

# Snakes, SAWs, and Symbolic Dynamics

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joint work with Nathalie Aubrun

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# Wang Tiles

- Our story begins with Wang tiles:

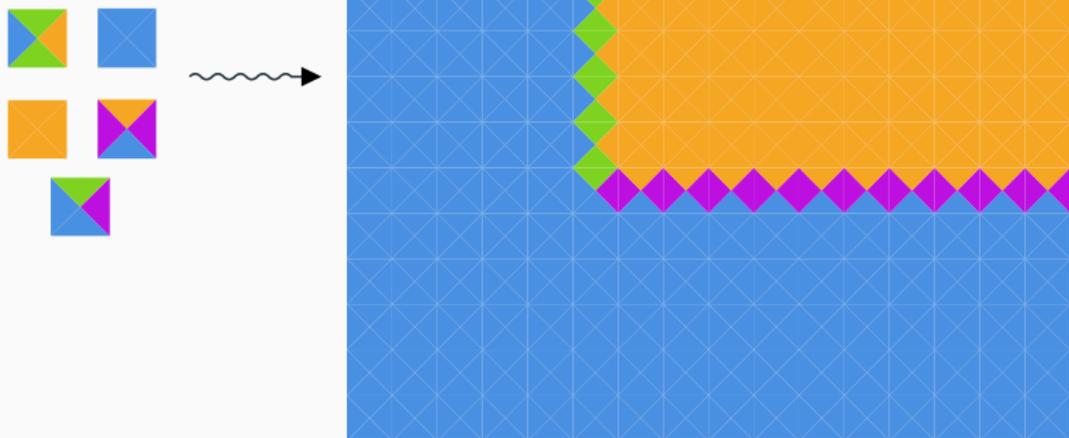


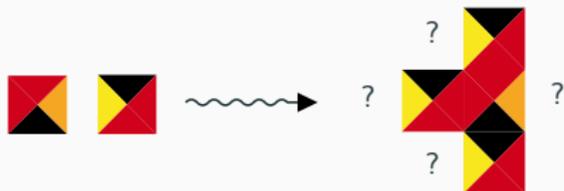
First introduced by Hao Wang in 1961.

- They can be placed side by side if they share the same color along their common border.



# Tilings





# Tiling?



## Domino Problem

Given a finite set  $\tau$  of Wang tiles, does  $\tau$  tile the plane?

# Tiling?



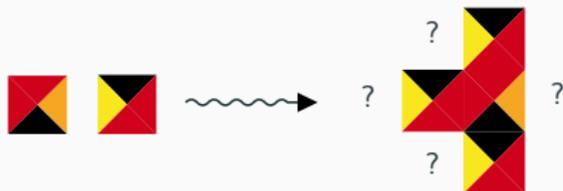
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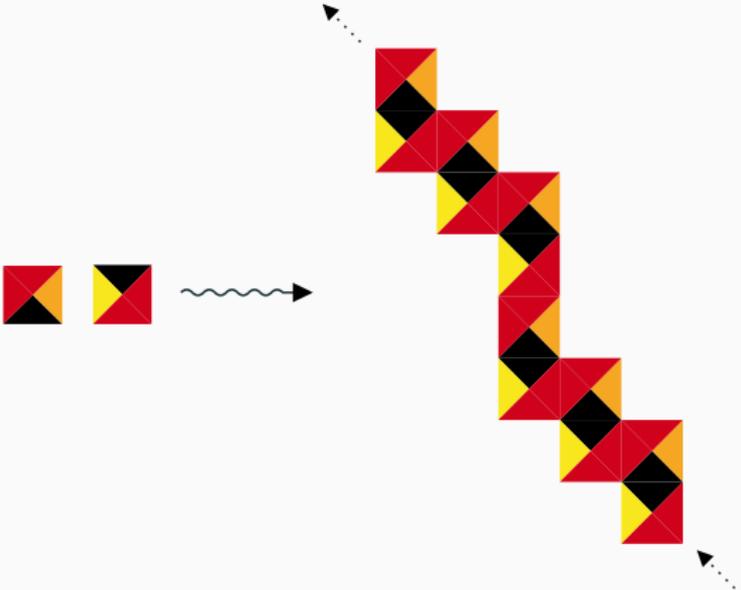
## Theorem (Berger '64)

The Domino Problem is undecidable ( $\Pi_1^0$ -comp.).

