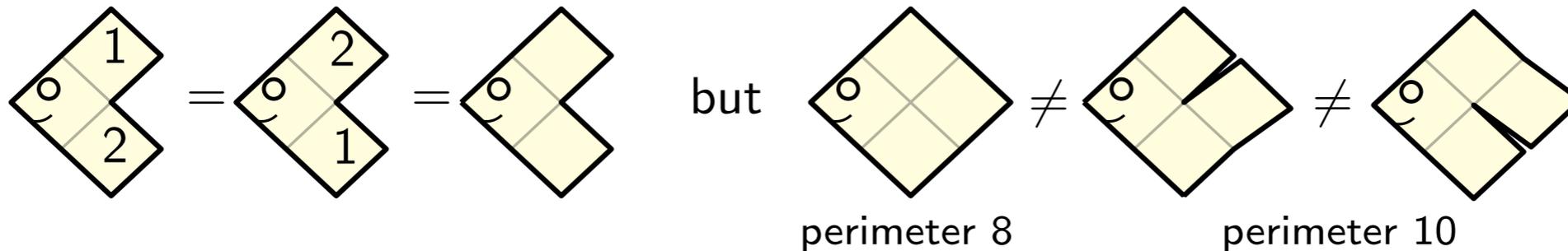
# Fighting fish, definition

## Fighting fish

A fighting fish is a surface that can be obtained from a single cell by a sequence of directed cell agregations.

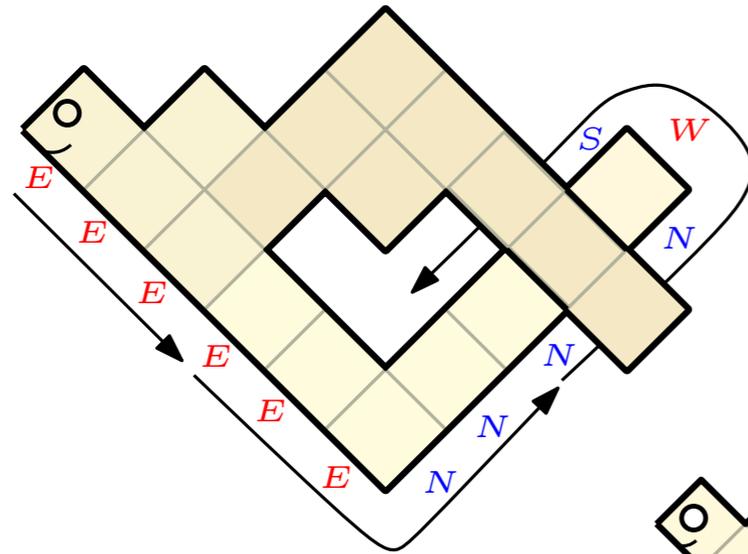


We are interested only in the resulting surface, not in the aggregation order (but type of aggregation matters)

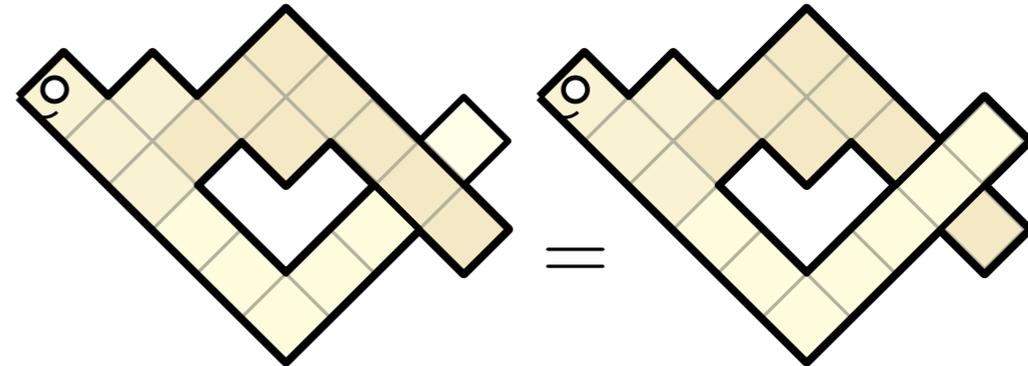
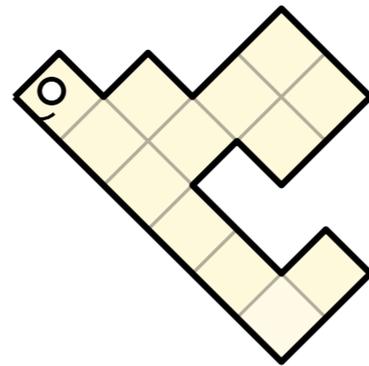
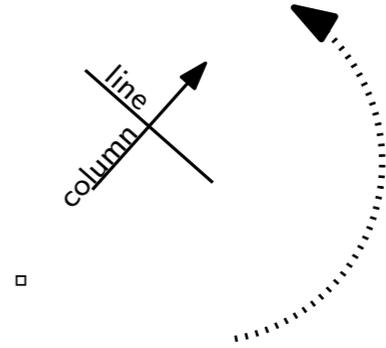


# Fighting fish

Fighting fish are simply connected surfaces that can be recovered from the walk around their boundary



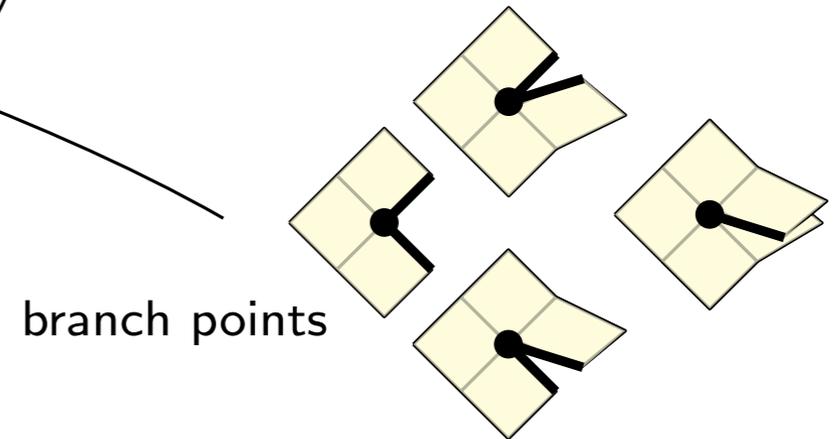
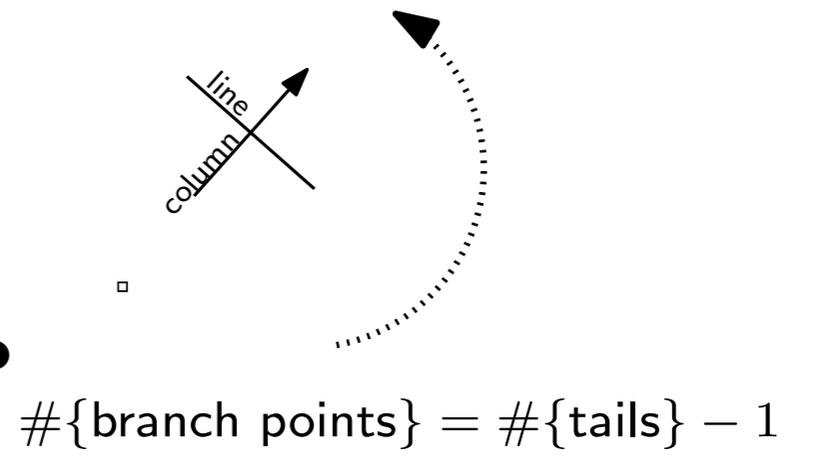
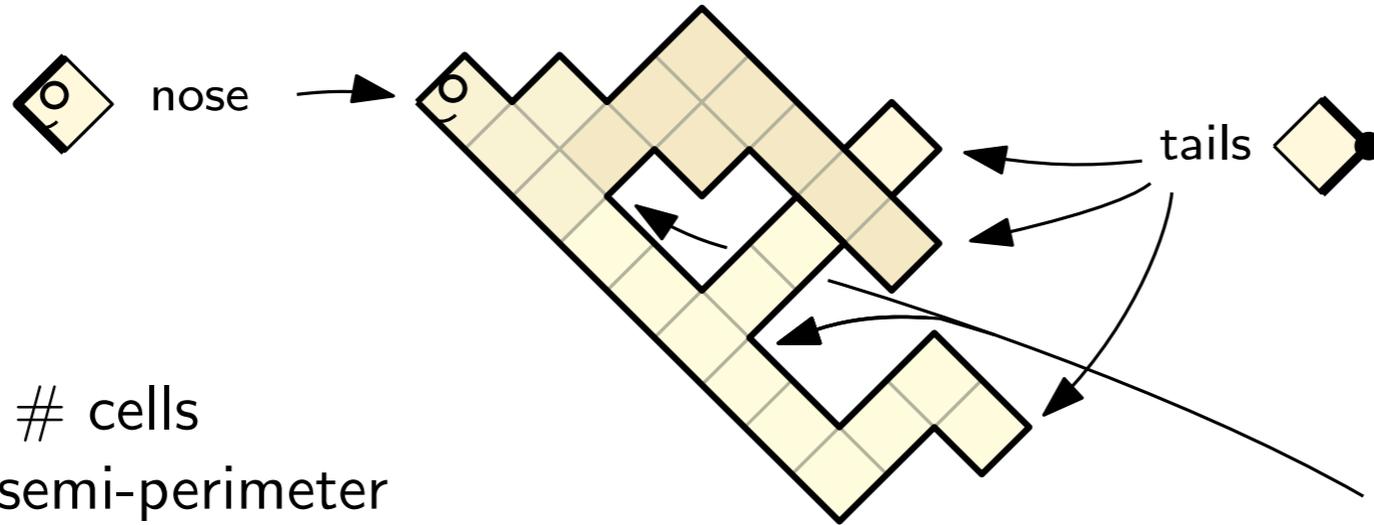
*EEEEEE NNNNN W S ...*



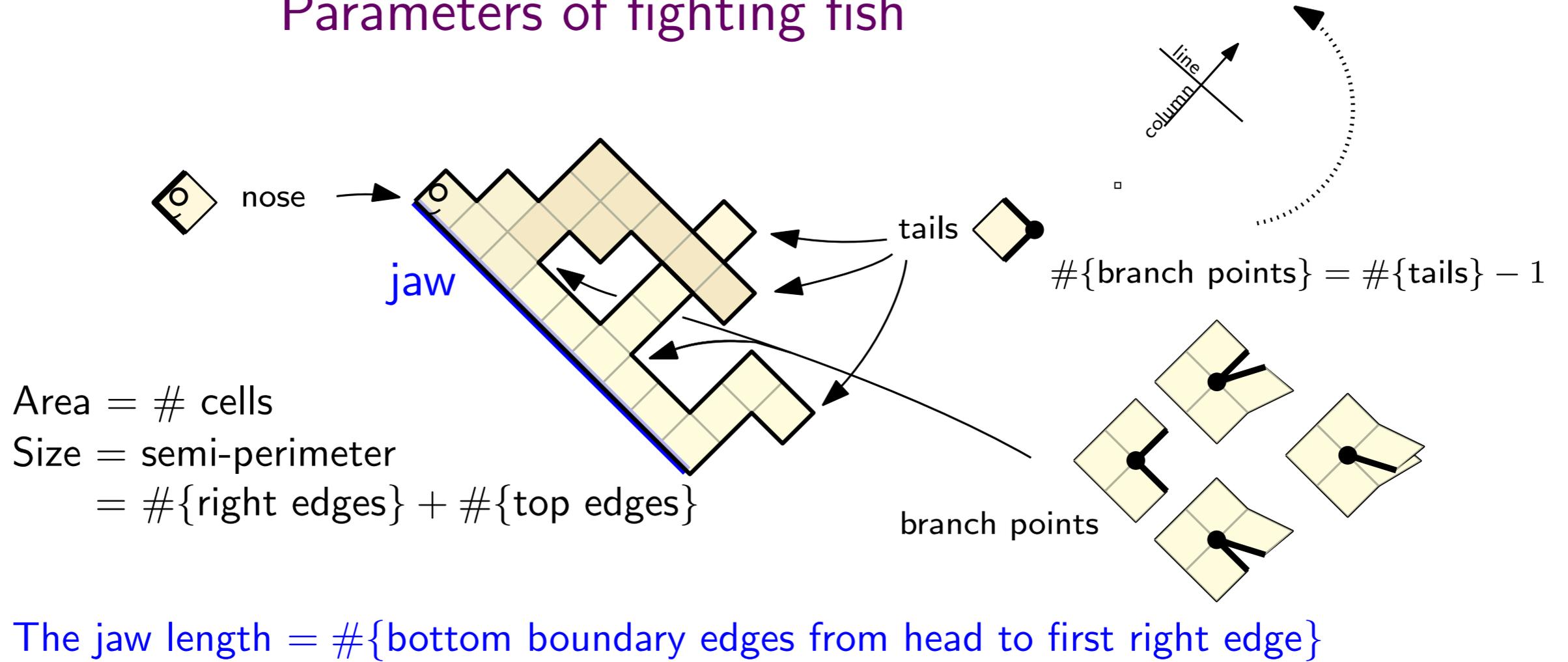
Fighting fish do not necessarily fit in the plane.

# Parameters of fighting fish

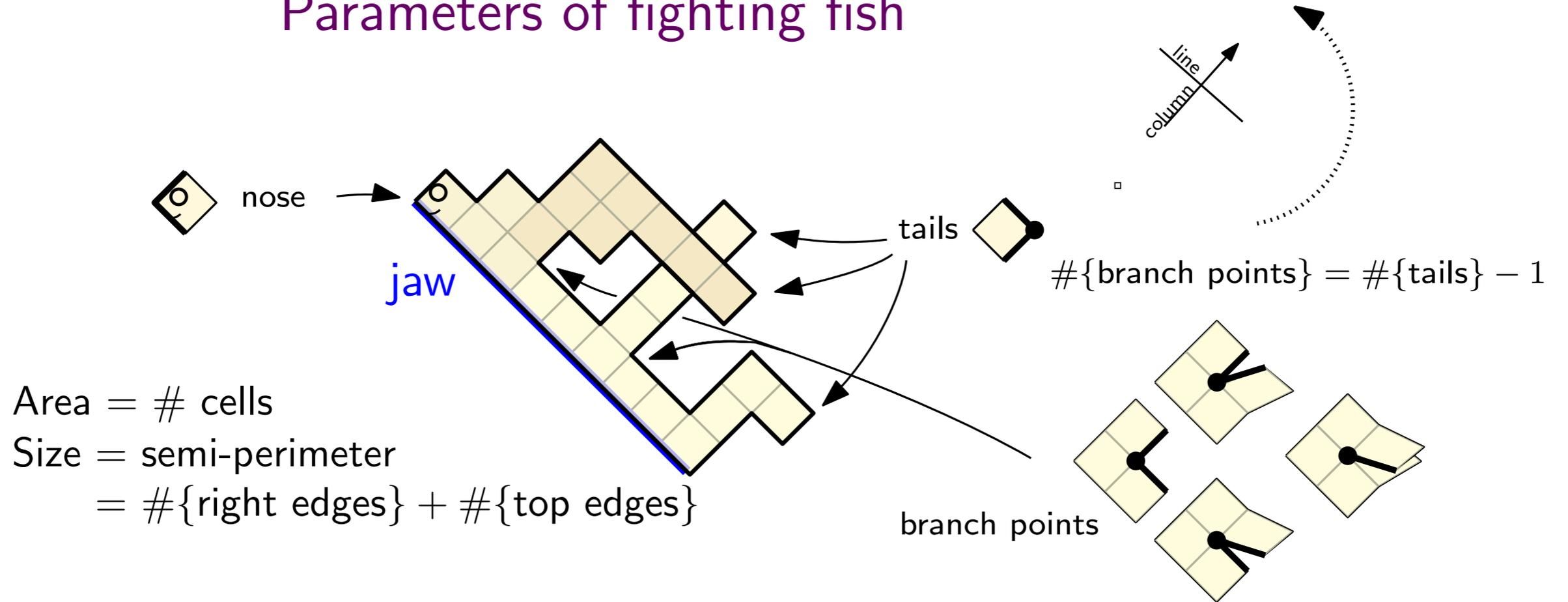
Area = # cells  
Size = semi-perimeter  
= # {right edges} + # {top edges}



# Parameters of fighting fish



# Parameters of fighting fish



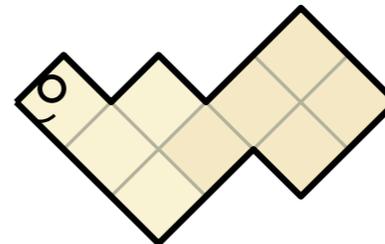
Area = # cells

Size = semi-perimeter

= # {right edges} + # {top edges}

The jaw length = # {bottom boundary edges from head to first right edge}

Fighting fish with exactly 1 tail  
 = parallelogram polyominoes  
 aka staircase polygons



# Enumerative results and bijections with other classes

**Theorem** (D.–Guerini–Rinaldi-Schaeffer 2016)

$$\# \left\{ \begin{array}{l} \text{fighting fish} \\ \text{with semi-perimeter } n + 1 \end{array} \right\} = \frac{2}{(n + 1)(2n + 1)} \binom{3n}{n}$$

1, 2, 6, 91, 408, 1938...

Number of fighting fish of size  $n + 1$  is equal to numbers of

- Two-stack sortable permutations of  $\{1, \dots, n\}$   
(West, Zeilberger, Bona, 90's)
- Synchronized Tamari intervals of length  $2n$   
(Fang, Préville-Ratelle, 2017)
- Rooted non separable planar maps with  $n + 1$  edges  
(Tutte, Mullin and Schellenberg, 60's)
- **Left ternary trees with  $n - 1$  edges**  
(Del Lungo, Del Ristoro, Penaud, 1999++)

$n = 1$

**Bijections**



1

*Fang(2018), Cioni, Ferrari, Henriët (2024)*



*D., Henriët (2023)*

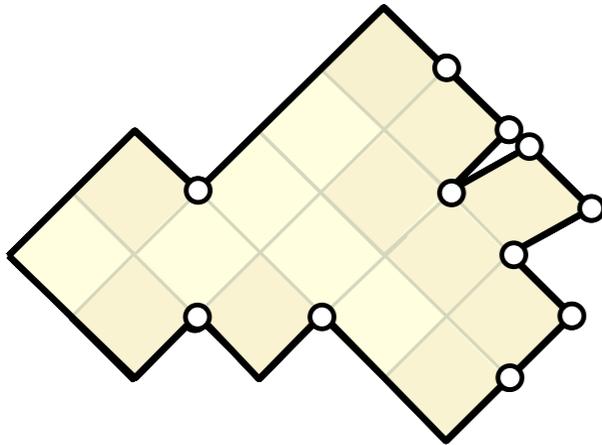
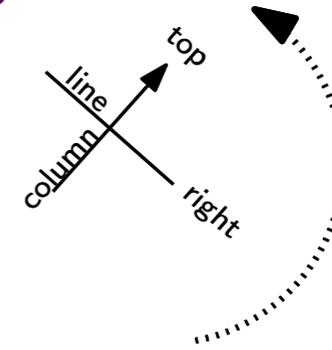


*D., Henriët (2022)*

0

*Eu, Fu et Pan (2025)*

# Fighting fish and ternary trees



A stem cell is a cell of the fish such that its top-right corner is on the boundary



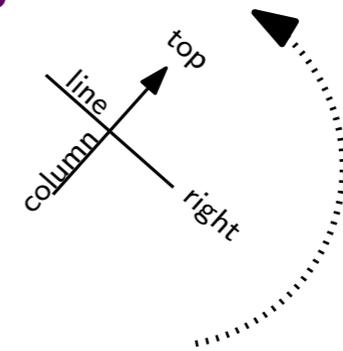
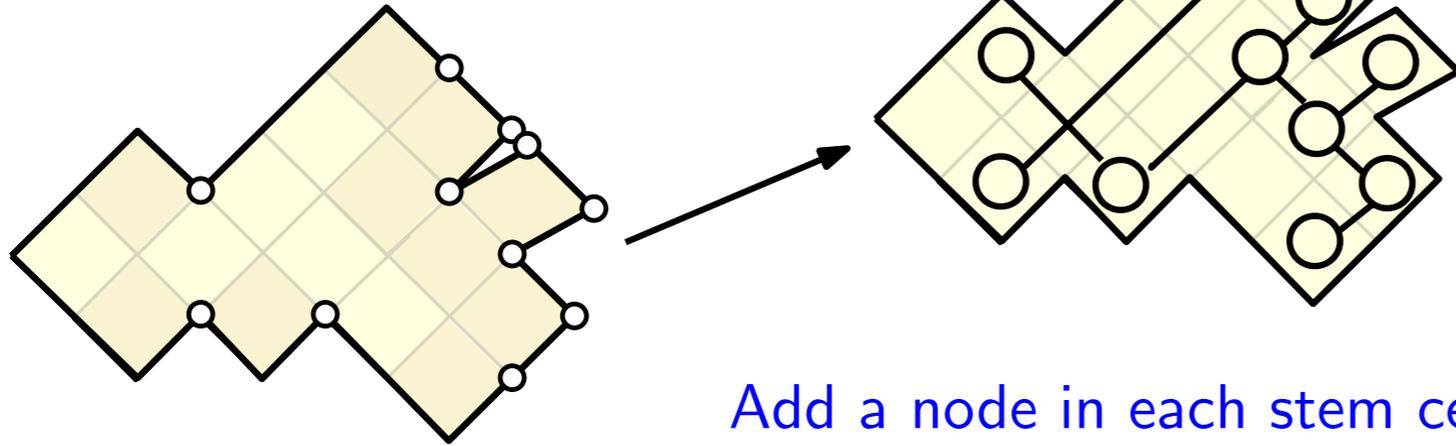
$$\#\{\text{stem cells}\} = s_0 + s_1 + s_2 = s_1 + 2s_2 - 1 = n$$

$$\#\{\text{stem cells}\} = s_0 + s_1 + s_2$$

$s_i$  contributes  $i$  to the semiperimeter of the fish, so that  $0 \cdot s_0 + 1 \cdot s_1 + 2 \cdot s_2 = n + 1$

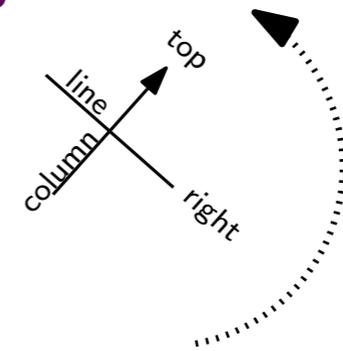
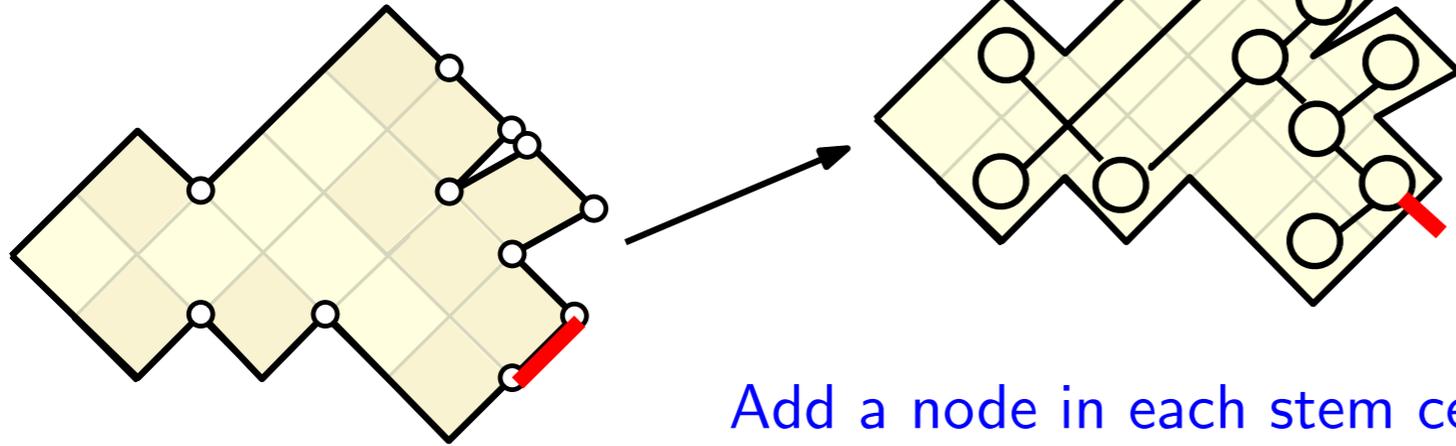
$s_0 = \#\{\text{branching points}\}$  and  $s_2 = \#\{\text{tails}\}$ , so that  $s_0 = s_2 - 1$

# Fighting fish and ternary trees



- Add a node in each stem cell
- Connect each node to the next one in each column
- Connect each node to the next one in each row

# Fighting fish and ternary trees



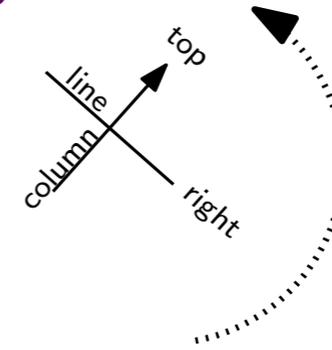
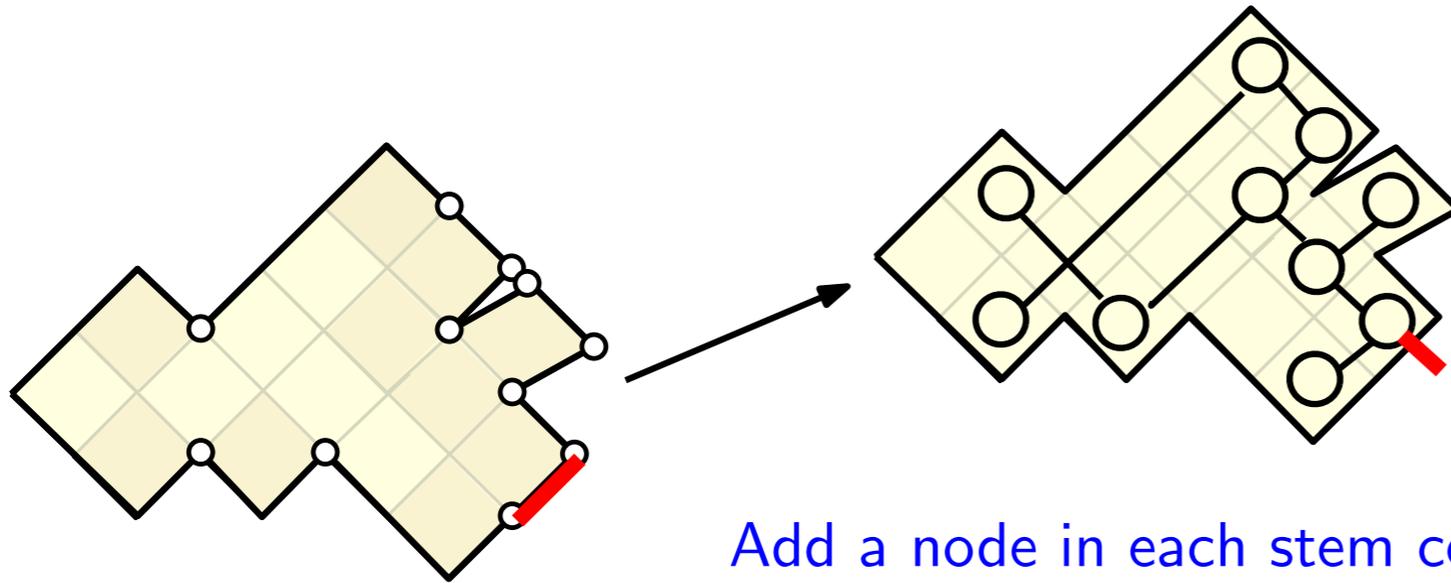
Add a node in each stem cell

Connect each node to the next one in each column

Connect each node to the next one in each row

Mark a right or top edge, and root the tree accordingly.