# Snakes?



# Snakes!



# New Problem: Snakes on a Plane

An *a priori* weaker version of the Domino Problem:

## Infinite Snake

Given a finite Wang tileset  $\tau$ , does there exist a snake tiled by  $\tau$ ?

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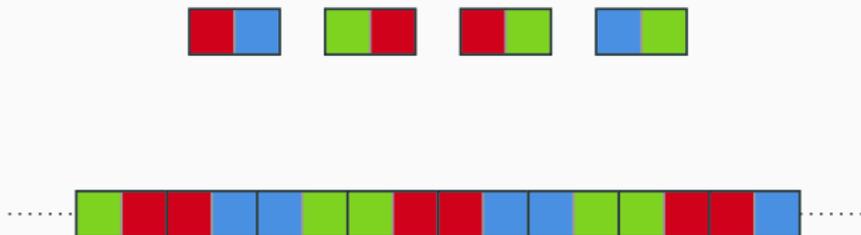
## Infinite Snake

Given a finite Wang tileset  $\tau$ , does there exist a snake tiled by  $\tau$ ?

## Theorem (Adleman, J. Kari, L. Kari, Reishus '02)

The infinite snake problem in  $\mathbb{Z}^2$  is undecidable ( $\Pi_1^0$ -comp).

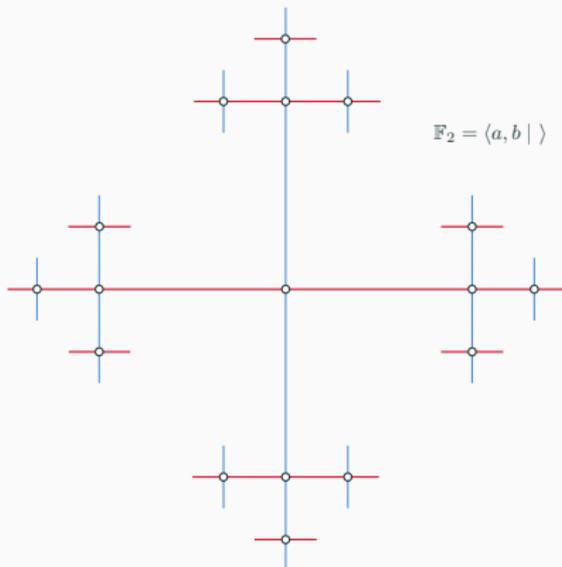
Nevertheless, in  $\mathbb{Z}$  both the Domino Problem and Infinite Snake Problem are decidable.



What accounts for this difference?

# Where is the Undecidability?

We study our problem in a particular class of graphs: **Cayley graphs** of finitely generated infinite groups.

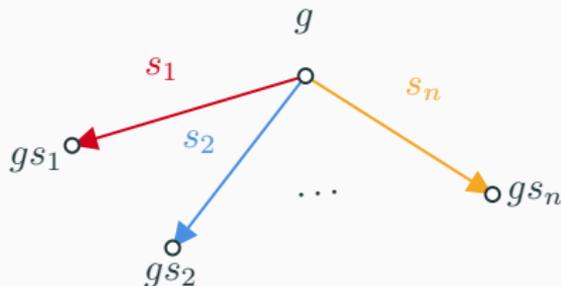


These graphs are infinite, regular, transitive, deterministically edge labelled.

# Groups as Graphs

A Cayley graph is defined from a group  $G$  along with a finite symmetric generating set  $S$ :

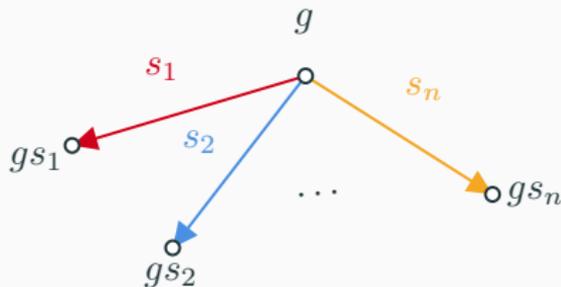
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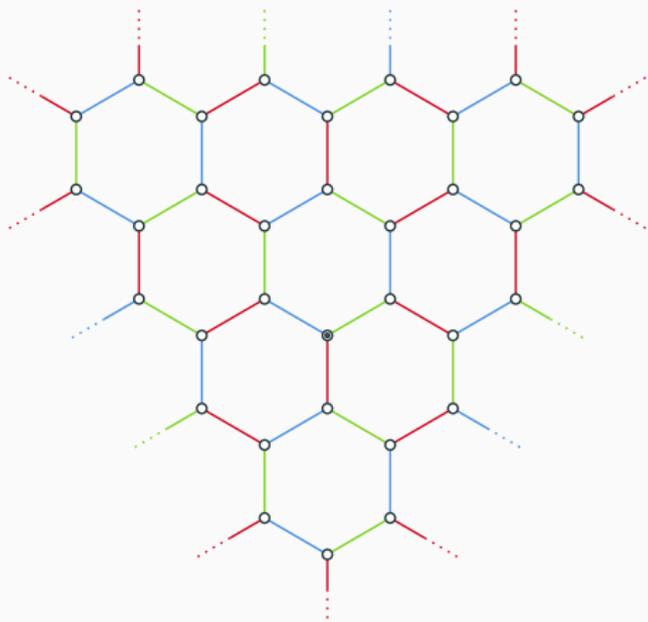


$$G = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle t, s \mid s^2 \rangle:$$

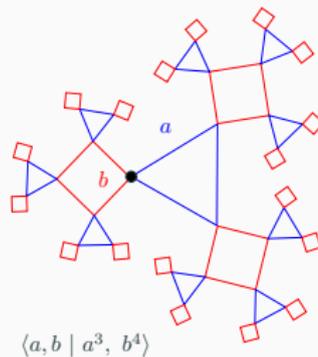
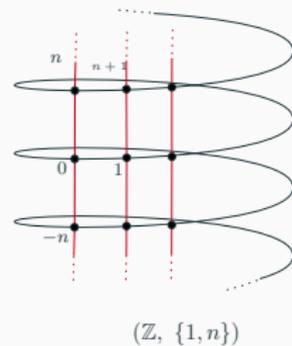
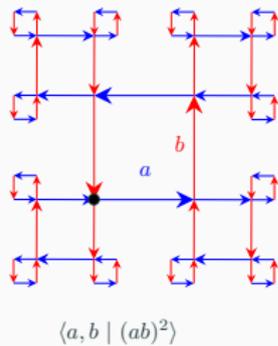
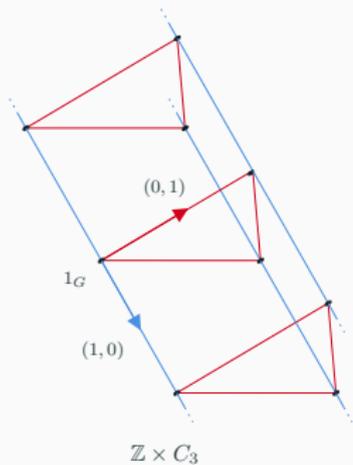


# Examples

$$\mathbb{H} = \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (bc)^3, (ac)^3 \rangle$$



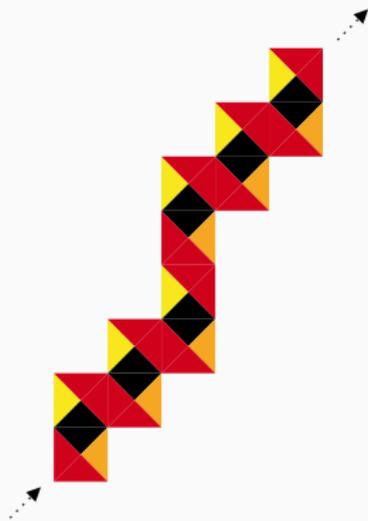
# More Examples



# Formal Snakes

Let  $\tau$  be a tileset. A  $\tau$ -snake is a pair of functions  $(\omega, \zeta)$ :

- (Skeleton)  $\omega : \mathbb{Z} \rightarrow G$  injective  
s.t.  $\omega(i)^{-1}\omega(i+1) \in S$ ,
- (Scales)  $\zeta : \mathbb{Z} \rightarrow \tau$  respecting  
local adjacency rules.