# Fighting fish and ternary trees



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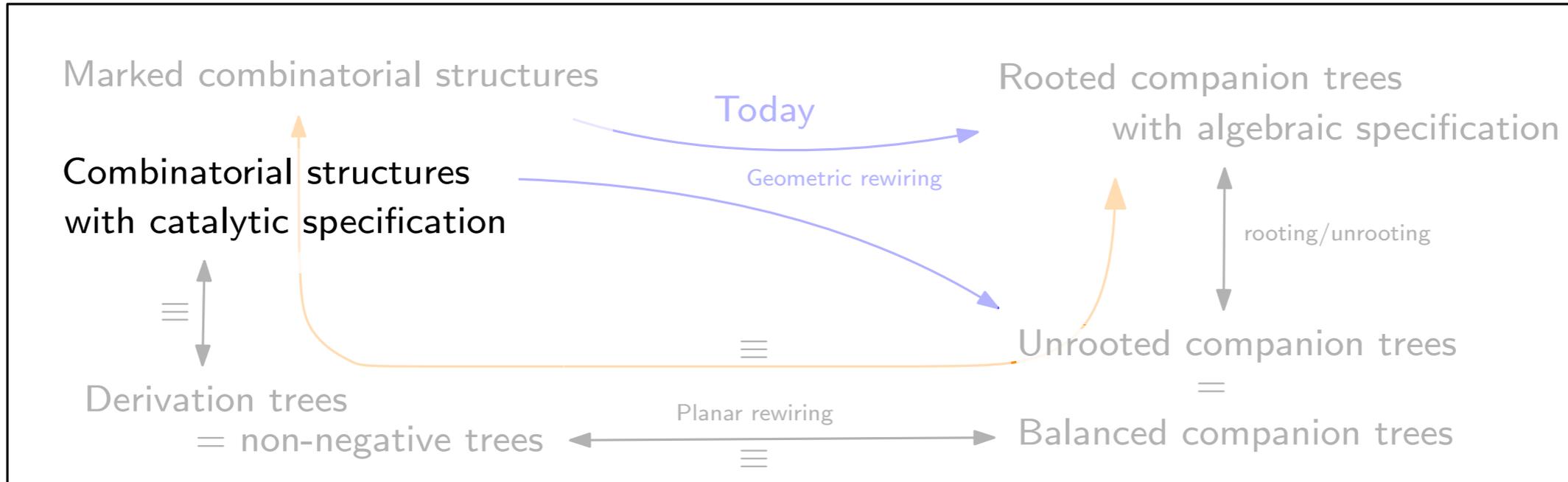
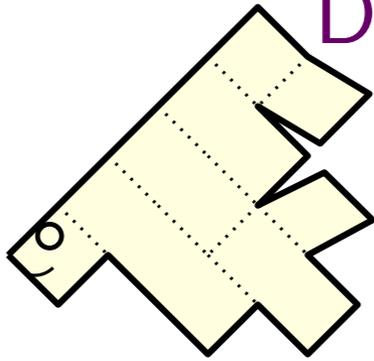
Mark a right or top edge, and root the tree accordingly.

**Theorem** (Eu, Fu and Pan 2025). This construction is a 2-to-1 correspondence between fighting fish with a marked right or top edge and rooted ternary trees.

**Corollary** . The number of fighting fish with semi perimeter  $n + 1$  is  $\frac{2}{n+1} \cdot \frac{1}{2n+1} \binom{3n}{n}$

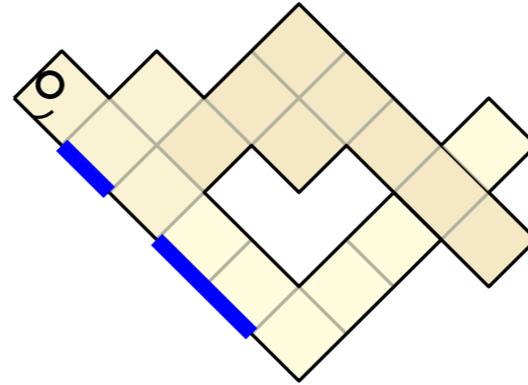
# Derivation trees of fighting fish

$\mathcal{F}$

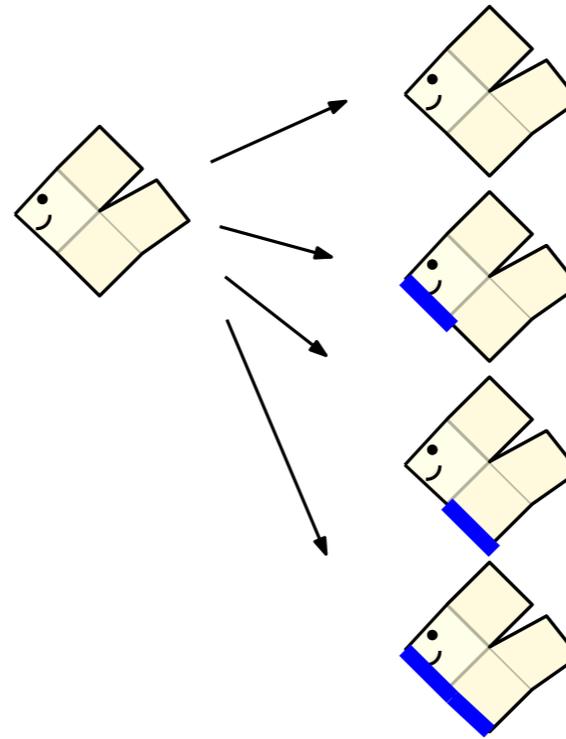


# Sticky fighting fish

Sticky fish = fighting fish with marked edges on the jaw

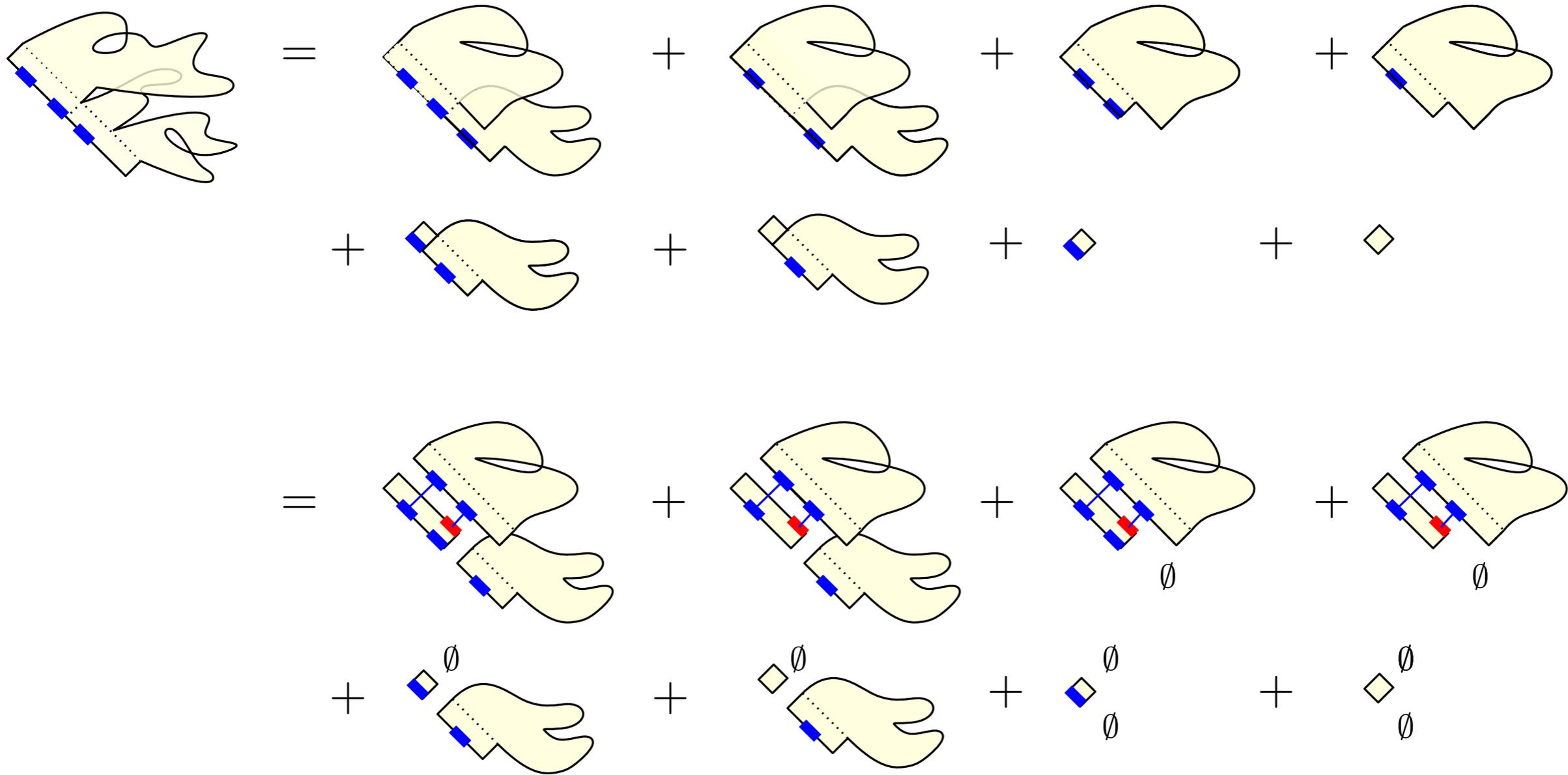


One fighting fish with jaw length  $k$   
 $\rightarrow 2^k$  sticky fighting fish



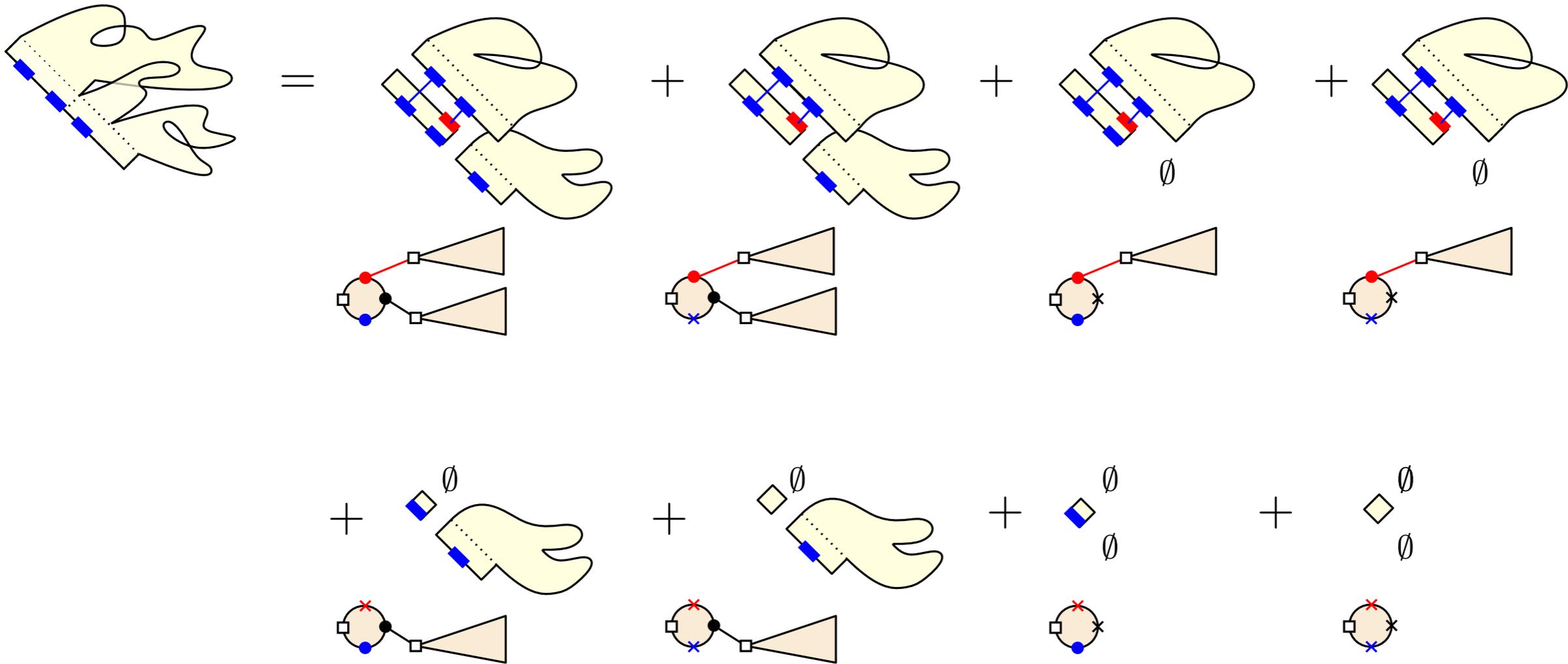
# A catalytic specification of sticky fighting fish

The following decomposition is a simple adaptation to sticky fighting fish of the decomposition of fighting fish from [D.-Henriet, 22]



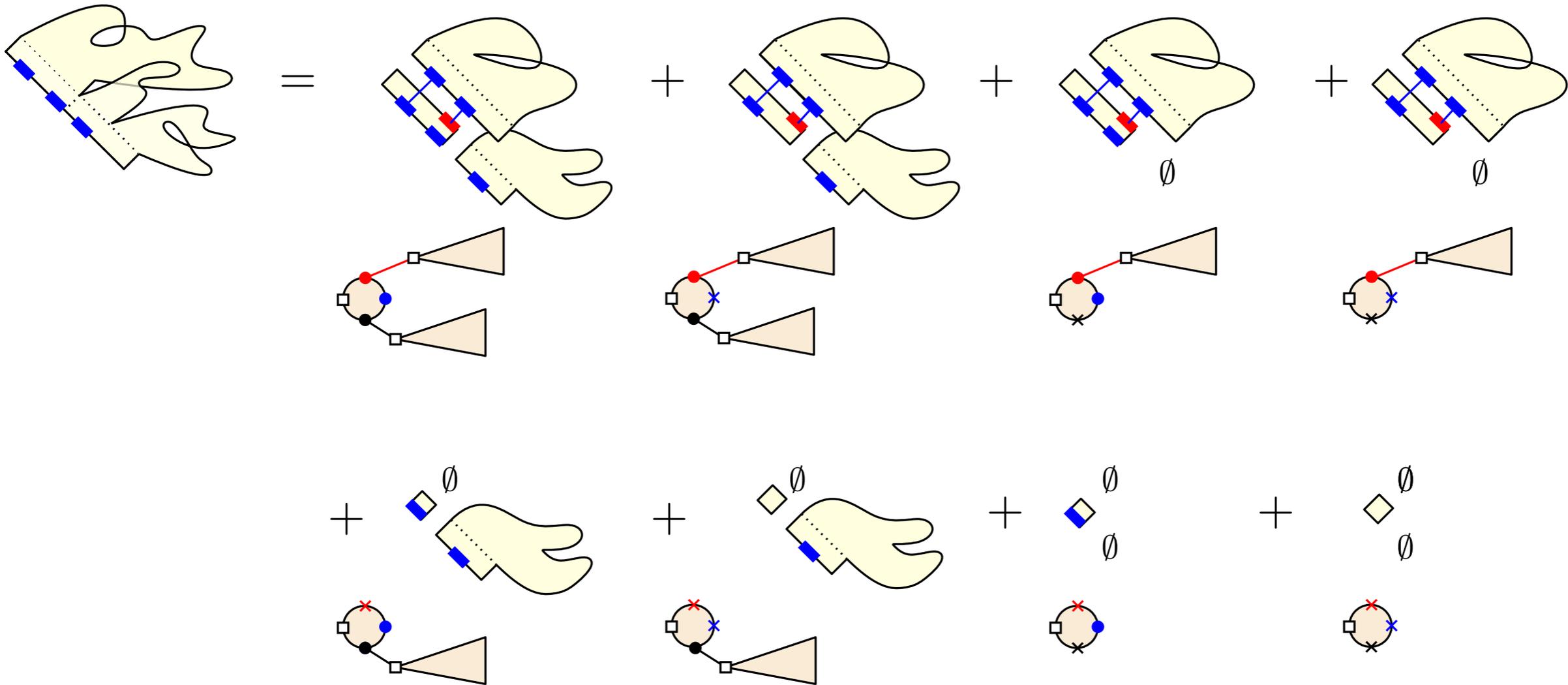
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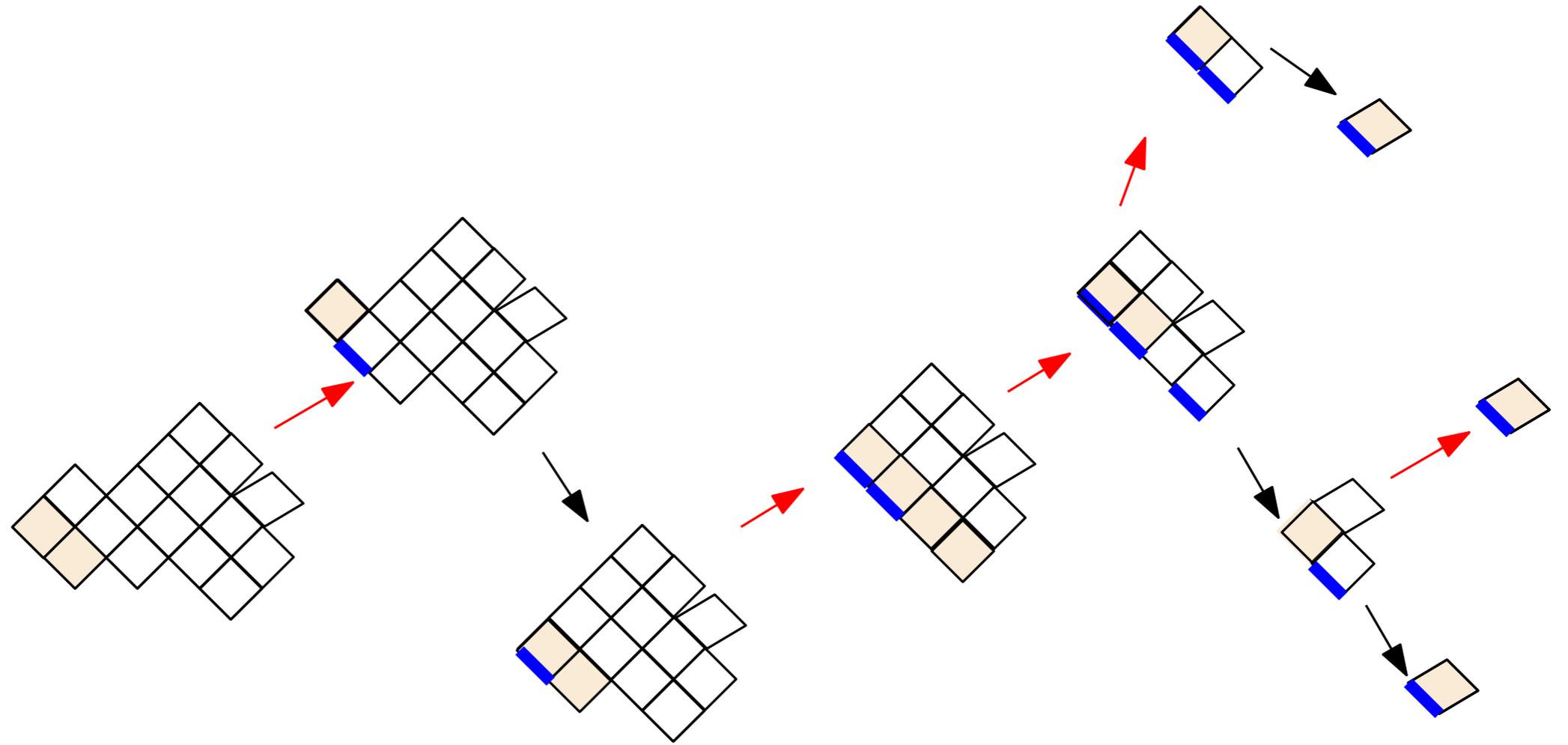
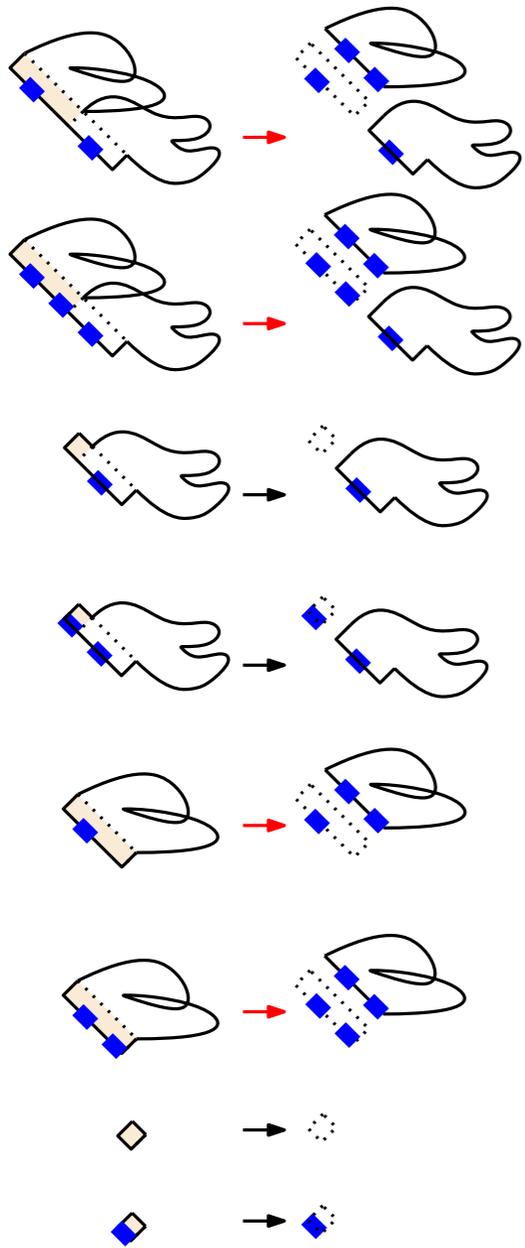