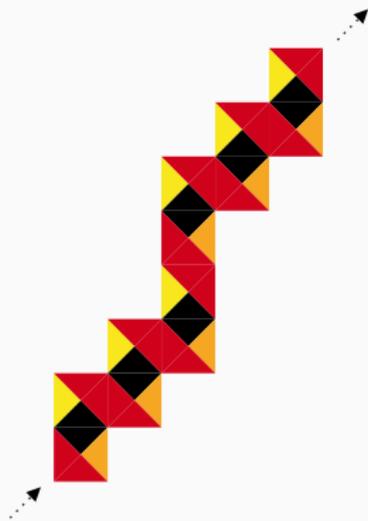




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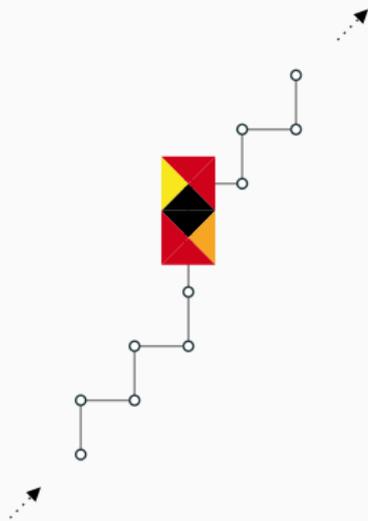
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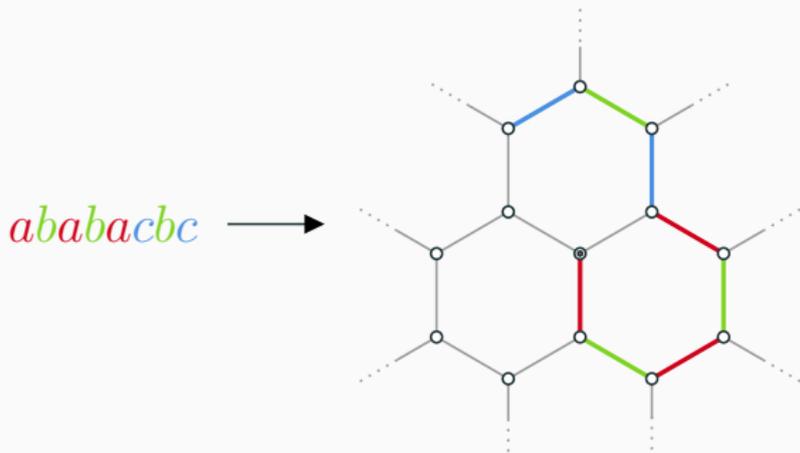
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# Words and Paths

Given a Cayley graph for a group  $G$  with generating set  $S$ , there is a correspondence between paths and words in  $S^*$ .



For instance, cycles are described by the set

$$\text{WP}(G, S) = \{w \in S^* \mid w =_G \varepsilon\}.$$





**Full-shift:**  $A^{\mathbb{Z}} = \{x : G \rightarrow A\}$ ,  
 $A$ : Finite alphabet

**Shift:**  $\sigma : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ ,  
 $\sigma(x)(k) = x(k+1)$

## Definition

A **subshift** is a subset  $X \subseteq A^{\mathbb{Z}}$  such that there exists  $\mathcal{F} \subseteq A^*$  with

$$X = \mathcal{X}_{\mathcal{F}} = \{x \in A^{\mathbb{Z}} \mid \forall w \sqsubseteq x, w \notin \mathcal{F}\}.$$

We say  $X \subseteq A^{\mathbb{Z}}$  is

- a subshift of finite (**SFT**) if  $\exists \mathcal{F}$  finite s.t.  $X = \mathcal{X}_{\mathcal{F}}$ ,
- **sofic** if  $\exists \mathcal{F}$  regular s.t.  $X = \mathcal{X}_{\mathcal{F}}$ .