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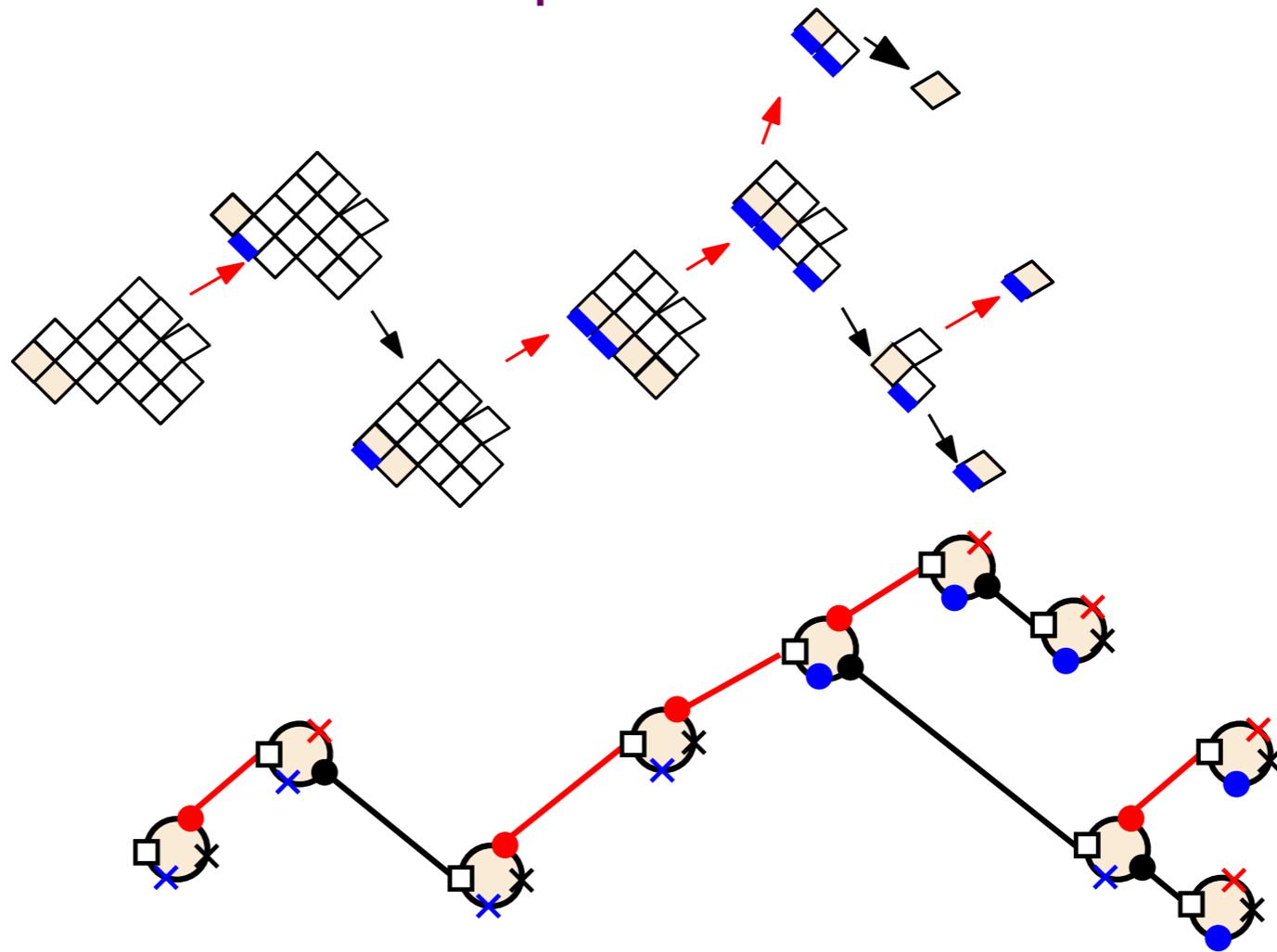
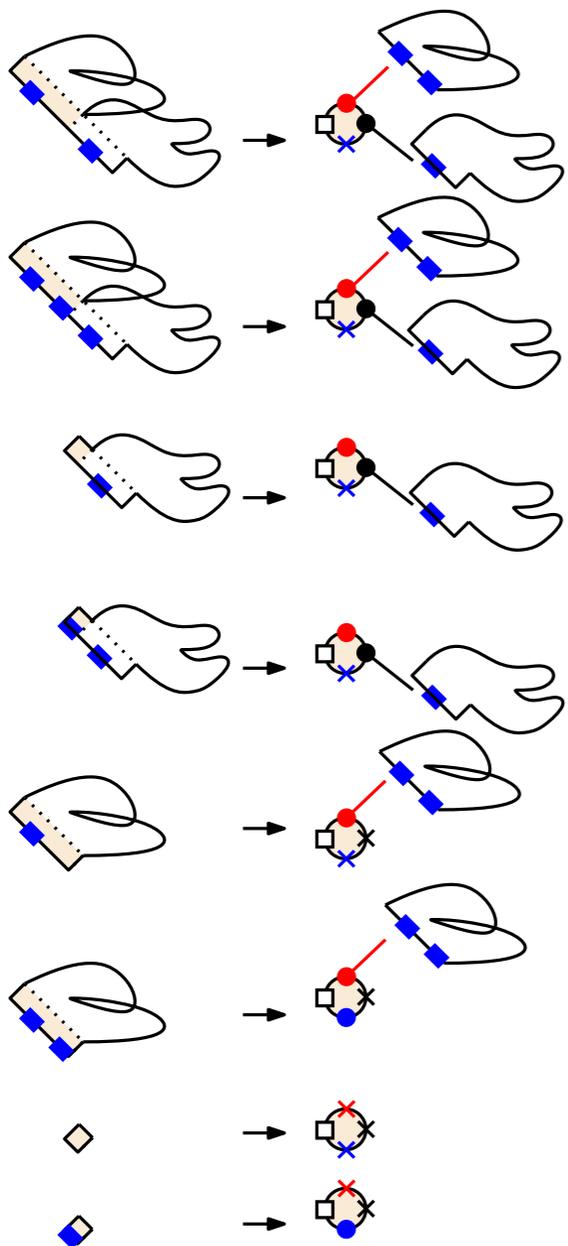
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# An example of sticky decomposition of fighting fish



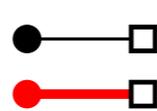
# Conditions on the decomposition tree



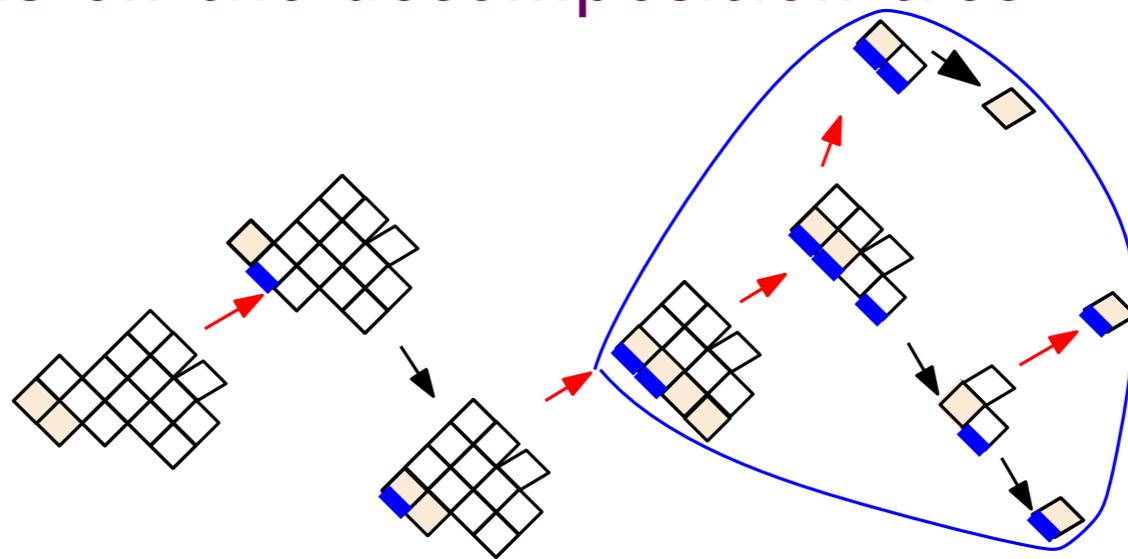
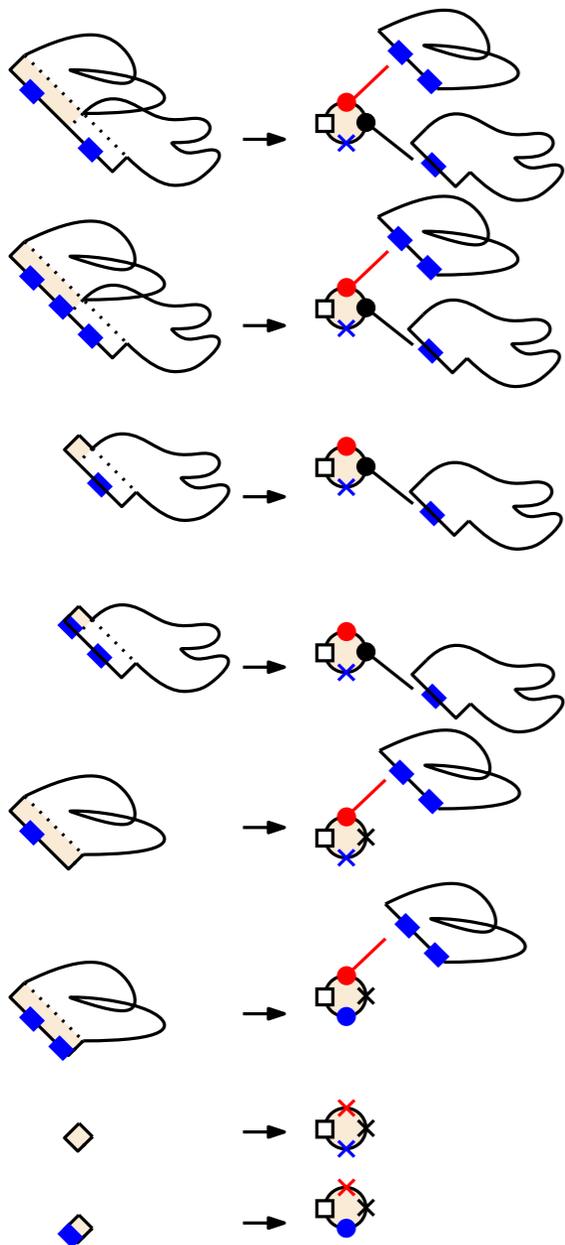
vertices

$$Q = \left\{ \begin{array}{cccc} \begin{array}{c} \text{red dot} \\ \text{black dot} \\ \text{blue dot} \\ \text{blue 'x'} \end{array} & \begin{array}{c} \text{red dot} \\ \text{black dot} \\ \text{blue dot} \\ \text{blue 'x'} \end{array} & \begin{array}{c} \text{red dot} \\ \text{black dot} \\ \text{blue dot} \\ \text{blue 'x'} \end{array} & \begin{array}{c} \text{red dot} \\ \text{black dot} \\ \text{blue dot} \\ \text{blue 'x'} \end{array} \\ \begin{array}{c} \text{red dot} \\ \text{black dot} \\ \text{blue dot} \\ \text{blue 'x'} \end{array} & \begin{array}{c} \text{red dot} \\ \text{black dot} \\ \text{blue dot} \\ \text{blue 'x'} \end{array} & \begin{array}{c} \text{red dot} \\ \text{black dot} \\ \text{blue dot} \\ \text{blue 'x'} \end{array} & \begin{array}{c} \text{red dot} \\ \text{black dot} \\ \text{blue dot} \\ \text{blue 'x'} \end{array} \end{array} \right\}$$

edges



# Conditions on the decomposition tree

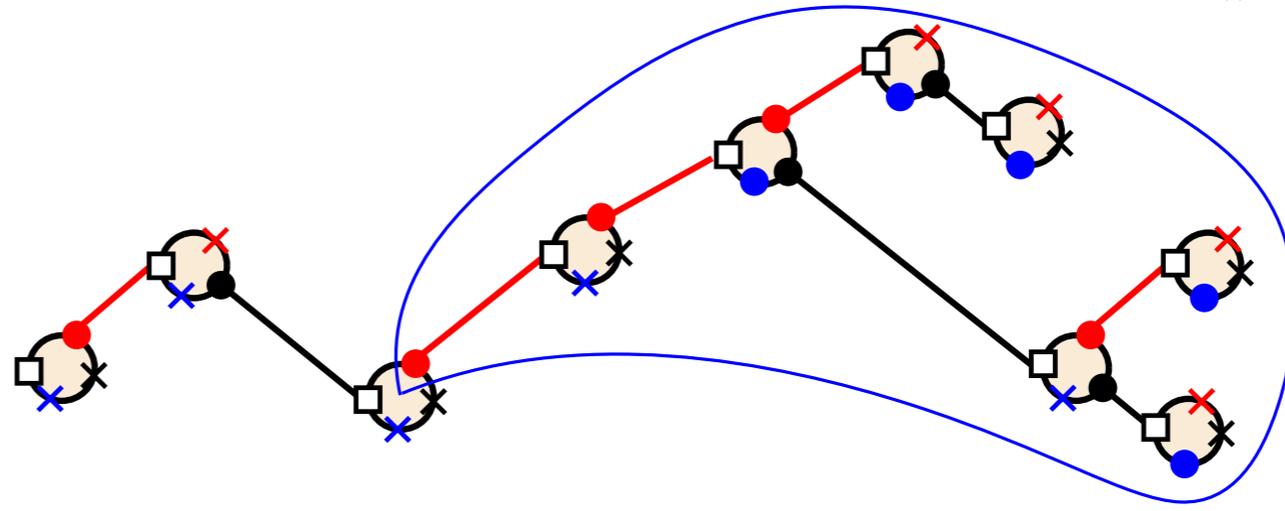


# of sticky edges  
in each fish

$$= \# \bullet - \# \bullet$$

in corresponding  
planted subtree

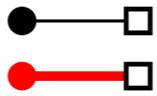
(root pearl  
included)



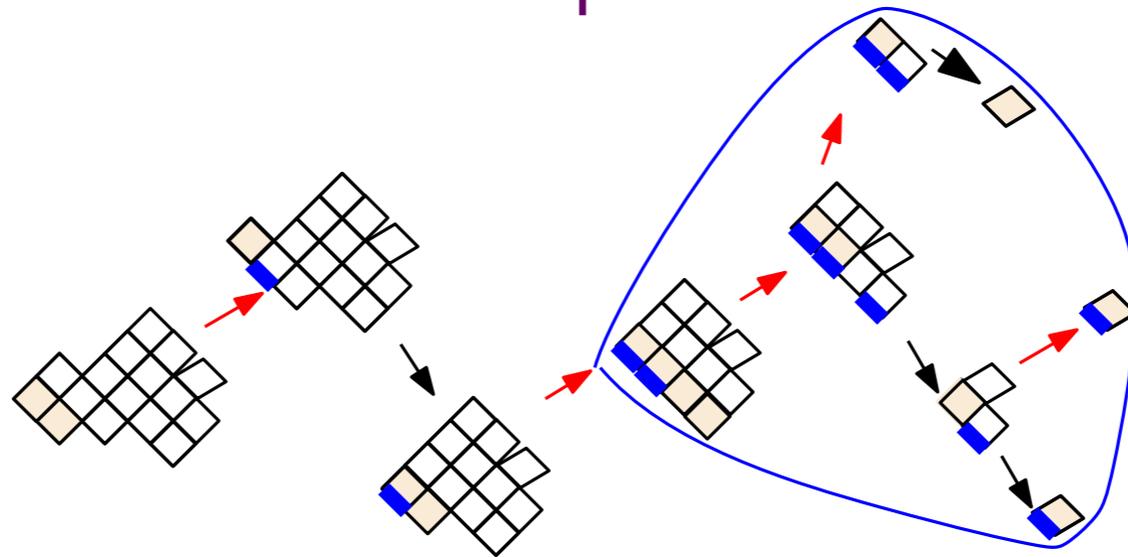
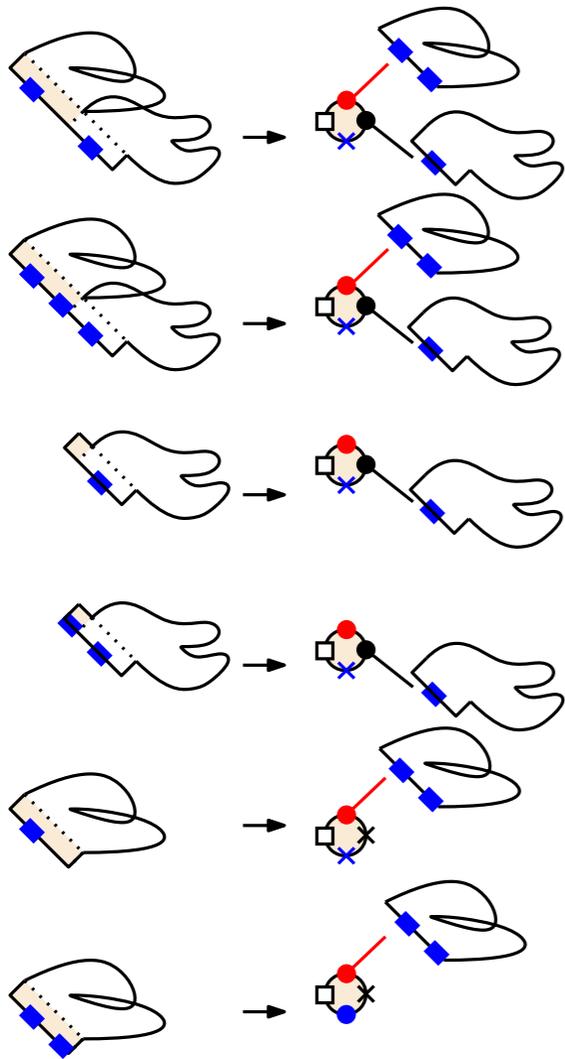
vertices

$$Q = \left\{ \begin{array}{cccc} \begin{array}{c} \times \\ \square \\ \bullet \\ \bullet \end{array} & \begin{array}{c} \times \\ \square \\ \bullet \\ \times \end{array} & \begin{array}{c} \times \\ \square \\ \bullet \\ \times \end{array} & \begin{array}{c} \times \\ \square \\ \bullet \\ \times \end{array} \\ \begin{array}{c} \bullet \\ \square \\ \bullet \\ \bullet \end{array} & \begin{array}{c} \bullet \\ \square \\ \bullet \\ \times \end{array} & \begin{array}{c} \bullet \\ \square \\ \bullet \\ \times \end{array} & \begin{array}{c} \bullet \\ \square \\ \bullet \\ \times \end{array} \end{array} \right\}$$

edges



# Conditions on the decomposition tree

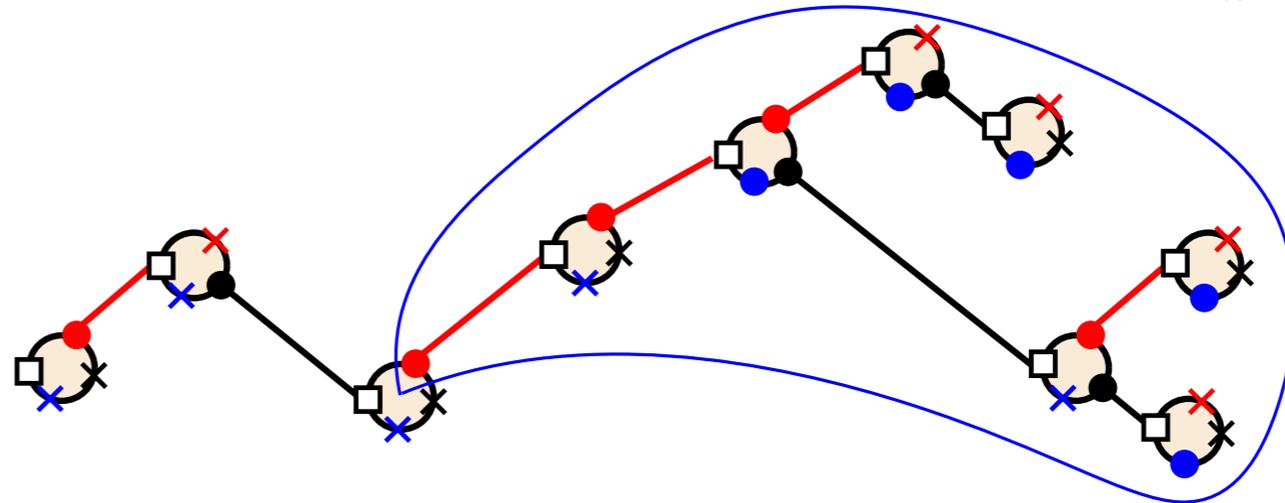


# of sticky edges  
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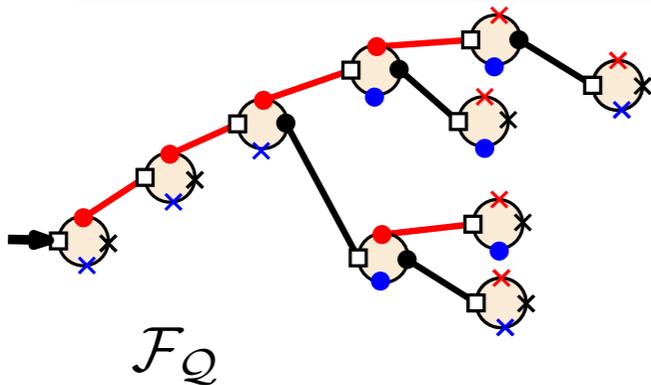
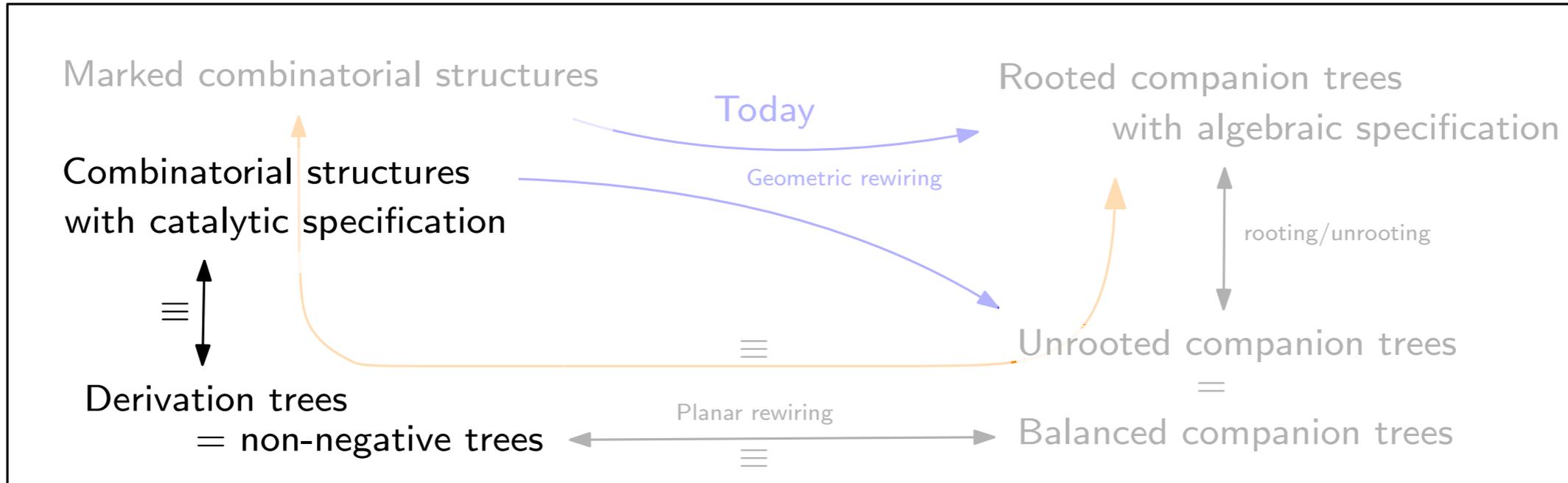
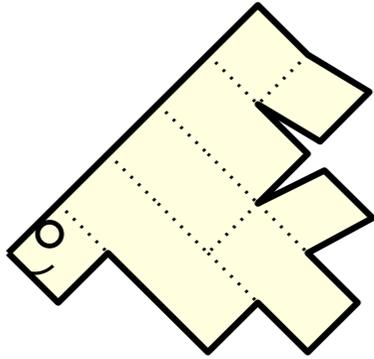


## non-negative Q-tree

vertices	edges	•, • are all matched.
$Q = \left\{ \begin{array}{cccc} \begin{array}{c} \text{fish with } \bullet, \times \\ \text{fish with } \bullet, \times \end{array} & \begin{array}{c} \text{fish with } \bullet, \times \\ \text{fish with } \bullet, \times \end{array} & \begin{array}{c} \text{fish with } \bullet, \times \\ \text{fish with } \bullet, \times \end{array} & \begin{array}{c} \text{fish with } \bullet, \times \\ \text{fish with } \bullet, \times \end{array} \\ \begin{array}{c} \text{fish with } \bullet, \times \\ \text{fish with } \bullet, \times \end{array} & \begin{array}{c} \text{fish with } \bullet, \times \\ \text{fish with } \bullet, \times \end{array} & \begin{array}{c} \text{fish with } \bullet, \times \\ \text{fish with } \bullet, \times \end{array} & \begin{array}{c} \text{fish with } \bullet, \times \\ \text{fish with } \bullet, \times \end{array} \end{array} \right\}$		$\#\{\bullet\} \geq \#\{\bullet\}$ <p>in planted subtrees</p>

# Rewiring for fighting fish

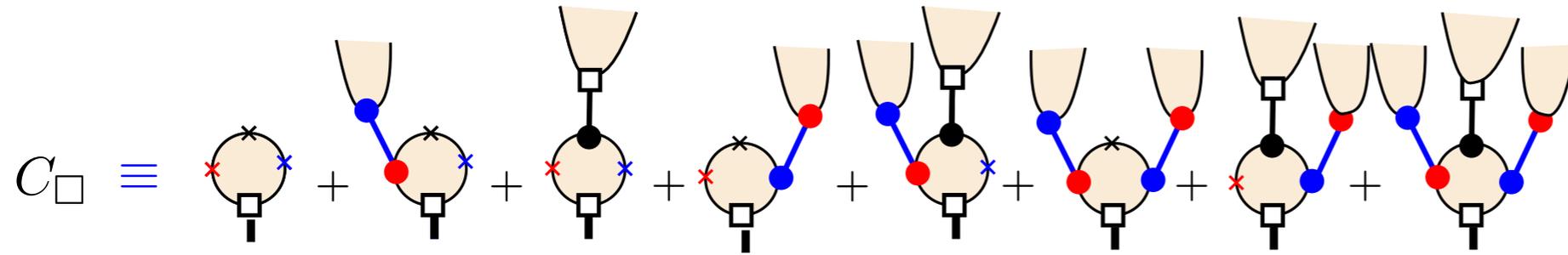
$\mathcal{F}$



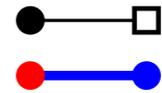




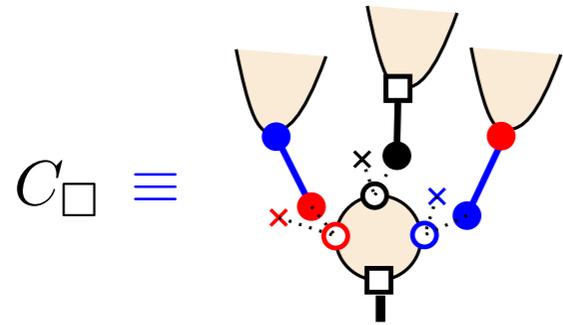
# A grammar for rooted companion trees



edges

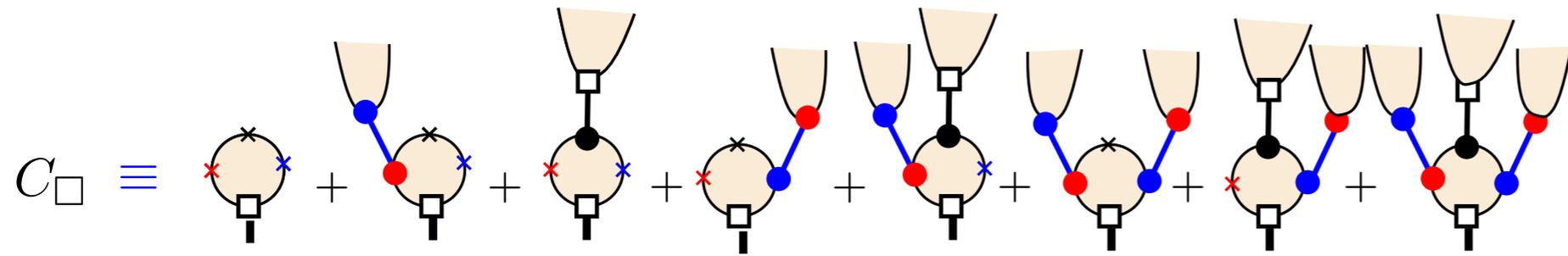


•, •, • are all matched.

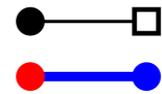


$$\Rightarrow C_{\square} \equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\square}) \times (1 + C_{\bullet})$$

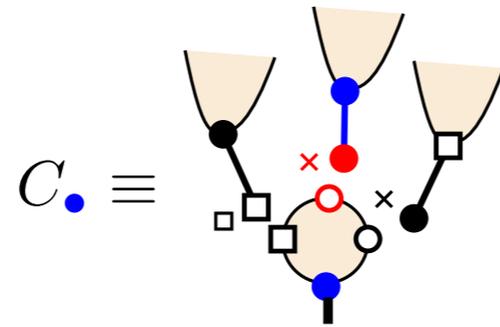
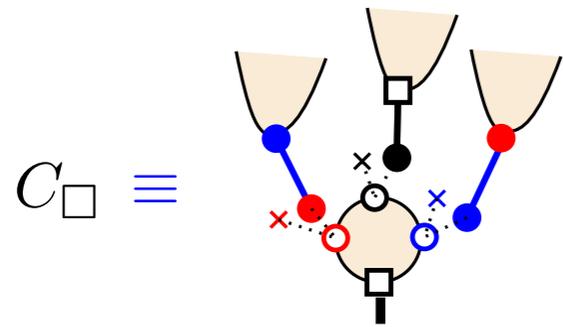
# A grammar for rooted companion trees



edges



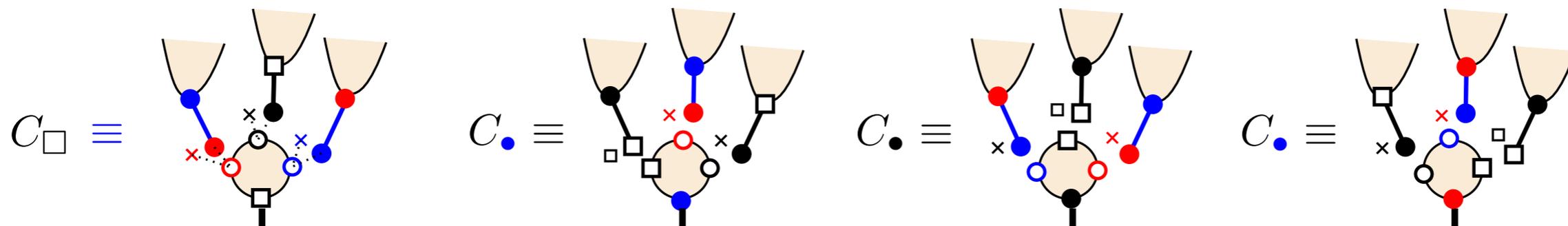
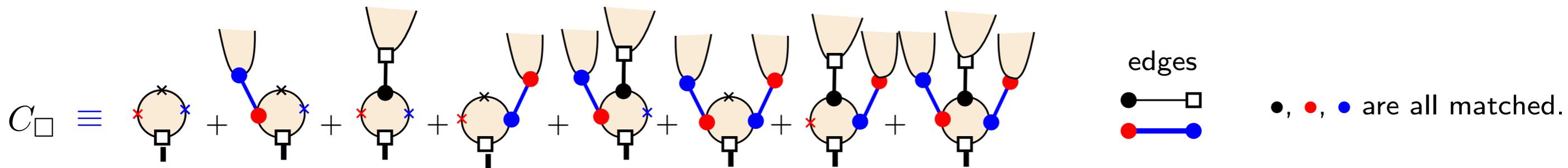
$\bullet, \bullet, \bullet$  are all matched.



$$\Rightarrow C_{\square} \equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\square}) \times (1 + C_{\bullet})$$

$$C_{\bullet} \equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\bullet}) \times (1 + C_{\square})$$

# A grammar for rooted companion trees



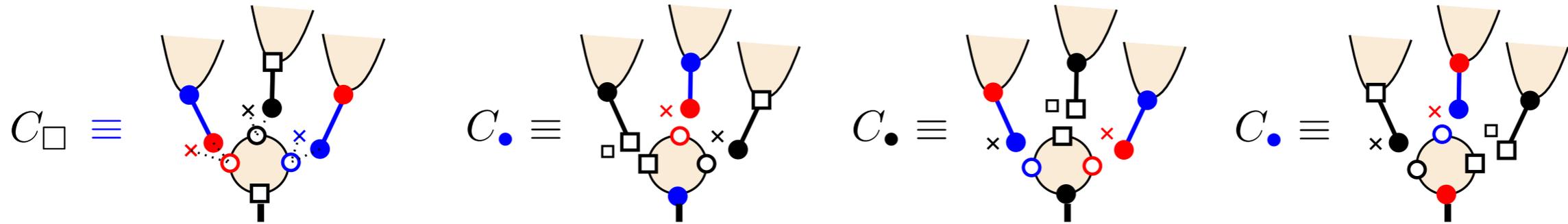
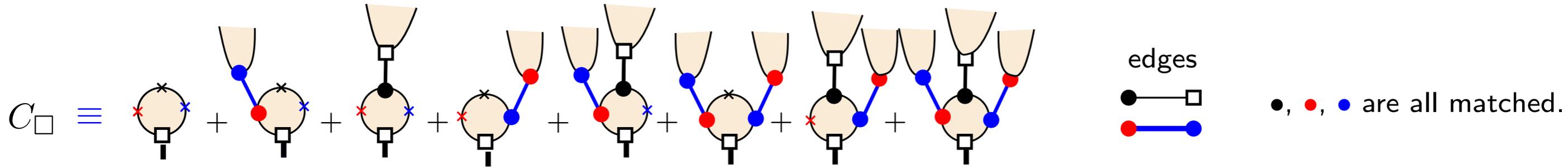
$$\Rightarrow C_{\square} \equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\square}) \times (1 + C_{\bullet})$$

$$C_{\bullet} \equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\bullet}) \times (1 + C_{\square})$$

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$$C_{\bullet} \equiv \mathcal{Z} \times (1 + C_{\square}) \times (1 + C_{\bullet}) \times (1 + C_{\bullet})$$

# A grammar for rooted companion trees



$$\Rightarrow \begin{aligned} C_{\square} &\equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\square}) \times (1 + C_{\bullet}) \\ C_{\bullet} &\equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\bullet}) \times (1 + C_{\square}) \\ C_{\bullet} &\equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\bullet}) \times (1 + C_{\square}) \\ C_{\bullet} &\equiv \mathcal{Z} \times (1 + C_{\square}) \times (1 + C_{\bullet}) \times (1 + C_{\bullet}) \end{aligned}$$

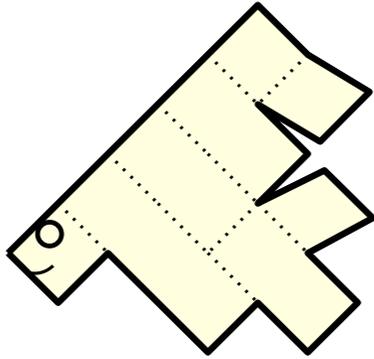
$$\Rightarrow C_{\square} \equiv C_{\bullet} \equiv C_{\bullet} \equiv C_{\bullet} \equiv \mathcal{T}$$

with  $\mathcal{T} = \mathcal{Z} \times (1 + \mathcal{T})^3$

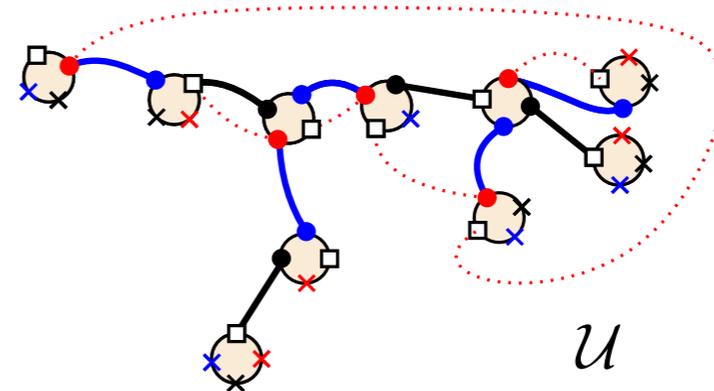
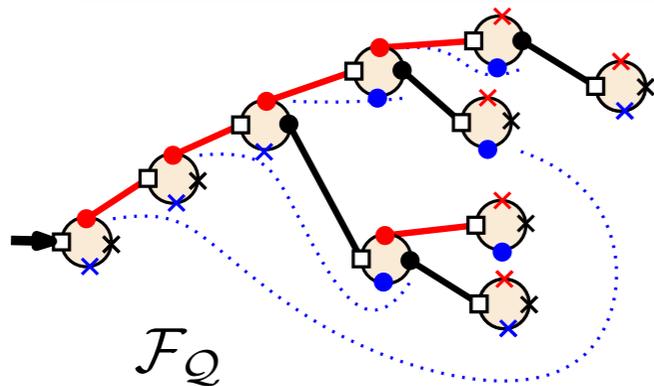
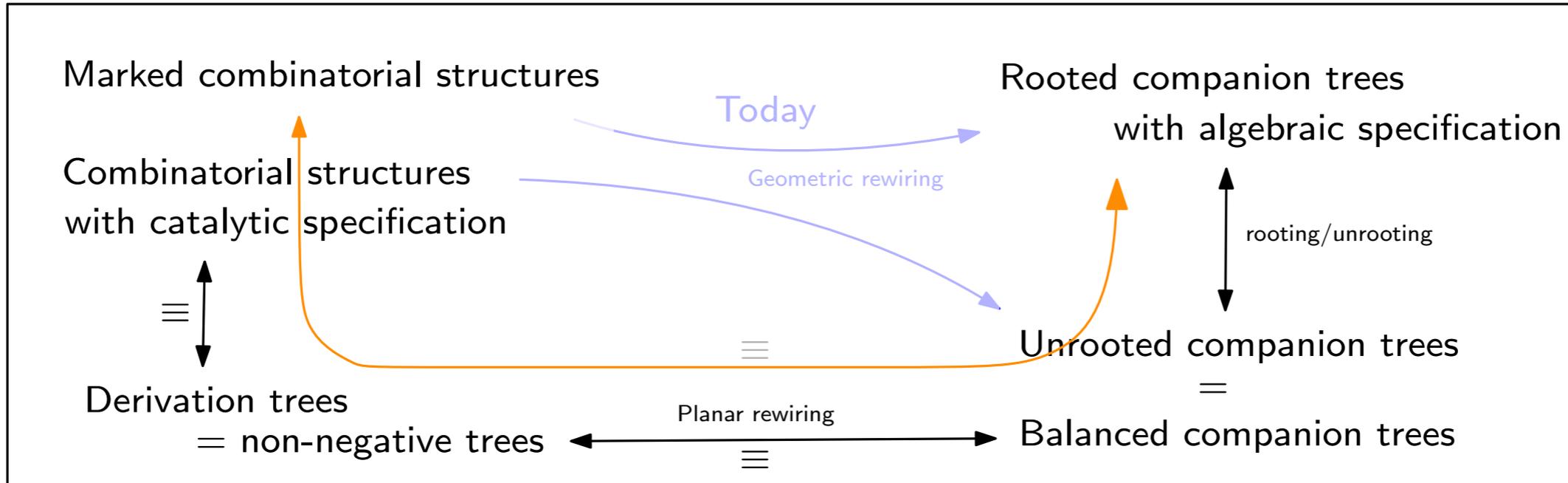
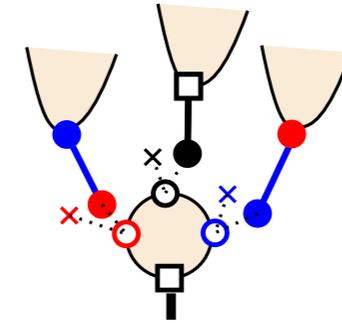
Rooted companion trees associated to fighting fish are rooted ternary trees

# Rooted companion trees

$\mathcal{F}$

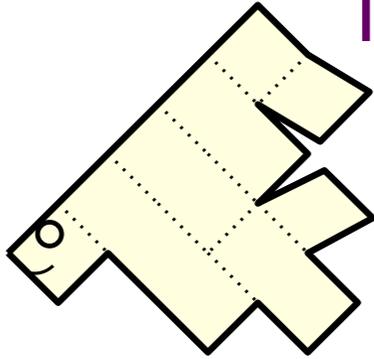


$C_{\square} \equiv$



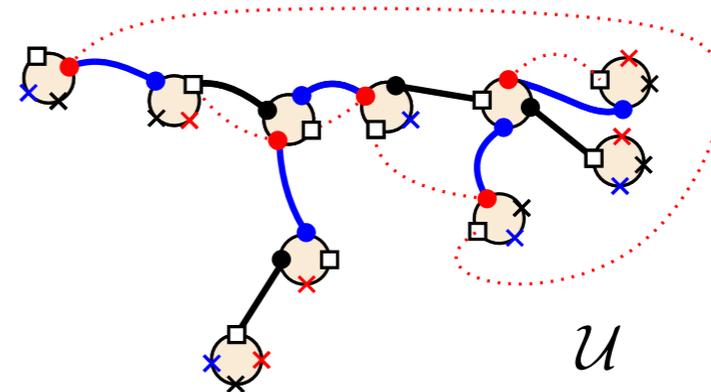
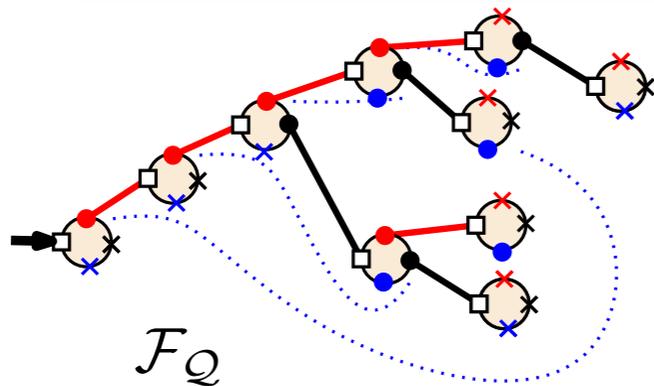
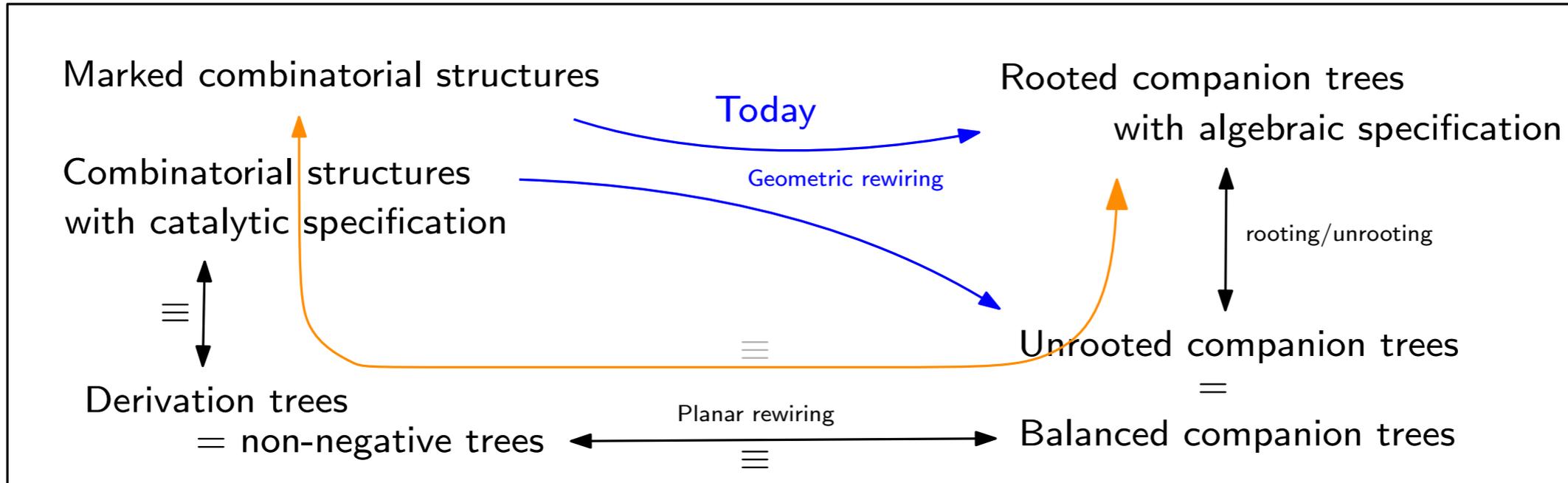
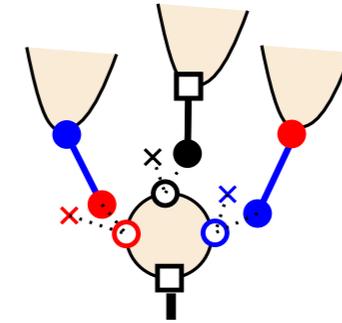
# The Geometric wiring principle

$\mathcal{F}$

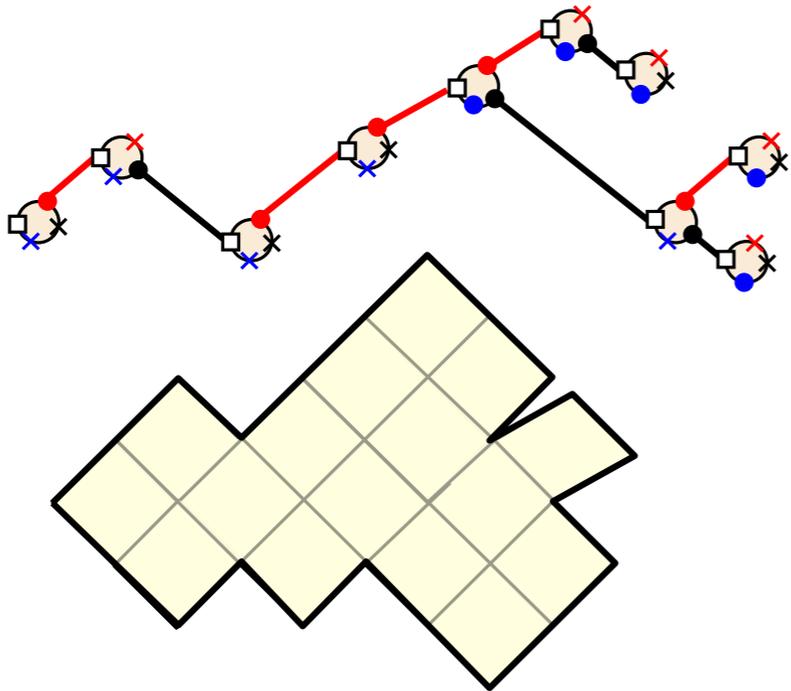


Each  edge should connect an operation creating a -pearl to the -pearl operation using it.