**Remark:** SFT  $\implies$  sofic

- $x \in A^{\mathbb{Z}}$  is **periodic** if  $\exists k \in \mathbb{Z}$  s.t.  $\sigma^k(x) = x$ ,
- $X \subseteq A^{\mathbb{Z}}$  is **aperiodic** if it contains no periodic configs.

The **language** of  $X \subseteq A^{\mathbb{Z}}$ :

$$\mathcal{L}(X) = \{w \in A^* \mid \exists x \in X, w \sqsubseteq x\}.$$

## Proposition (Folk.)

$X$  is sofic iff  $\mathcal{L}(X)$  is a regular language.

In particular, non-empty sofic shifts contain periodic points.

**Obs:**  $X = \mathcal{X}_{\mathcal{L}(X)^c}$ .

The **complexity function** of  $X$  is

$$p_X(n) = |\mathcal{L}(X) \cap A^n|$$

The **entropy** of  $X$  is

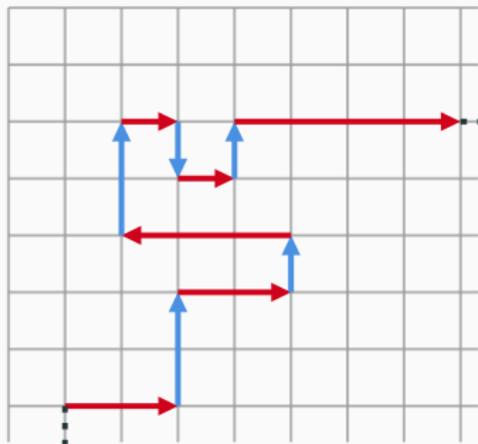
$$h_{\text{top}}(X) = \lim_{n \rightarrow \infty} \frac{\log(p_X(n))}{n}$$

## Theorem (Lind '84)

For a sofic subshift  $X$ ,  $h_{\text{top}}(X) = q \log(\lambda)$ , where  $q \in \mathbb{Q}_+$  and  $\lambda$  is a Perron number.

# Skeletons

This allows us to understand skeletons as bi-infinite words without loops.



...*a a b b a a b a<sup>-1</sup> a<sup>-1</sup> a<sup>-1</sup> b b a b<sup>-1</sup> a b a a a a*...

# Spooky Skeletons

$G$  be a finitely generated group,  $S$  a symmetric set of generators.

The **skeleton** of  $G$  with respect to  $S$  is

$$\mathbb{X}_{G,S} = \mathcal{X}_{\text{WP}(G,S)} = \{x \in S^{\mathbb{Z}} \mid \forall w \sqsubseteq x, w \notin \text{WP}(G,S)\}$$

**Examples:**

$$\mathbb{X}_{\mathbb{Z}^2, \{a^{\pm 1}, b^{\pm 1}\}} = \{x \in \{a^{\pm 1}, b^{\pm 1}\}^{\mathbb{Z}} \mid \forall w \sqsubseteq x, |w|_a \neq |w|_{a^{-1}} \vee |w|_b \neq |w|_{b^{-1}}\}.$$

For the infinite dihedral group  $\mathcal{D}_{\infty} = \langle a, b \mid a^2, b^2 \rangle$ ,

$$\mathbb{X}_{\mathcal{D}_{\infty}, \{a, b\}} = \{(ab)^{\infty}, (ba)^{\infty}\}.$$

## Theorem (Aubrun, B. '23)

If  $X_{G,S}$  is sofic, then the infinite snake problem for  $(G, S)$  is decidable.

## Questions

When is  $X_{G,S}$  sofic? SFT? Effective?

What are its periodic points?

What is its entropy?

# Self-Avoiding Walks

- A **self-avoiding walk** (SAW) is a path on a graph that visits each vertex at most once.
- For  $c(n)$  = number of SAWs of length  $n$ ,

$$\mu(G, S) = \lim_{n \rightarrow \infty} \sqrt[n]{c(n)}$$

is known as the **connective constant** of the Cayley graph.