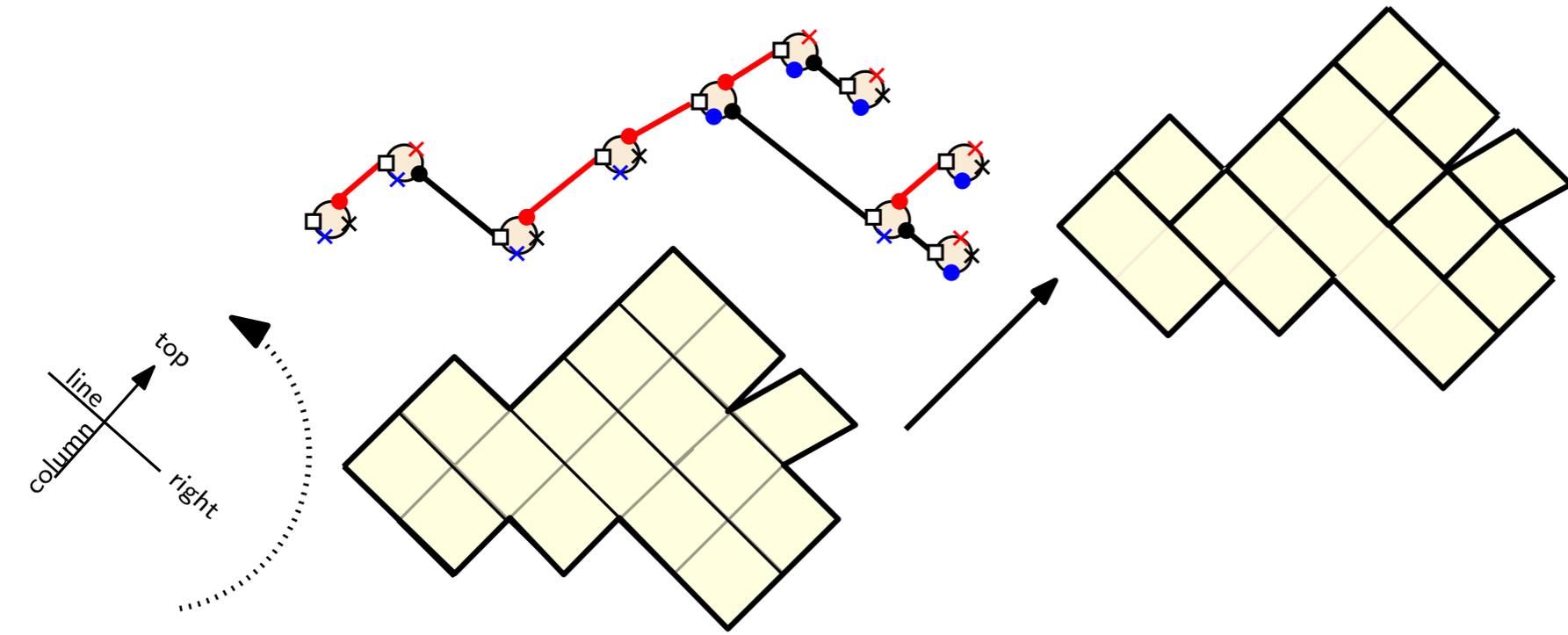
$C_{\square} \equiv$



# How to find the decomposition tree directly in the fish



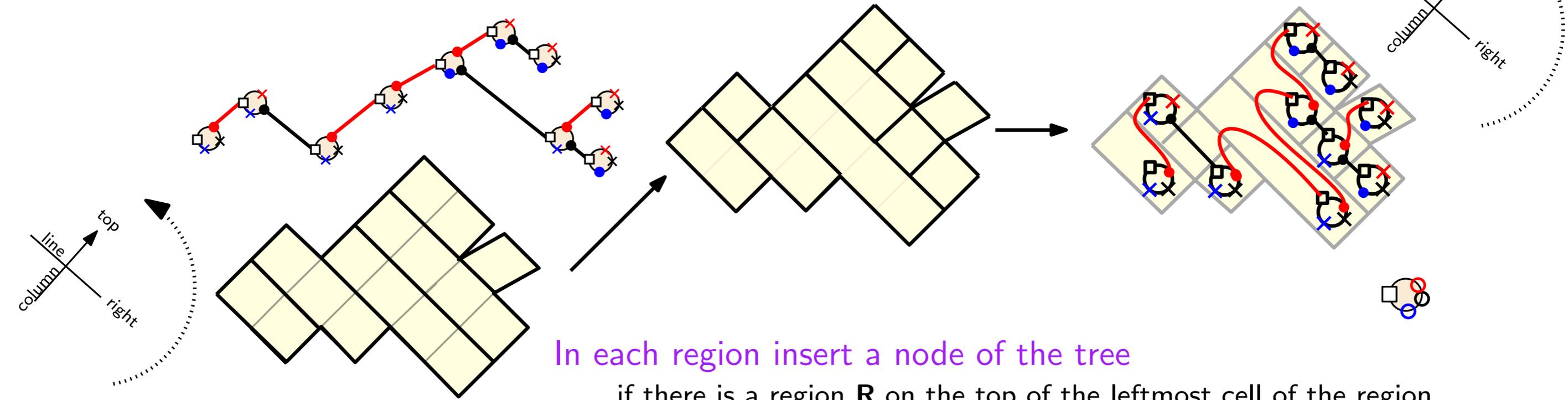
# How to find the decomposition tree directly in the fish



- Slice the fish into lines.
- Divide each line in rectangles after each stem cell

This divides the fish in regions corresponding to each step of the decomposition

# How to find the decomposition tree directly in the fish



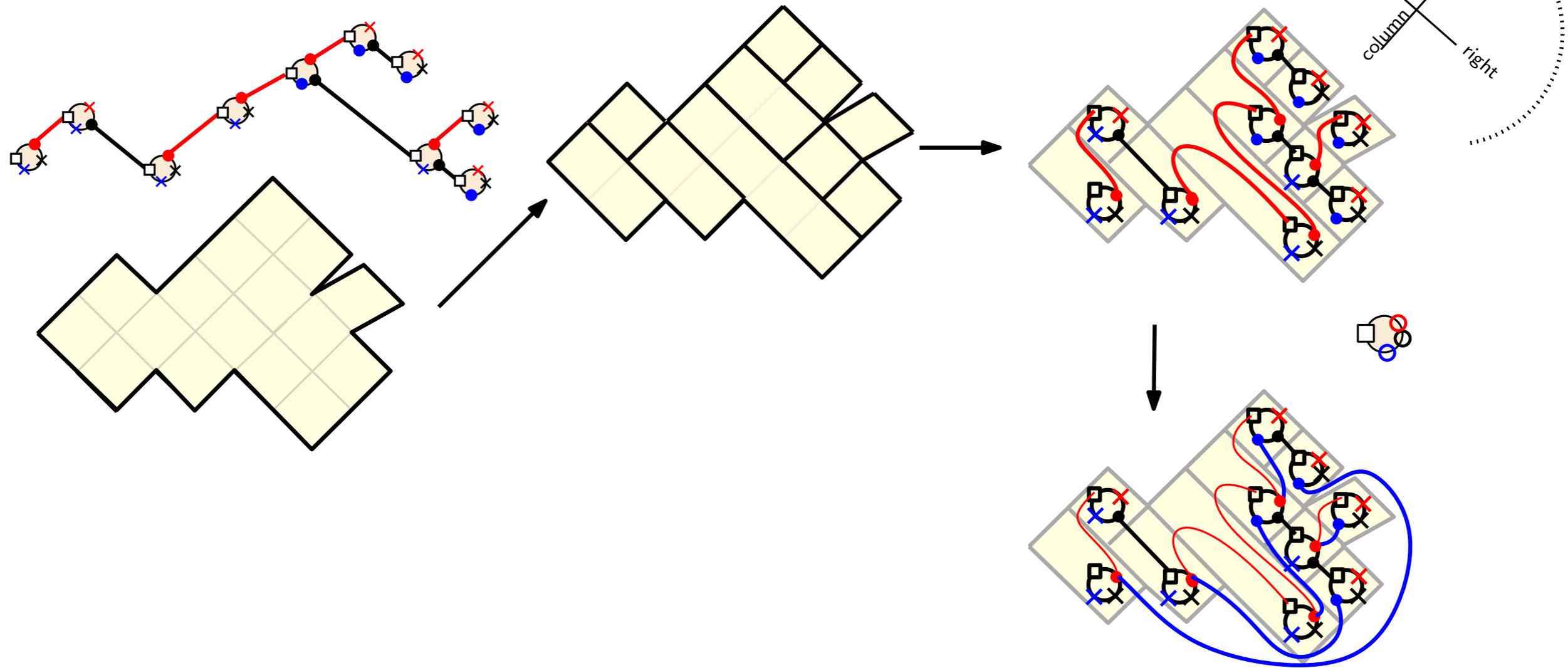
In each region insert a node of the tree

if there is a region  $R$  on the top of the leftmost cell of the region containing the node, then put a ●-pearl on the node, connected to □-pearl of the node in  $R$ .

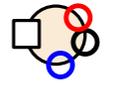
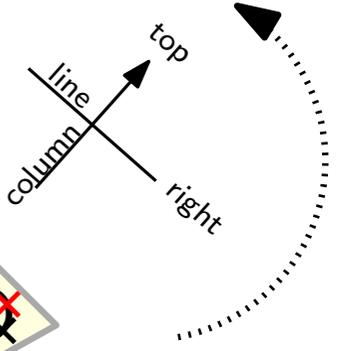
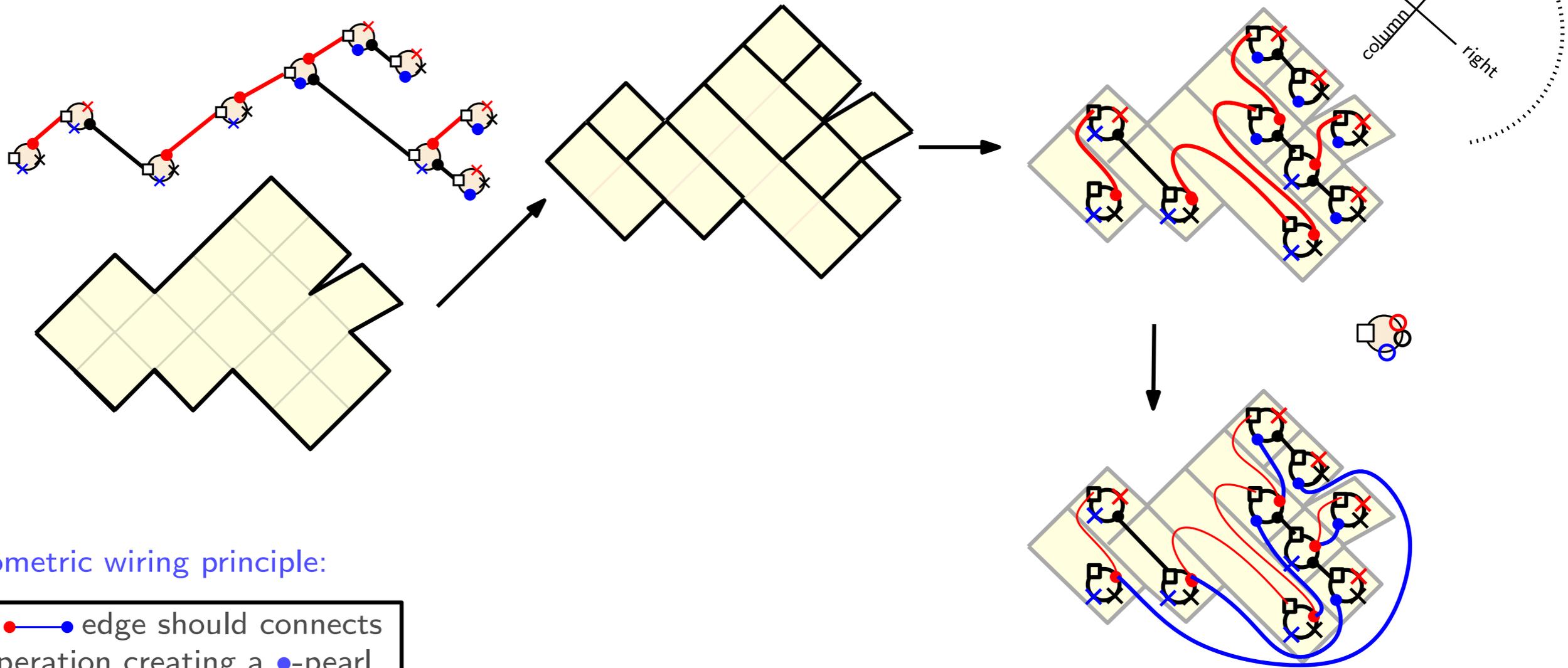
if there is a region  $R$  on the right of the rightmost cell of the region containing the node, then put a ●-pearl on the node, connected to the □-pearl of the node in  $R$ .

if, for a north-east step delimiting region there is another region  $R$  above it (not necessarily adjacent but connected) ending with a north east step in the same diagonal, then put a ●-pearl in the node of  $R$ .  
remaining unmatched positions are marked with ×, × and ×

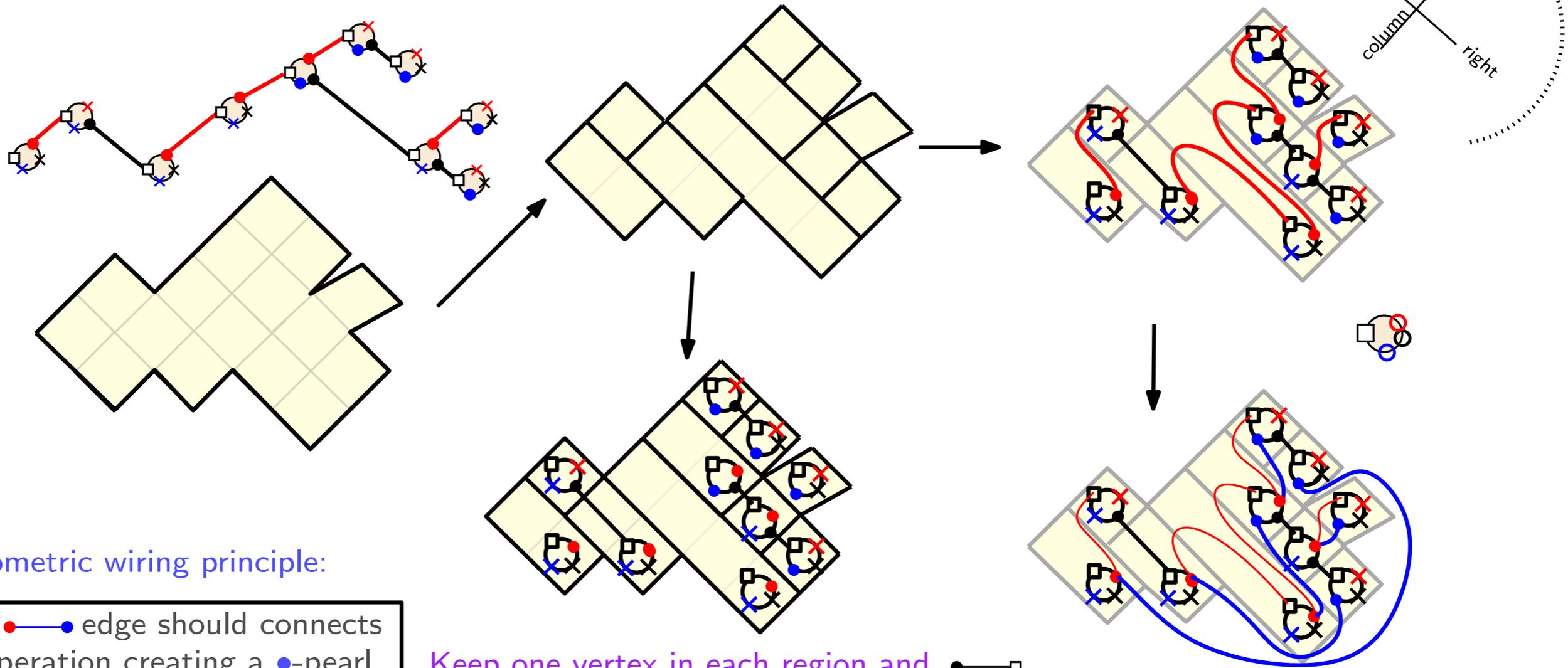
# Planar rewiring versus geometric wiring



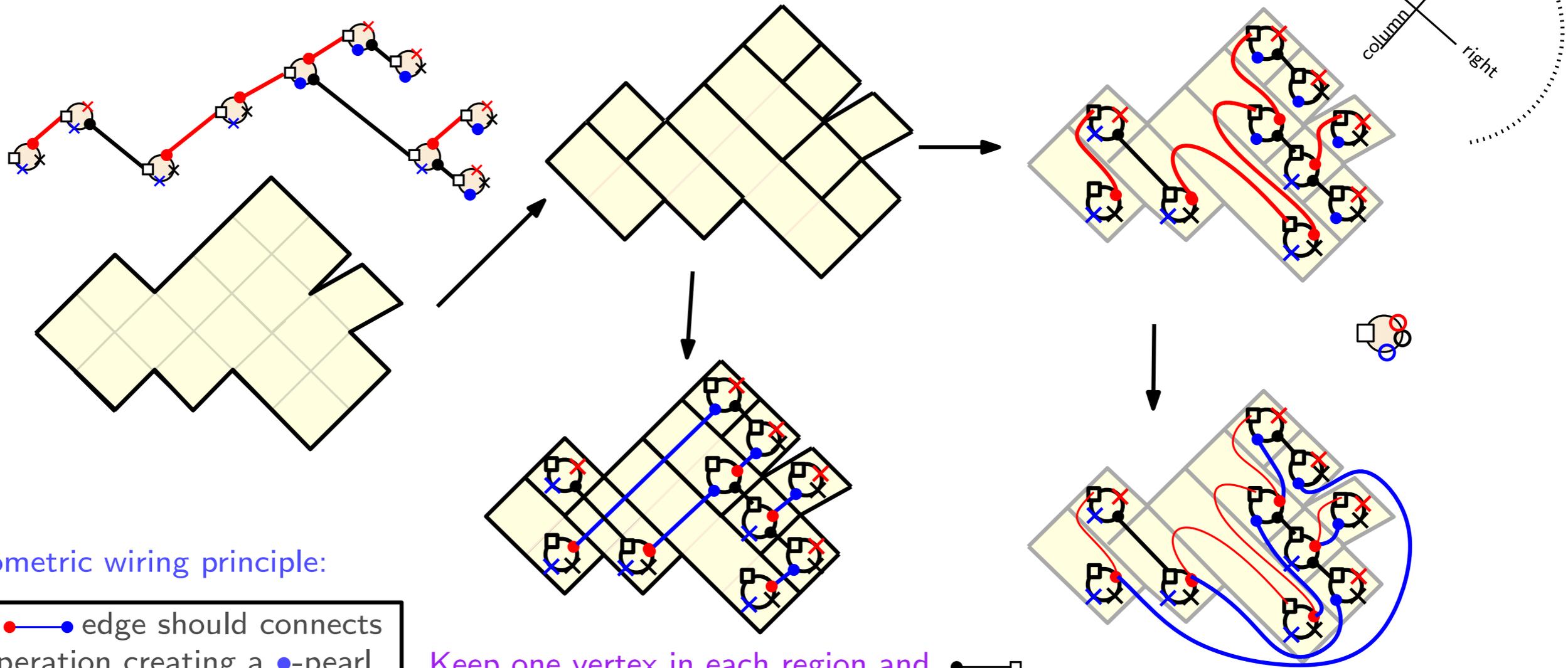
# Planar rewiring versus geometric wiring



# Planar rewiring versus geometric wiring



# Planar rewiring versus geometric wiring



Geometric wiring principle:

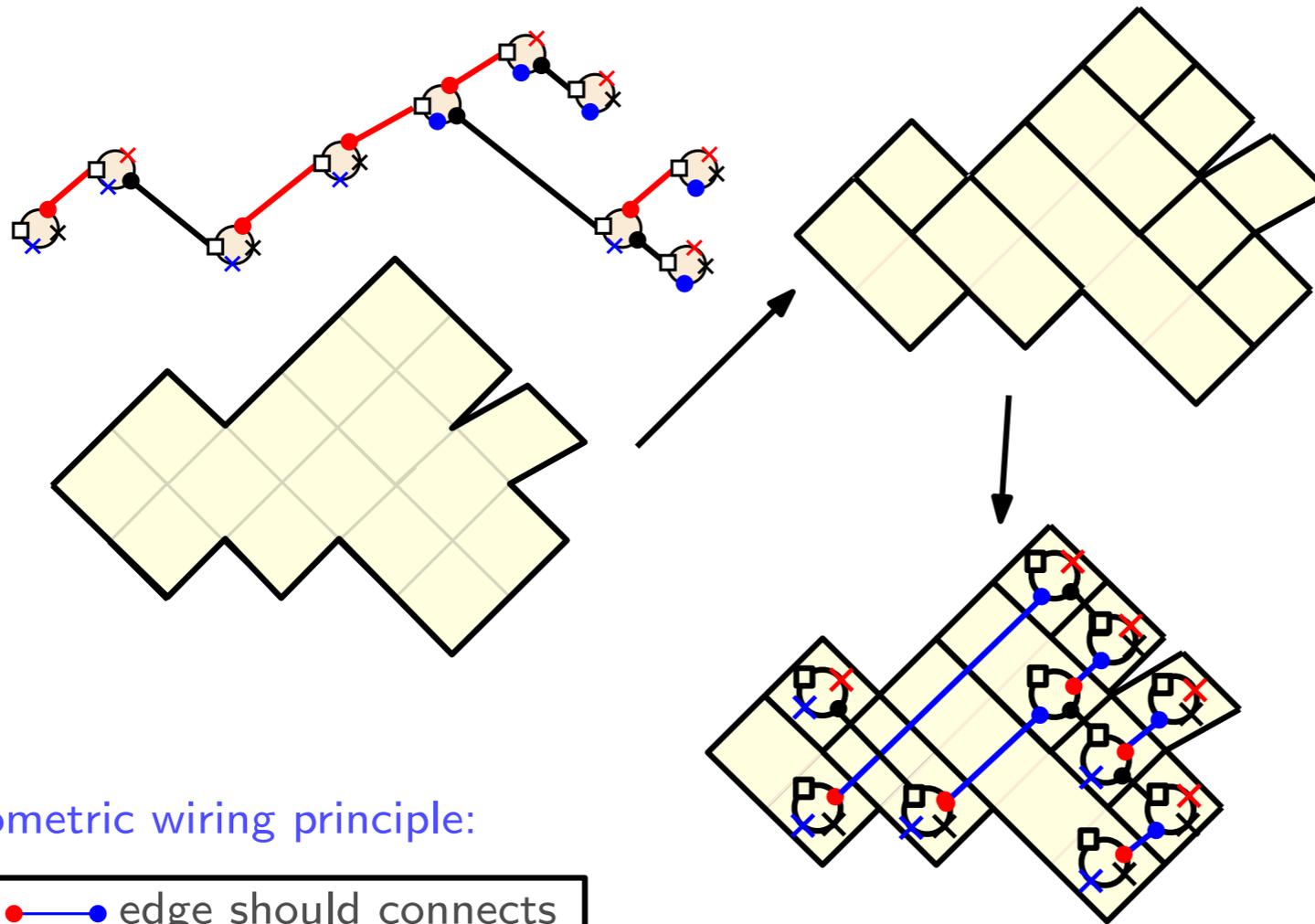
Each - edge should connect an operation creating a -pearl to the -pearl operation using it.

Keep one vertex in each region and edges between vertices in the same line 

- ⇒ Instead of putting the red edges, directly add a blue edge between each  pearl and the  pearl above it in the same column. This matches each red operation with the corresponding blue sticker.

However the planar rewiring of these decomposition trees do not respect the fish structure...

# Planar rewiring versus geometric wiring



**Theorem** (D. Schaeffer 26).  
**Geometric wiring** gives a bijection between fighting fish and unrooted companion ternary trees.

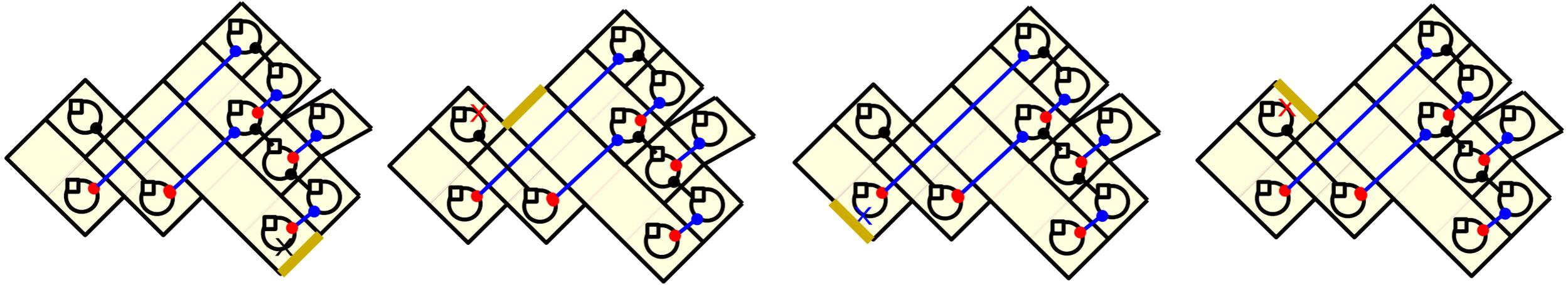
Geometric wiring principle:

Each ●—● edge should connect an operation creating a ●-pearl to the ●-pearl operation using it.

Keep one vertex in each region and ●—□ edges between vertices in the same line

- ⇒ Instead of putting the red edges, directly add a blue edge between each ● pearl and the ● pearl above it in the same diagonal.
- This matches each red operation with the corresponding blue sticker.

## The geometric wiring



**Theorem** (D. Schaeffer 26). **Geometric wiring** gives a bijection between fighting fish and unrooted companion ternary trees.

**Corollary** (Thm of Eu, Fu, Pan 25). There is a bijection between

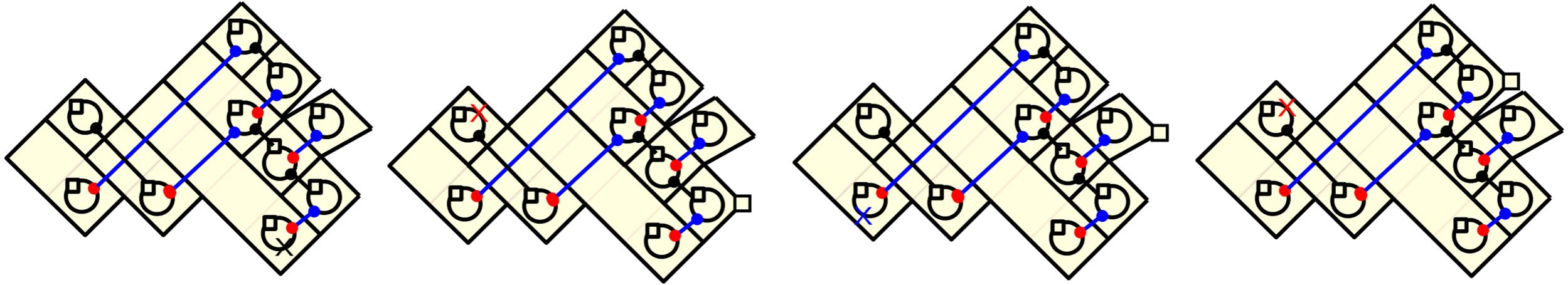
- fighting fish with a marked edge on the border,
- and companion ternary trees rooted on an unmatched  $\square$ -pearl, or  $\times$ -,  $\times$ - or  $\times$ -spot.

$$C_{\times} \equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\bullet}) \times (1 + C_{\square}) \equiv \mathcal{T}, \quad \text{same for } C_{\square}, C_{\times}, C_{\times}$$

**Corollary** (DGRS 16) The number of fighting fish with perimeter  $2n$  is

$$2(n+1) \cdot \mathcal{F}_n = 4 \cdot |\mathcal{T}_n| \quad \Rightarrow \quad |\mathcal{F}_n| = \frac{4}{2(n+1)} \cdot \frac{1}{2n+1} \binom{3n}{n}$$

## The geometric wiring



**Corollary** (Eu, Fu, and Pan 25). There is a bijection between fighting fish marked on a tail and ternary trees rooted on a vertex with a  $\times$  and a  $\times$ , or equivalently with couples of possibly empty ternary trees rooted respectively on a  $\bullet$  and on a  $\bullet$ .

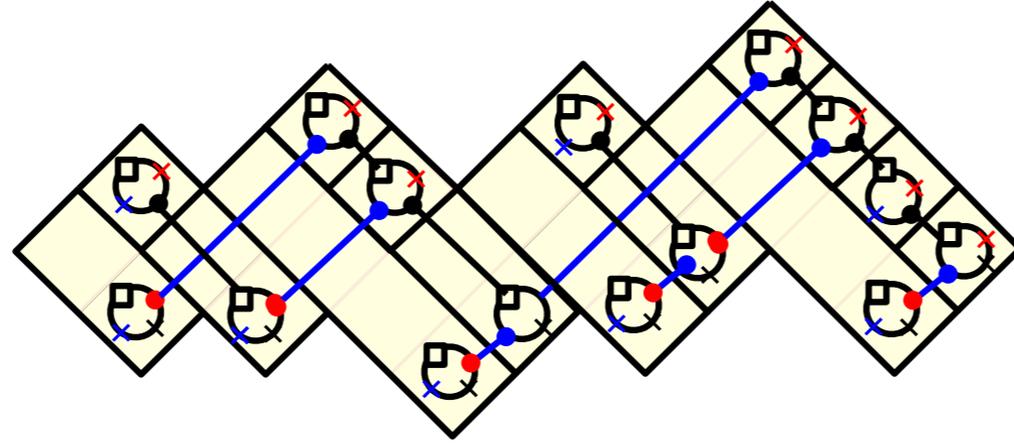
Hence the family  $\mathcal{F}^>$  of fish with a marked tail satisfies

$$\mathcal{F}^> \equiv C_{\times \times} \equiv \mathcal{Z} \times (1 + C_{\bullet}) \times (1 + C_{\bullet}) \equiv \mathcal{Z} \times (1 + \mathcal{T})^2.$$

**Corollary.** (DGRS 16) The number of fighting fish with semi perimeter  $n + 1$  with a marked tail is

$$\mathcal{F}_n^> = \frac{2}{2n+2} \binom{3n+1}{n}$$

# The geometric closure and Aval et al's bijection

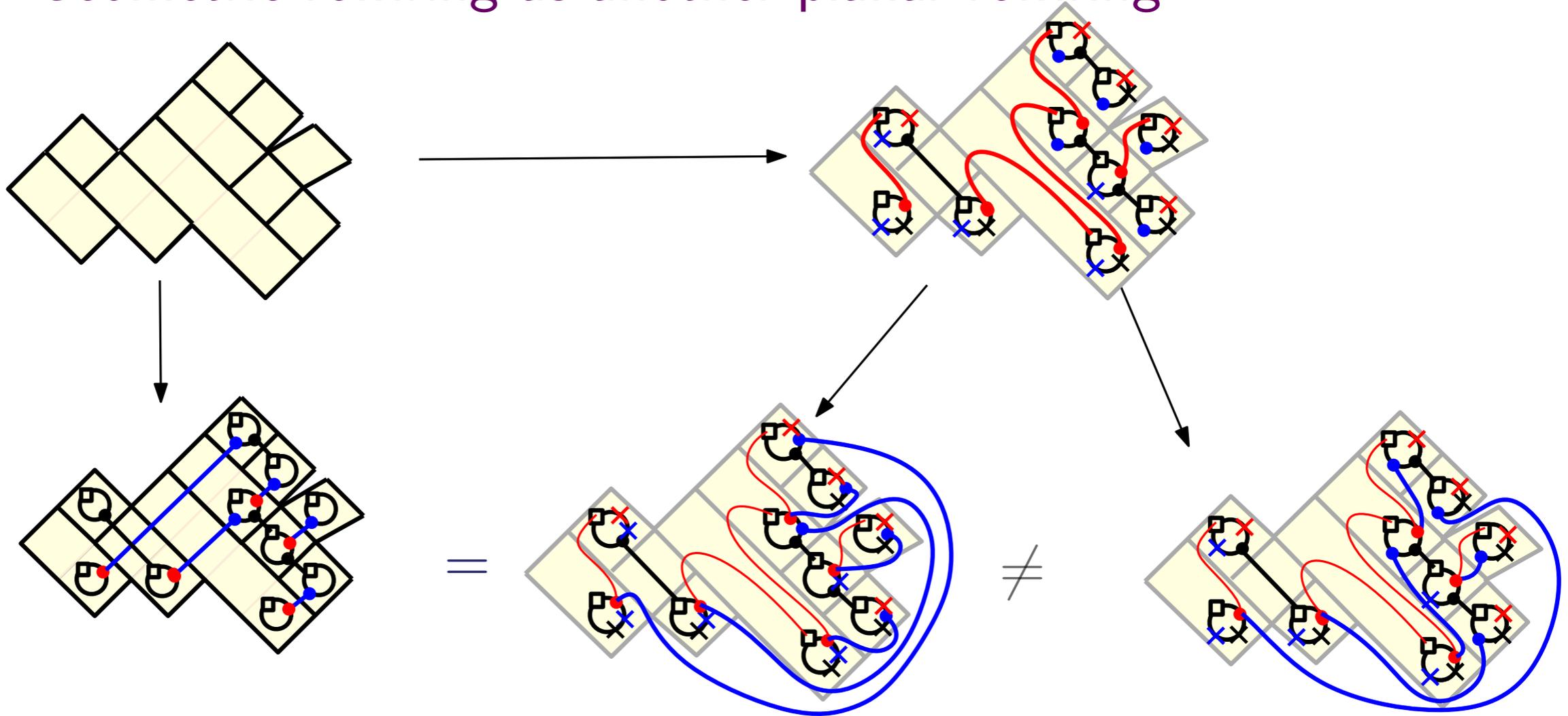


In the case of polyominoes (fish with a unique tail), the associated ternary trees are exactly ternary trees without vertices with both a  $\bullet$  and a  $\blacklozenge$  so that each branch from the tail vertex never comes back toward the right hand side: each of the two trees is thus a binary trees and we recover a bijection of Aval et al.

**Corollary.** The **geometric wiring** is a bijection between parallelogram polyominoes and ternary trees with a unique vertex with both a  $\times$  and a  $\times$ , or equivalently with couples of possibly empty binary trees rooted respectively on a  $\bullet$ -pearl and on a  $\bullet$ -pearl.

Why is geometric wiring a bijection?

# Geometric rewiring as another planar rewiring



The geometric companion tree

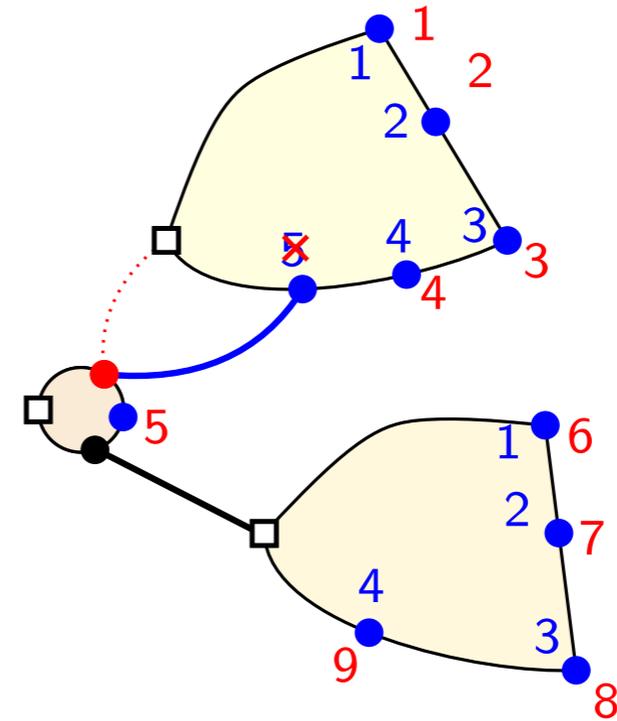
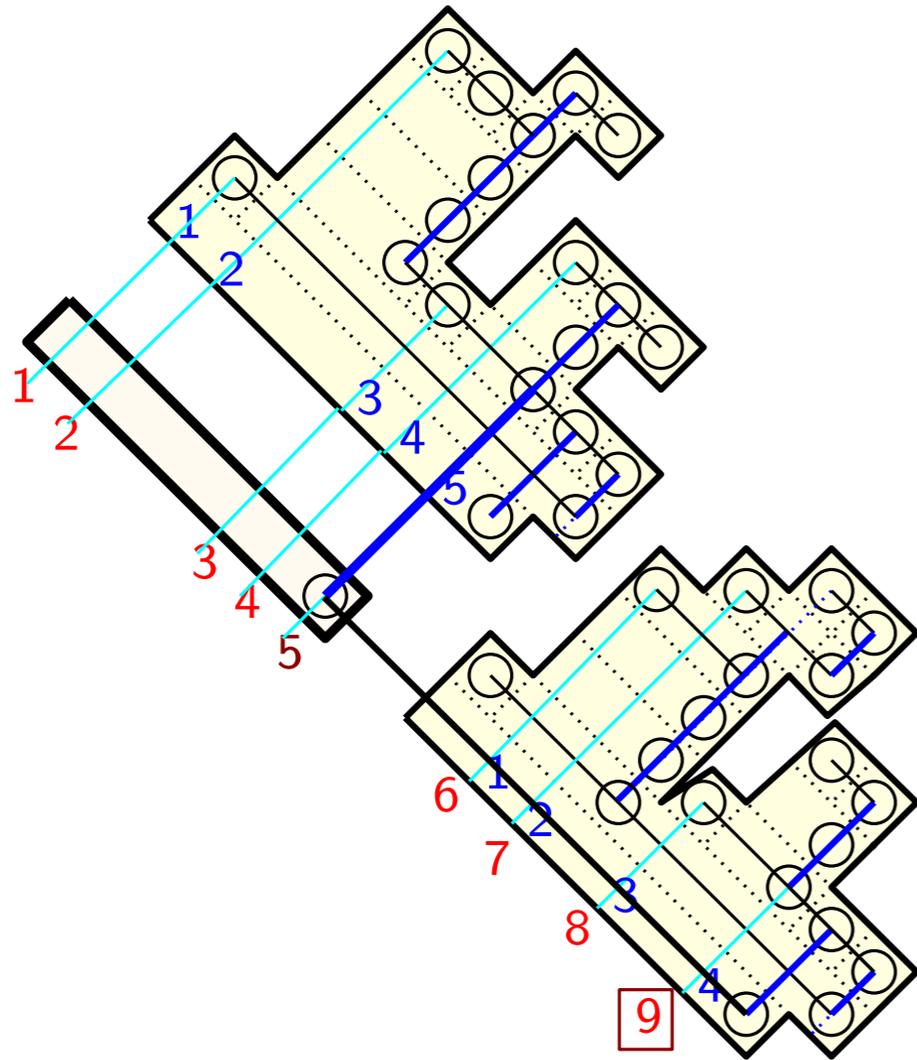
The  $\{\text{node}\}$ -companion tree

The  $\{\text{node}\}$ -companion tree

**Lemma:** Geometric companion tree and  $\{\text{node}\}$ -companion tree are the same.

**Corollary:** Geometric wiring is a bijection with unrooted geometric companion trees.

# A proof of the lemma by recurrence



At a given step the planar rewiring will be used and arranged ●-pearls in the same order as the geometric wiring

In this step of the decomposition, the geometric construction, gives the new sticker position 5 while the next sticker to be used is 9, the former sticker number 4 of the second fish.

# Proof of the main theorem

**PRINCIPLE** (D. Schaeffer 26). Provided **geometric wiring** can be realized as a planar rewiring, it is a bijection between your favorite combinatorial structures and unrooted companion  $\mathcal{Q}$ -trees

# Proof of the main theorem

**PRINCIPLE** (D. Schaeffer 26). Provided **geometric wiring** can be realized as a planar rewiring, it is a bijection between your favorite combinatorial structures and unrooted companion  $\mathcal{Q}$ -trees

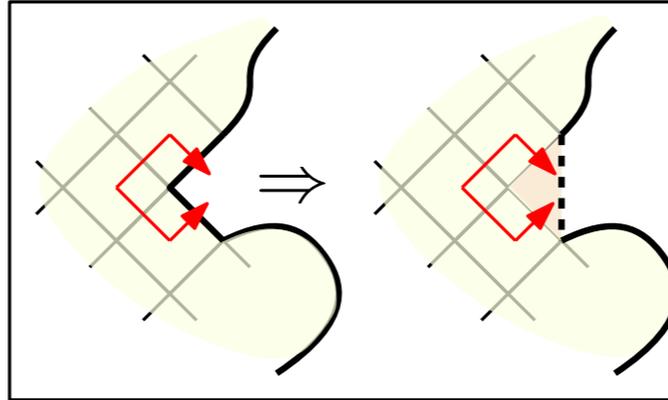
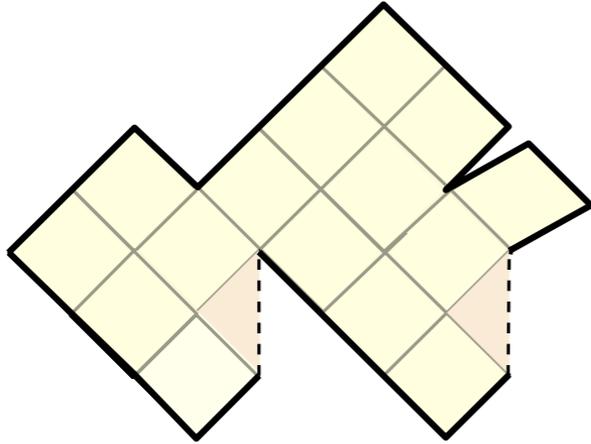
When can we hope to apply this principle?

*Hypotheses on the catalytic specification:*

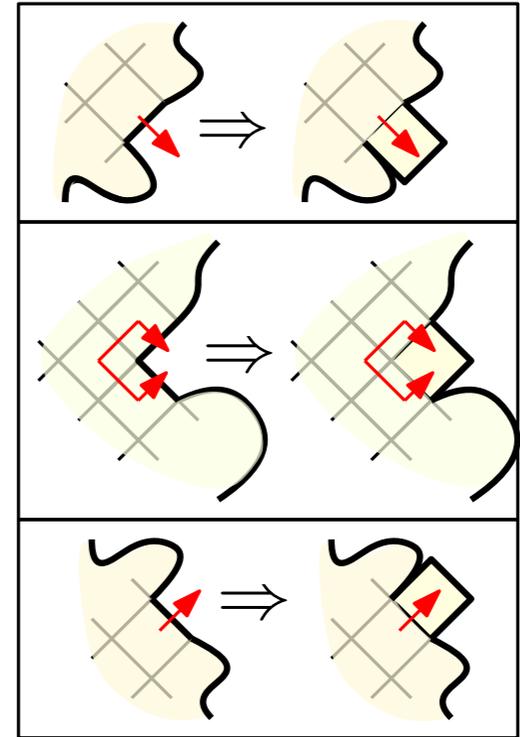
- **active sites** are *concatenated* in  $\times$  product constructions
- new active sites appear only on the **boundary** of substructures
- the red operations uses **boundary** active sites

Extended fighting fish and Tamari

# Extended fighting fish and Tamari lattices

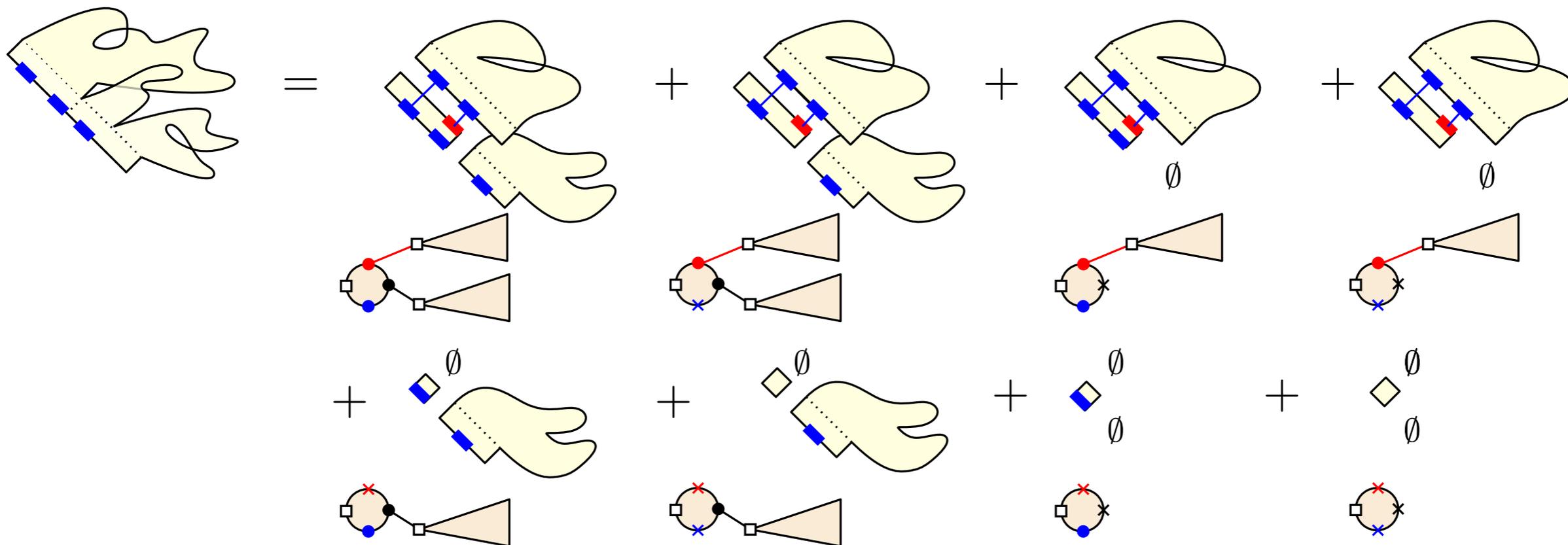


- extra rule to create boundary triangles
- vertical boundary edges cannot grow anymore
- vertical boundary edges do not count in size

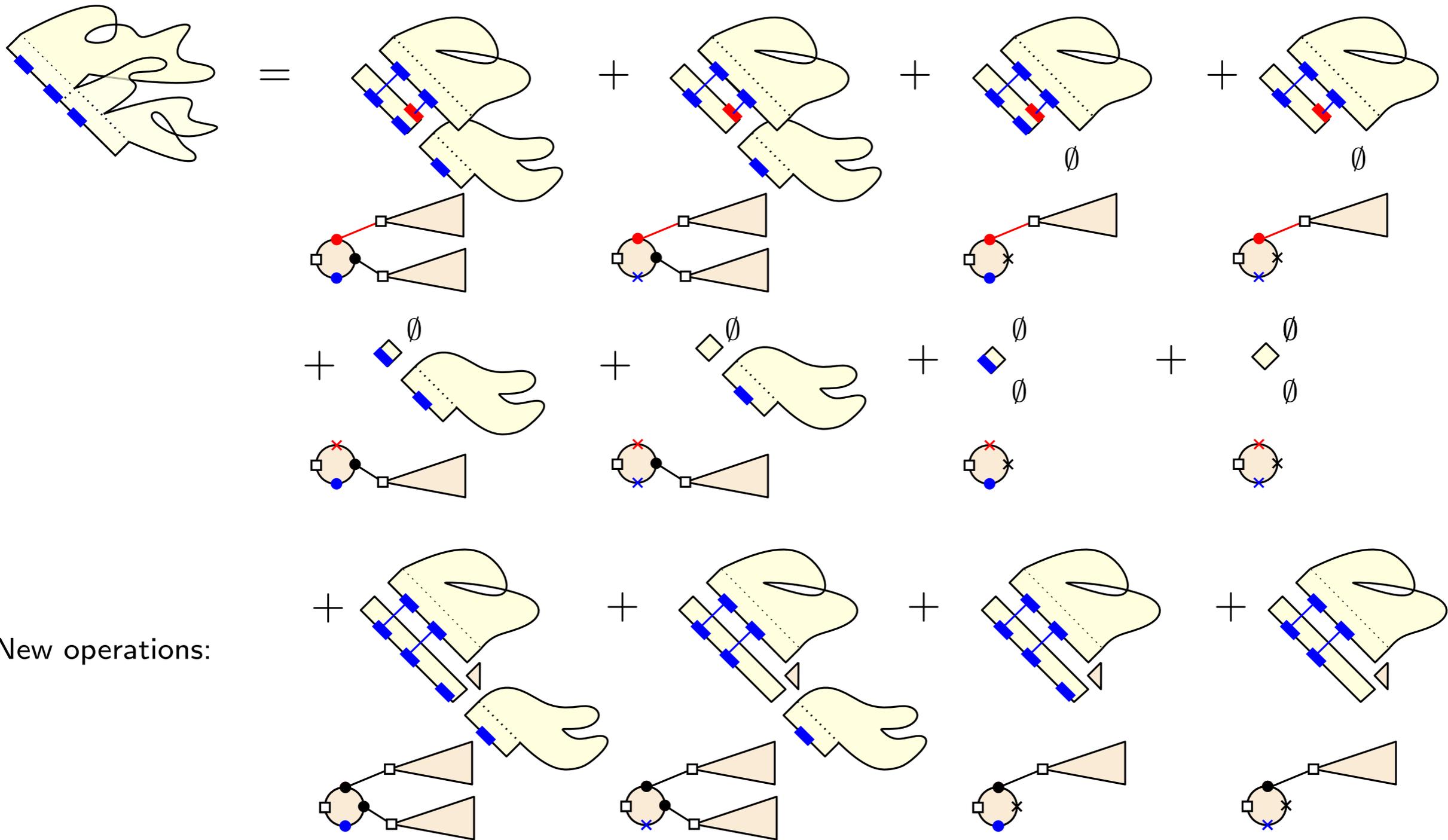


Extended fighting fish are in bijection with Tamari lattice intervals.

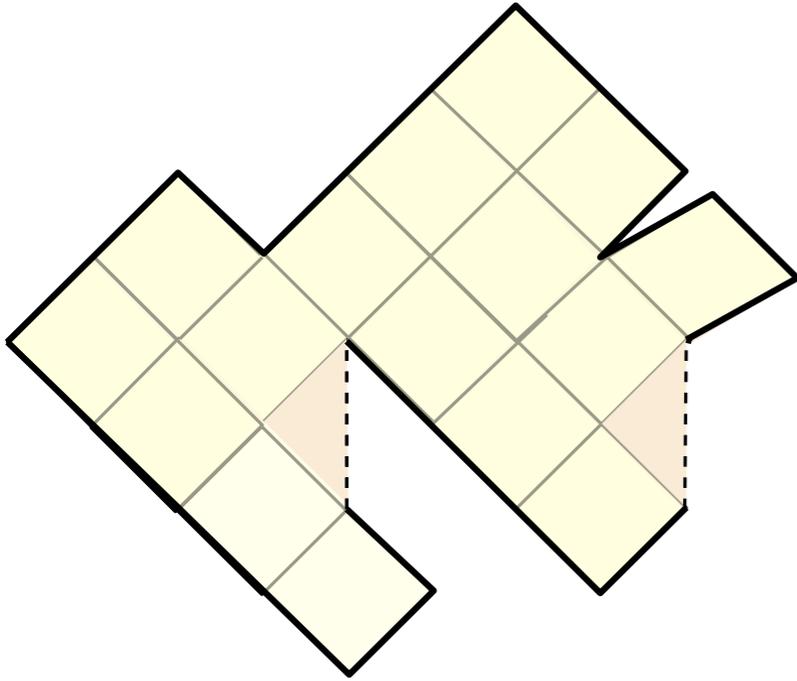
# Sticky extended fighting fish



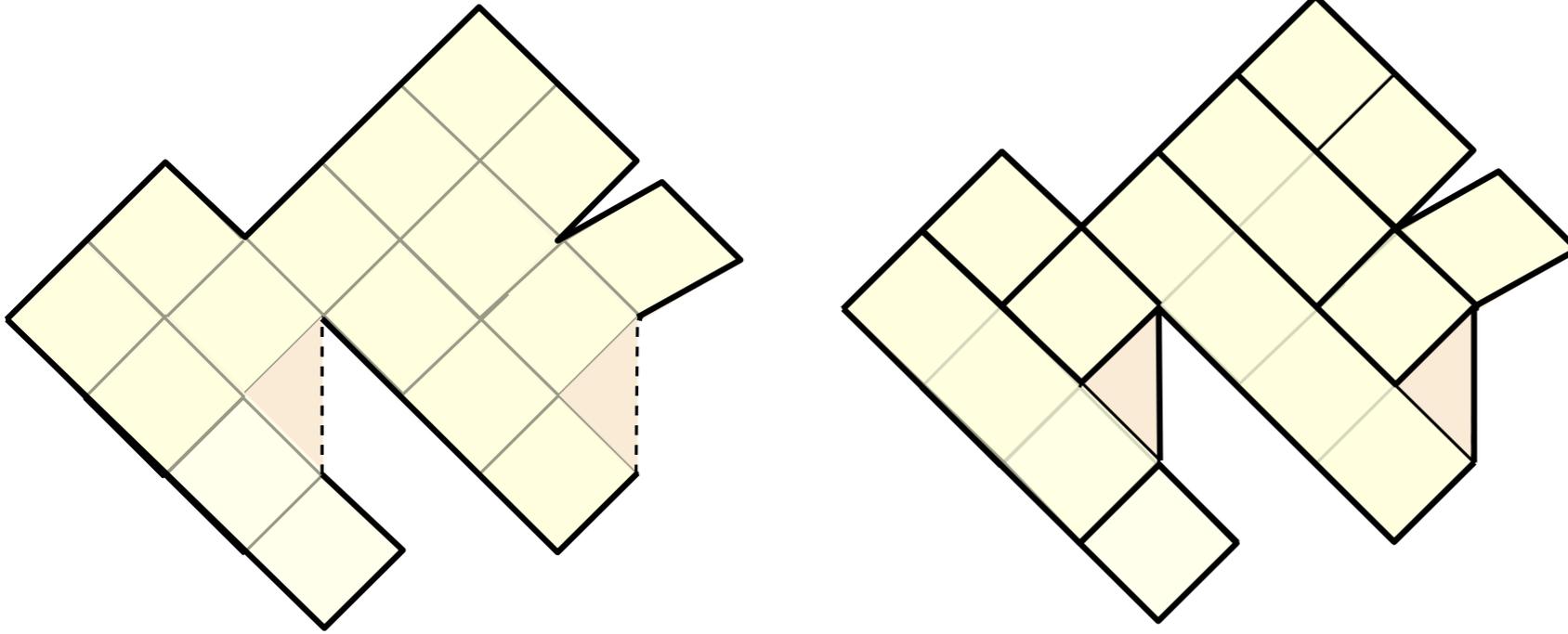
# Sticky extended fighting fish



# Geometric wiring for extended fish

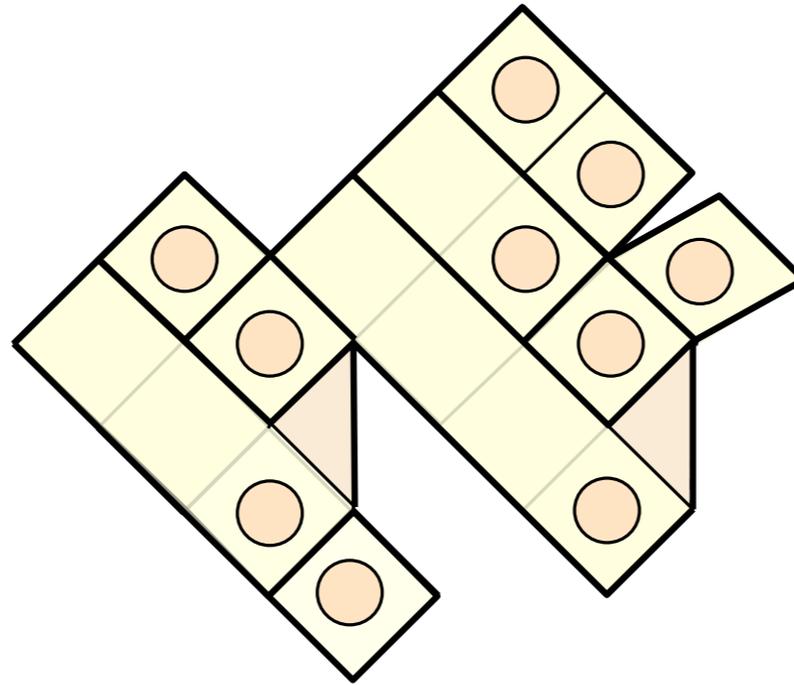
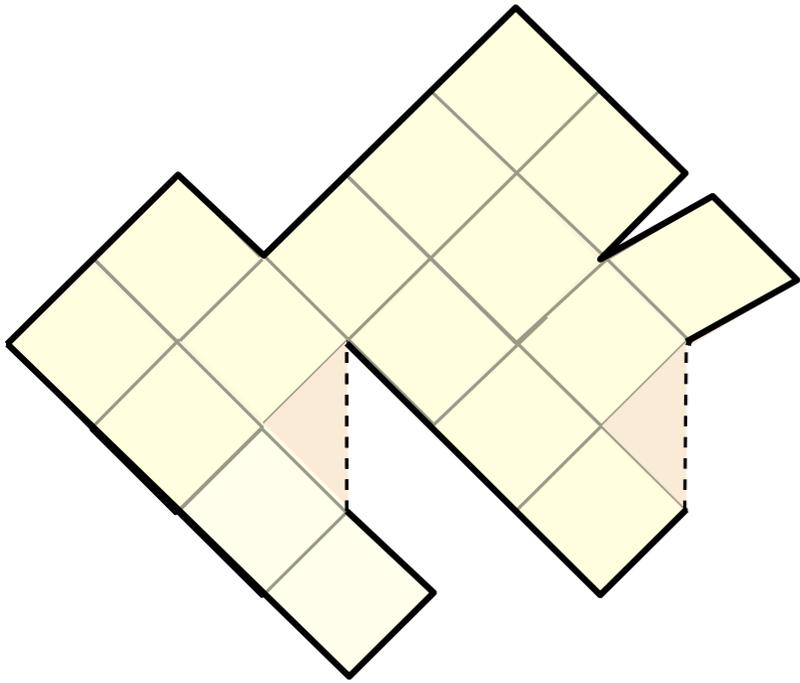


# Geometric wiring for extended fish



Identify the regions created by each operation

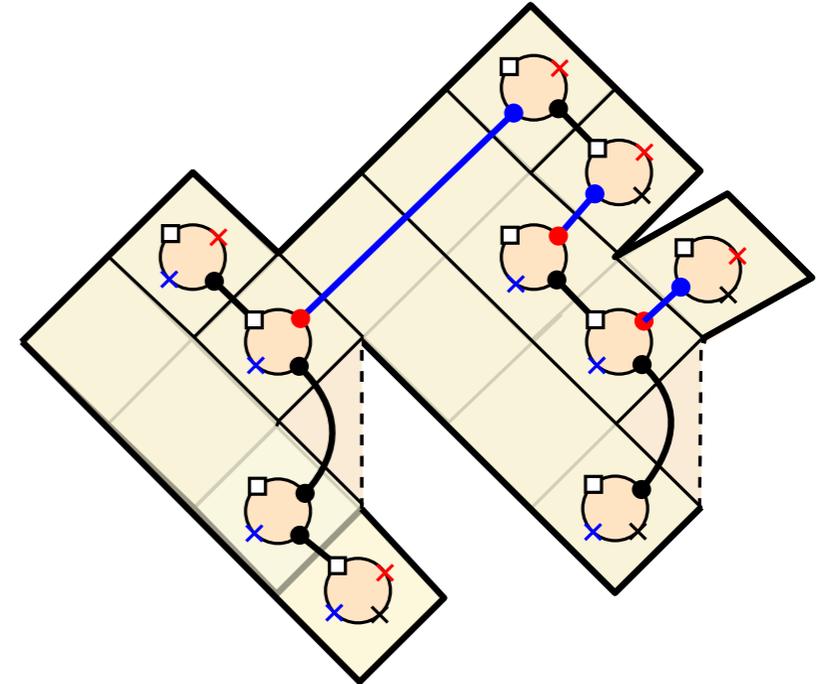
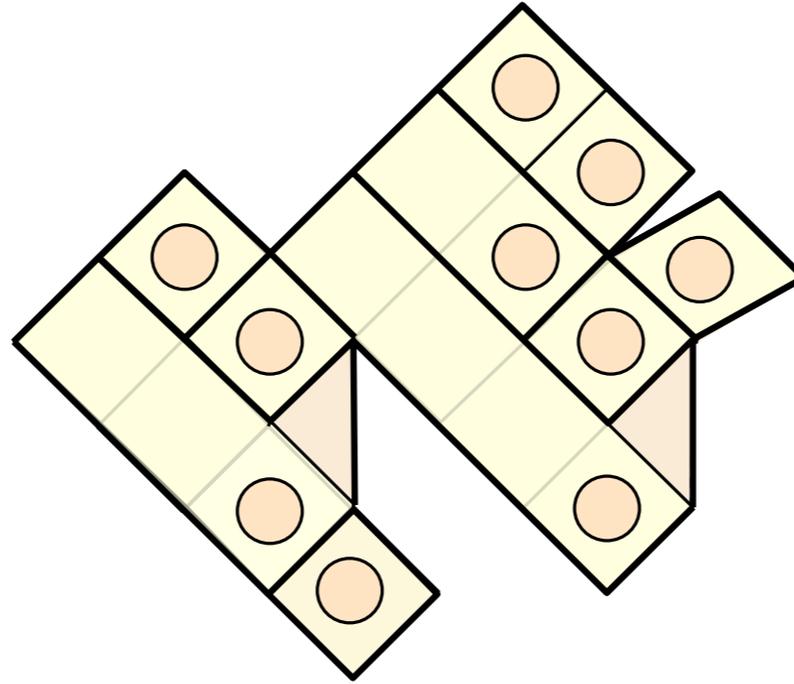
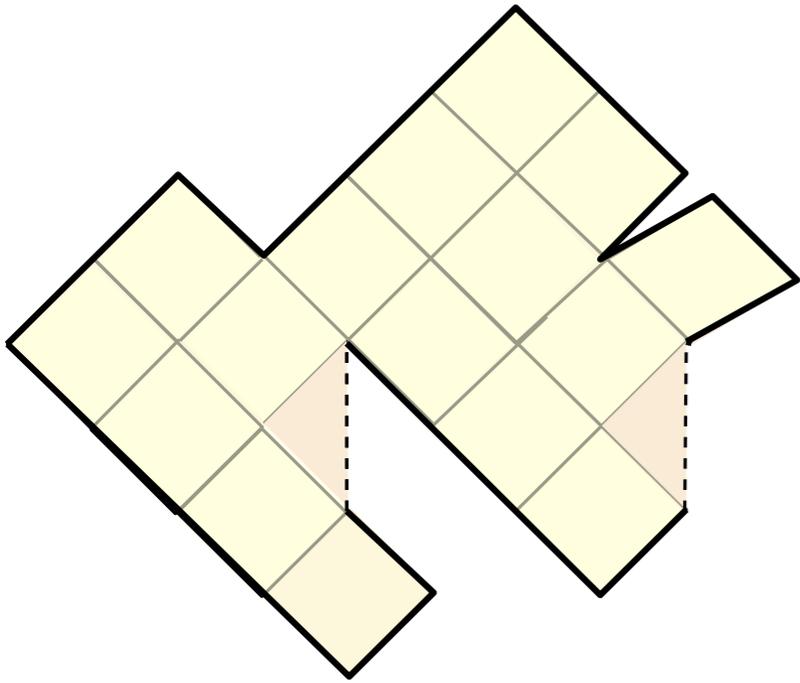
# Geometric wiring for extended fish



Identify the regions created by each operation

Each region corresponds to a vertex

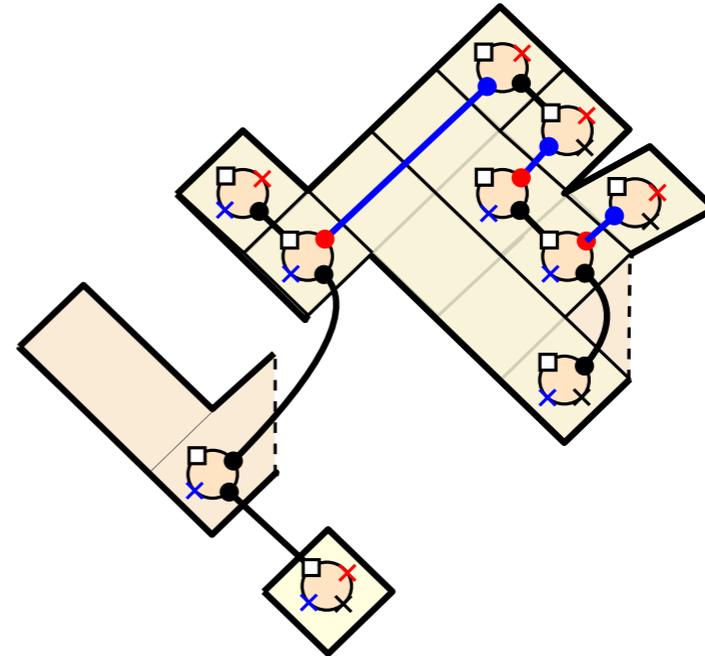
# Geometric wiring for extended fish



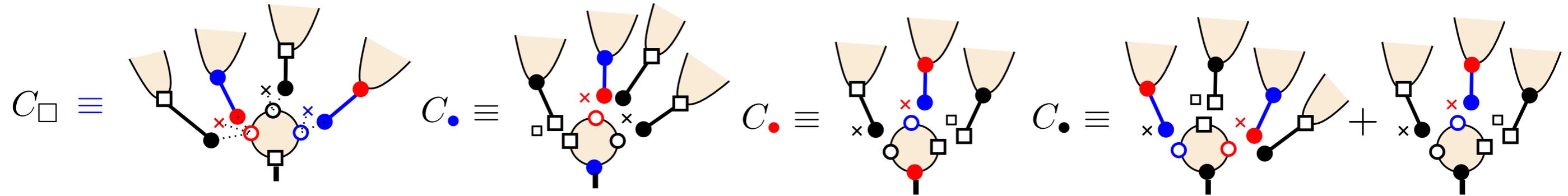
Identify the regions created by each operation

Each region corresponds to a vertex

Construct a tree by connecting regions according to the operations, using black and blue edges directly to connect subtrees



# A grammar for rooted companion trees



$$\Rightarrow C_{\square} \equiv \mathcal{Z} \times (1 + C_{\bullet} + C_{\square}) \times (1 + C_{\square}) \times (1 + C_{\circ})$$

$$C_{\bullet} \equiv \mathcal{Z} \times (1 + C_{\circ}) \times (1 + C_{\bullet} + C_{\square}) \times (1 + C_{\square})$$

$$C_{\circ} \equiv \mathcal{Z} \times (1 + C_{\square}) \times (1 + C_{\circ}) \times (1 + C_{\bullet})$$

$$C_{\circ} \equiv \mathcal{Z} \times (1 + C_{\circ}) \times (1 + C_{\bullet}) \times (1 + C_{\bullet} + C_{\square}) + C_{\circ}$$

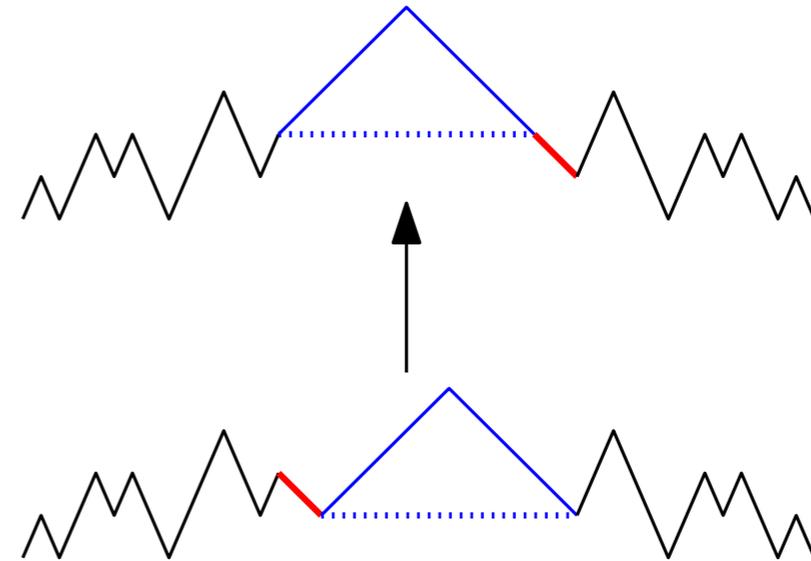
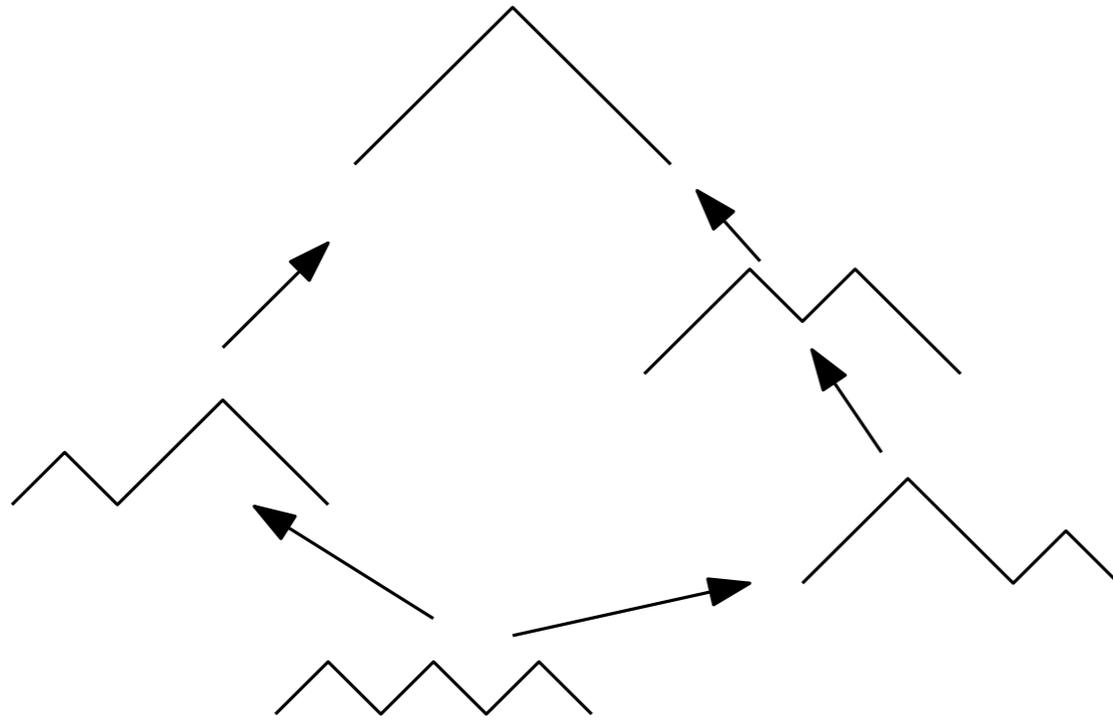
$$\Rightarrow C_{\square} \equiv \mathcal{Z} \times (1 + \mathcal{T})^4$$

Rooted companion trees associated to extended fighting fish are rooted quaternary trees

Intervals in the BKN poset

# The BKN poset

Take a Dyck path and endow it with the transitive closure  $\leq$  of the covering relation transforming an occurrence of  $DU^k D^k$  into an occurrence  $U^k D^k D$  with  $k \geq 1$ .

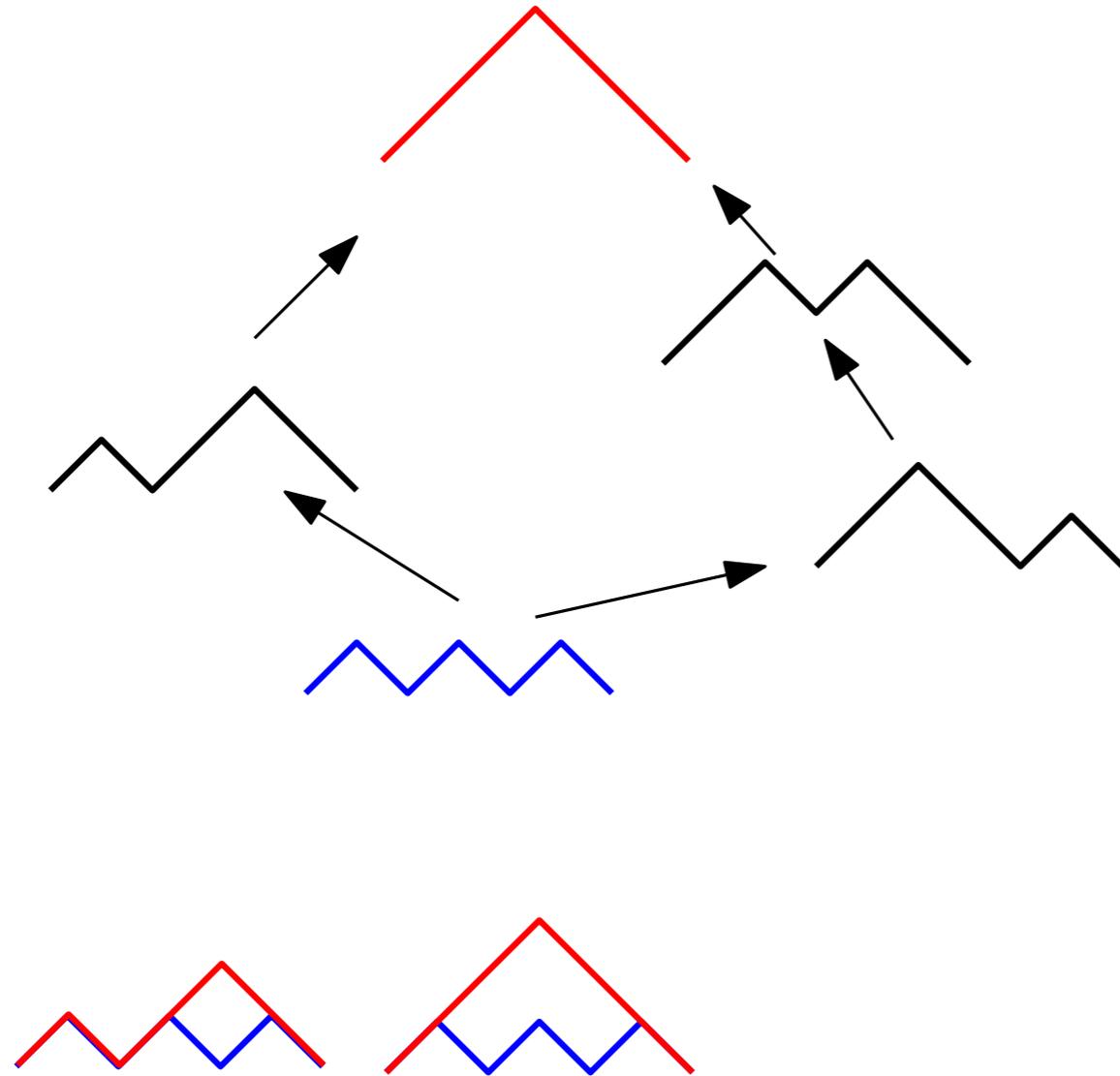


The BKN poset is a lattice

Introduced by Baril, Krizizov and Naima in 2023



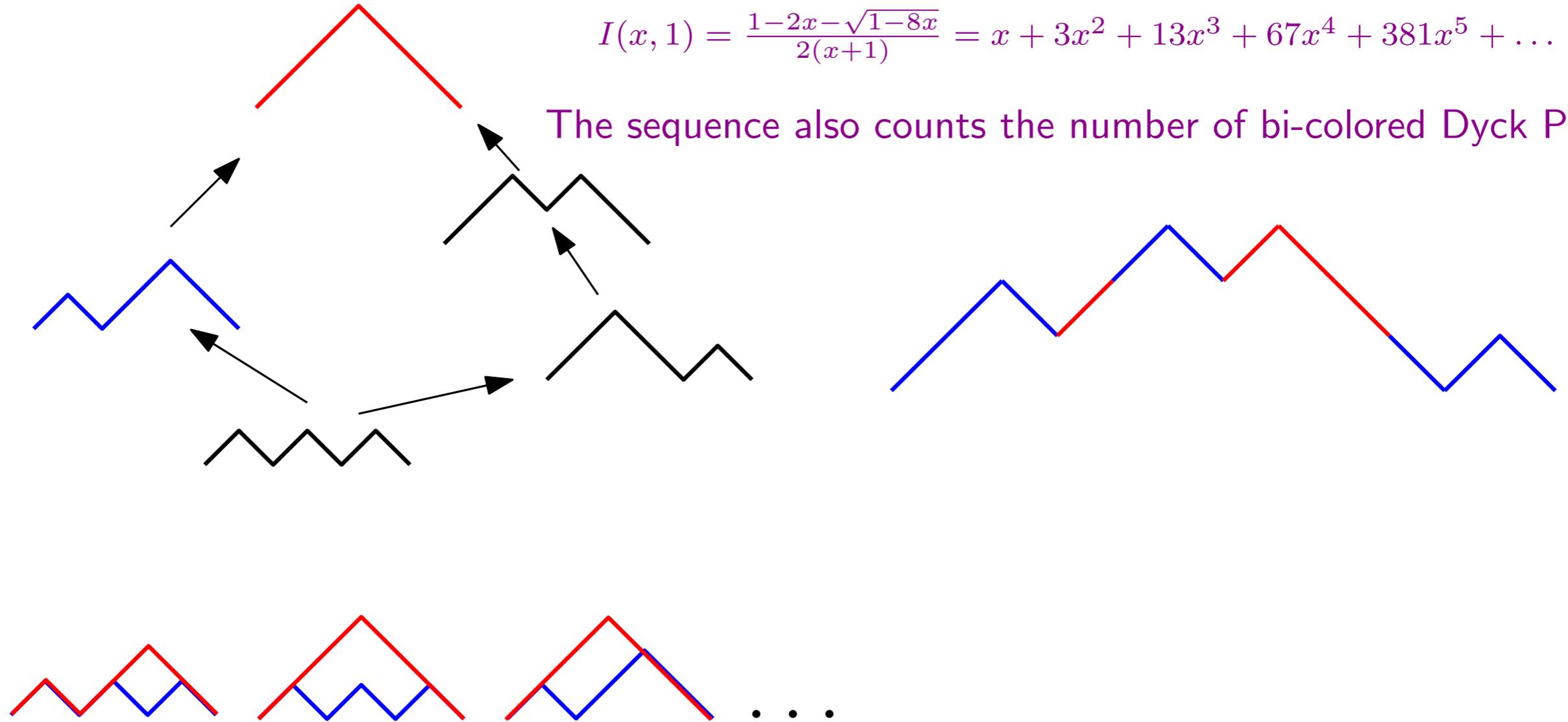
# The intervals of the BKN poset



# The intervals of the BKN poset

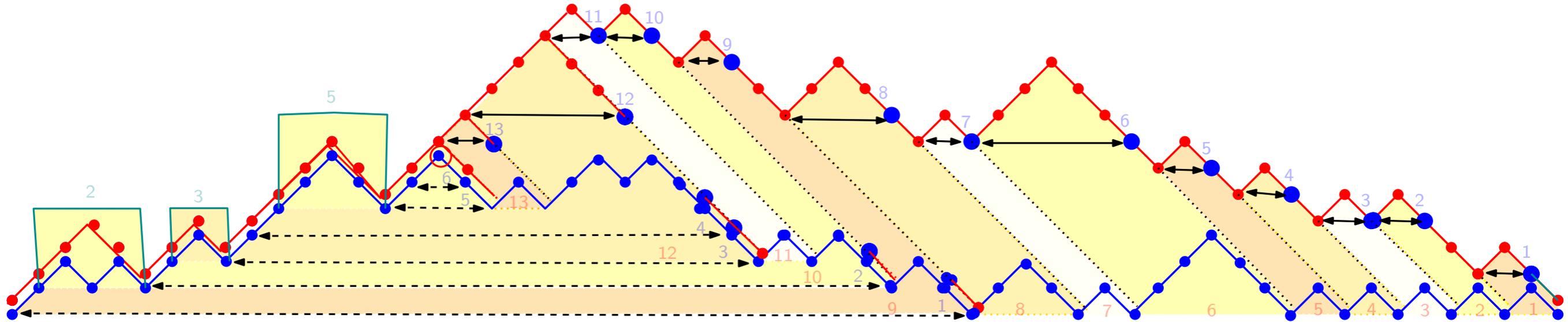
$$I(x, 1) = \frac{1-2x-\sqrt{1-8x}}{2(x+1)} = x + 3x^2 + 13x^3 + 67x^4 + 381x^5 + \dots$$

The sequence also counts the number of bi-colored Dyck Paths

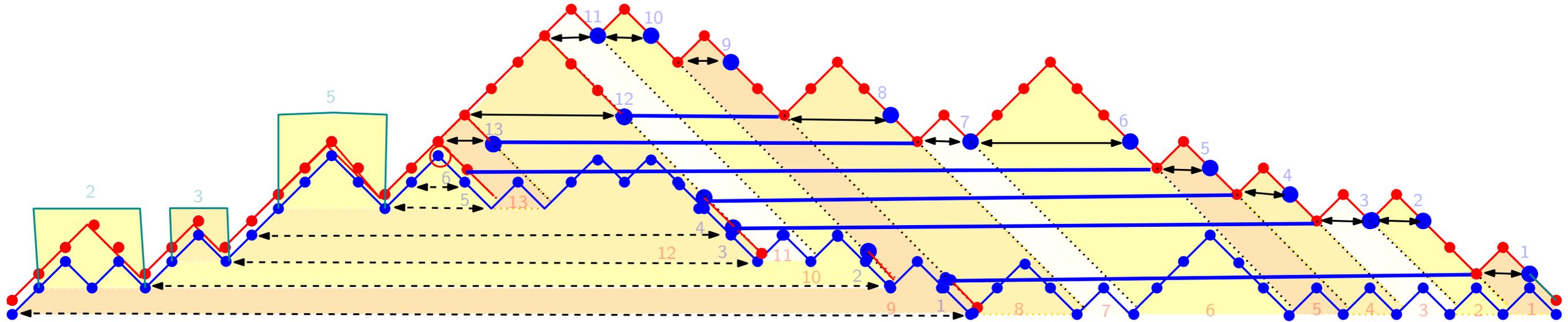




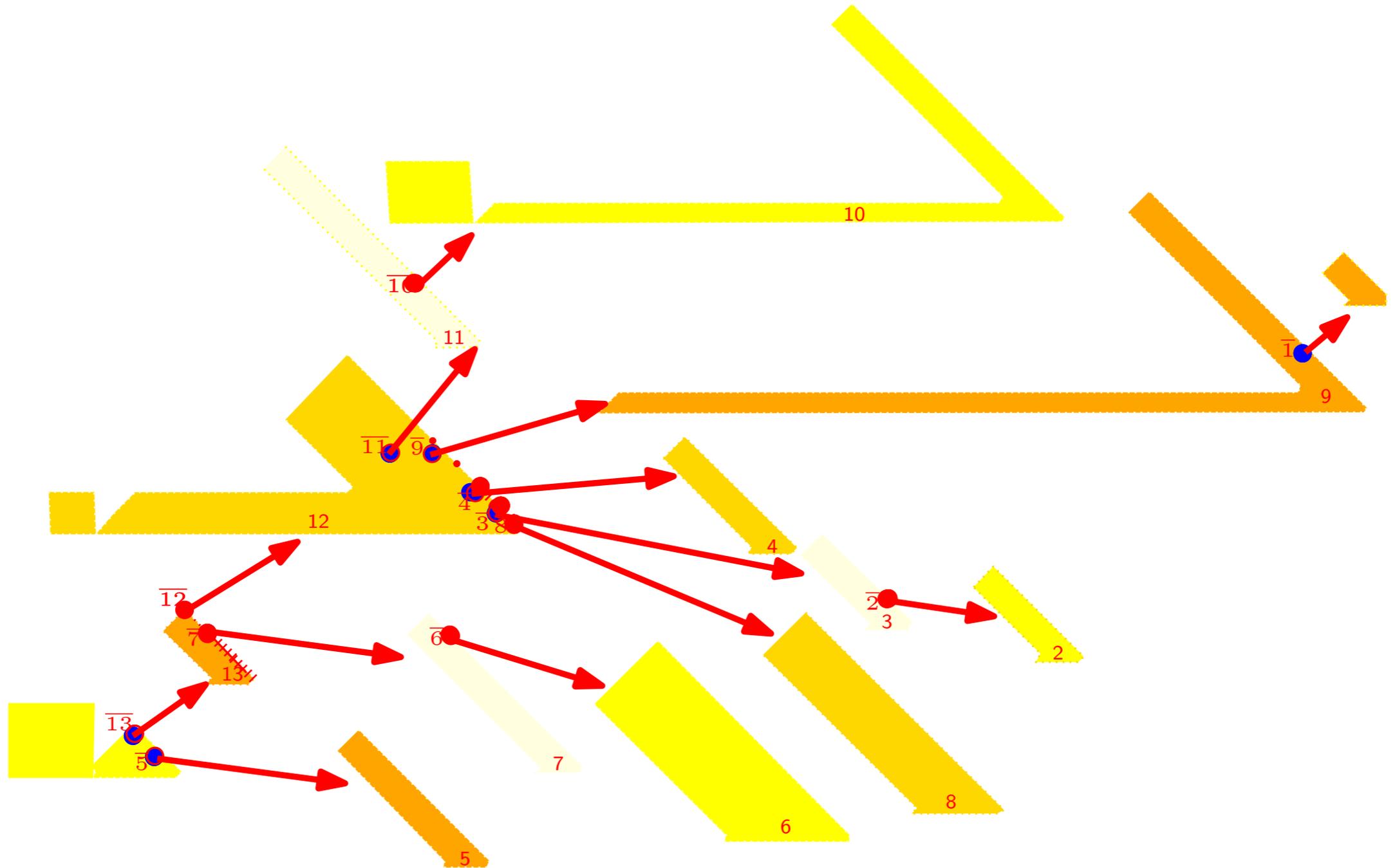
# Decomposition of a BKN interval in operations



# Decomposition of a BKN interval in operations

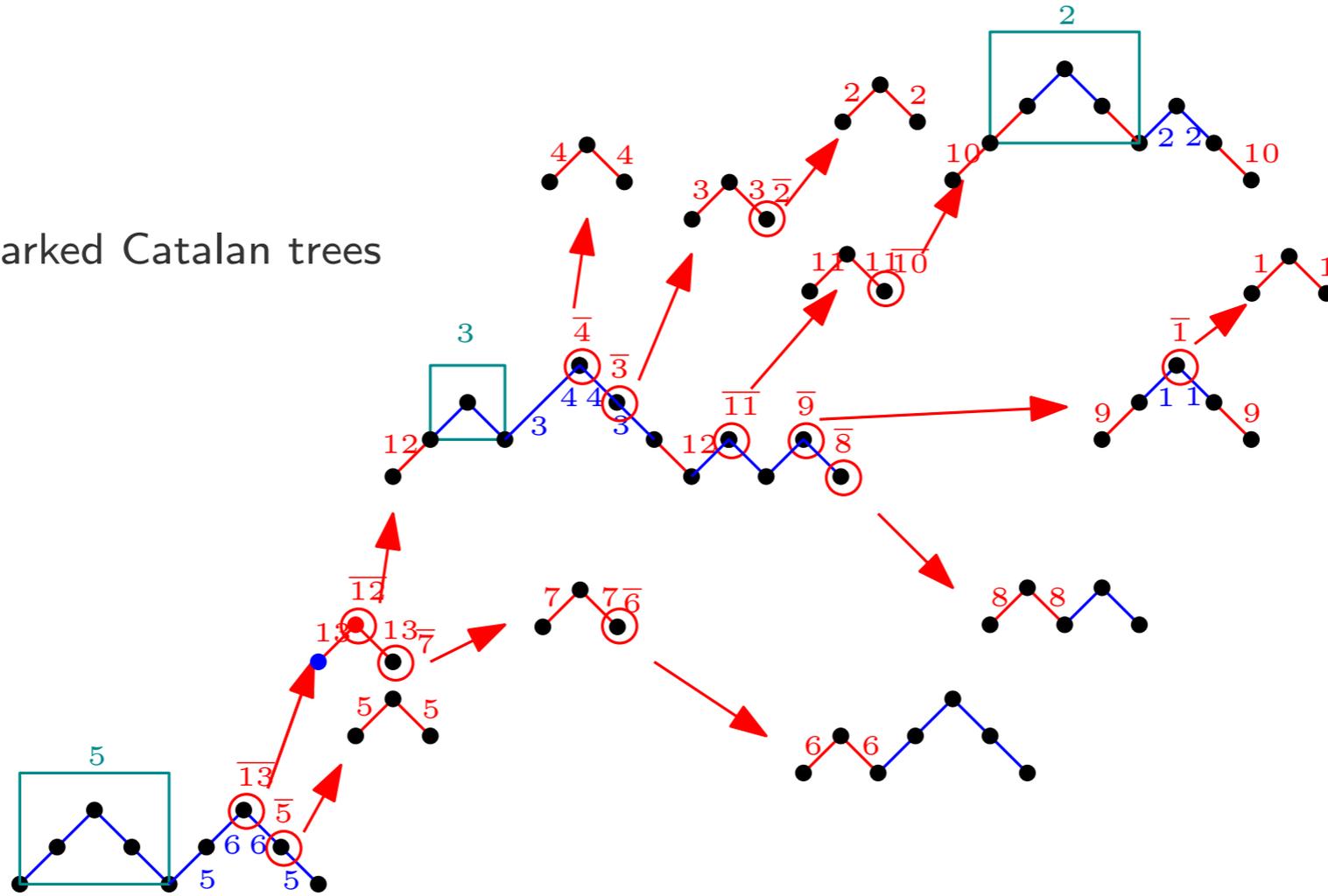


# The general shape of the companion tree

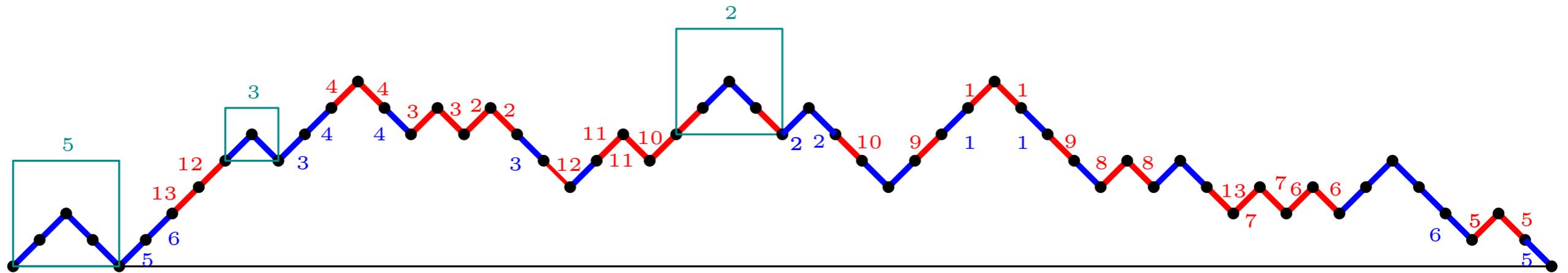


# The expanded form of the companion trees

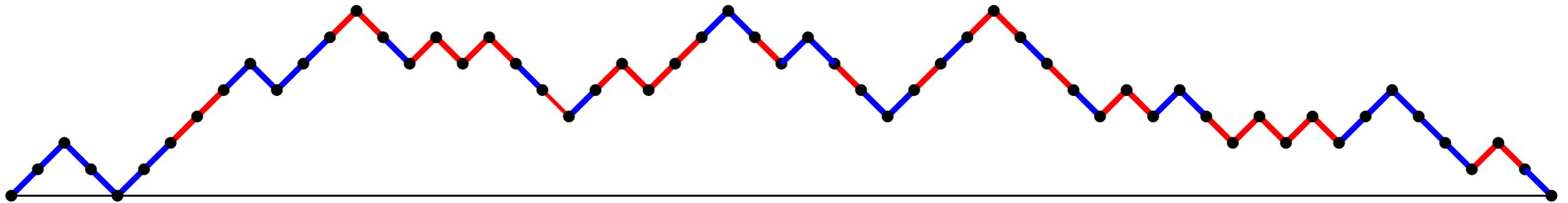
Nodes carry marked Catalan trees



# The recomposition in a bicolored tree



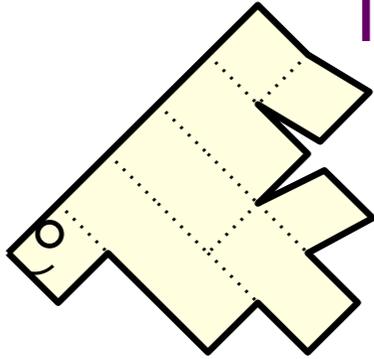
# The recomposition in a bicolored tree



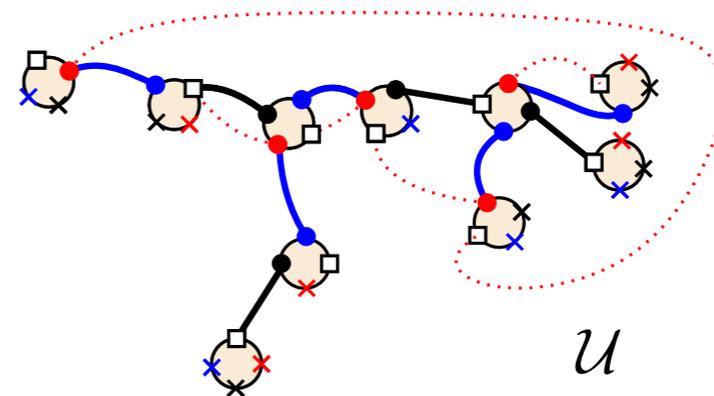
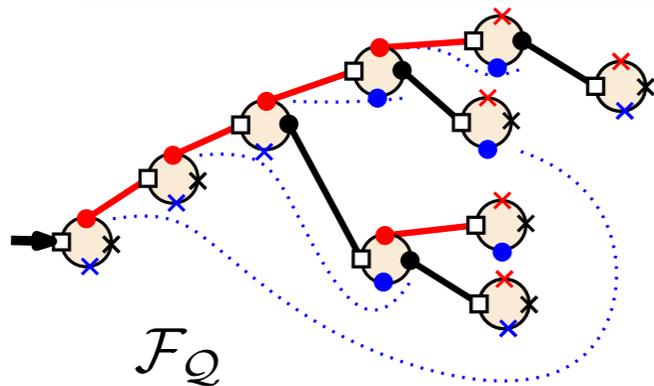
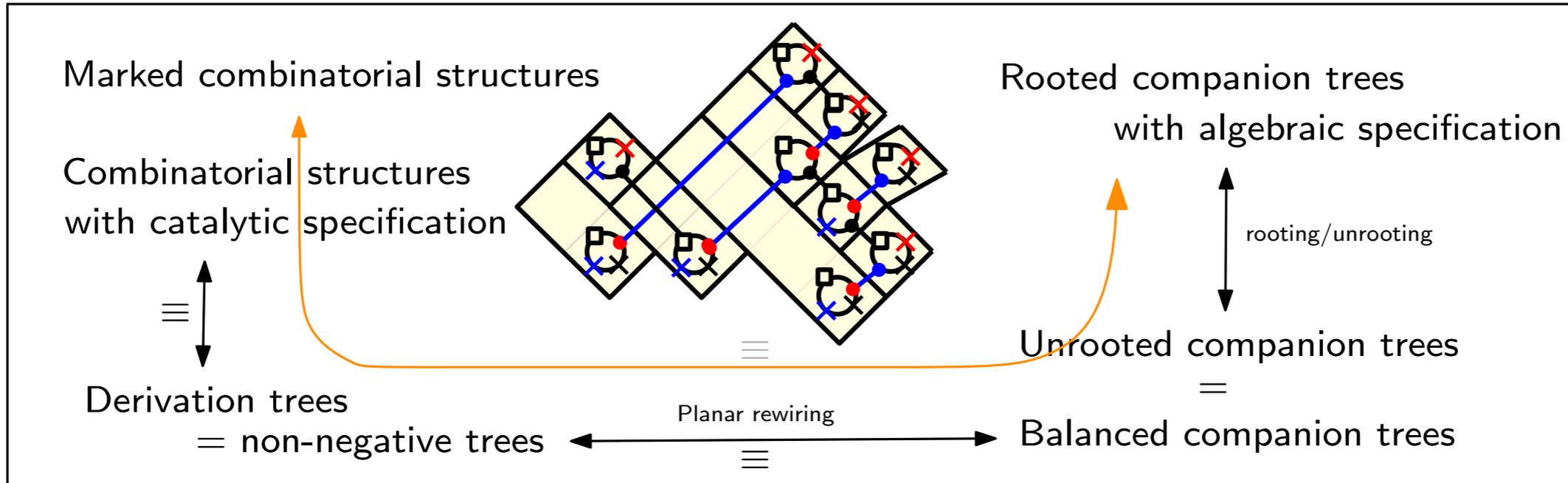
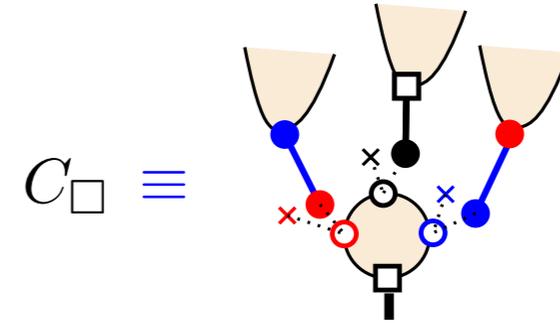


# The Geometric wiring principle

$\mathcal{F}$



Each  edge should connect an operation creating a -pearl to the -pearl operation using it.



Thank you!