Examples:

- (Duminil-Copin, Smirnov '12)  $\mu(\mathbb{H}) = \sqrt{2 + \sqrt{2}}$ ,
- (Alm, Janson '90)  $\mu(\mathbb{L}) = \frac{1}{2}(1 + \sqrt{5})$ ,

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Best approx so far (Jacobsen, Scullard, Guttman '16):

$$\mu(\mathbb{Z}^2) \approx 2.63815853032790(3)$$

$$L_{\text{SAW}}(G, S) = \{w \in S^* \mid w \text{ is the label of a SAW}\}.$$

### Lemma

$$\mathbb{X}_{G,S} = \{x \in S^{\mathbb{Z}} \mid x \text{ is the label of a bi-infinite SAW}\}.$$

- $\mathbb{X}_{G,S} = \mathcal{X}_{L_{\text{SAW}}(G,S)^c}$
- $\mathcal{L}(\mathbb{X}_{G,S}) = \{w \in S^* \mid w \text{ is the label of a bi-extendable SAW}\}.$
- $\mathcal{L}(\mathbb{X}_{G,S}) \subseteq L_{\text{SAW}}(G, S)$

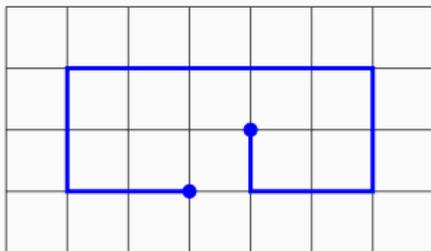
# Completing the title

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# A first look

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## Theorem (Aubrun, B. '24)

$G$  is a torsion group iff  $\bigtimes_{G,S}$  is aperiodic for (any) all generating sets  $S$ .

**Reminder:**  $G$  is a torsion group if  $\forall g \in G, \exists n \in \mathbb{N}$  s.t.  $g^n = 1_G$ .

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## Theorem (Halin '73, Aubrun, B. '24)

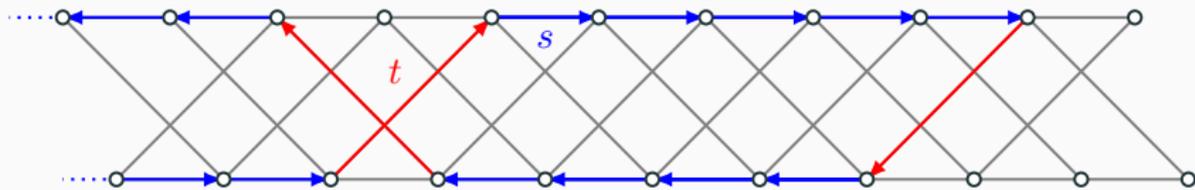
Consider  $S$ , a torsion-free element  $g \in G$ , and  $k \geq 1$  and  $g_0 \in G$  such that  $\|g_0^{-1}g^k g_0\|_S$  is minimal. Then, for any geodesic  $w \in S^*$  representing  $g_0^{-1}g^k g_0$ ,  $w^\infty \in \mathbb{X}_{G,S}$ .

## Proposition (Aubrun, B. '24)

For every group  $G$ , there exists  $S$  such that  $\mathbb{X}_{G,S}$  is not sofic.

We can use the **Pumping Lemma**:

For a torsion-free element  $g \in G$  we add  $s = g^2$  and  $t = g^3$  to  $S$  so we can find a copy of  $\text{Cay}(\mathbb{Z}, \{\pm 2, \pm 3\})$ .

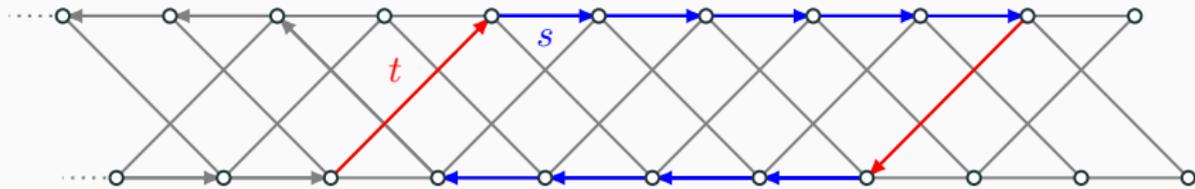


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$\forall n \in \mathbb{N}$ ,  $\mathcal{L}(\mathbb{X}_{G,S})$  contains the configuration  $ts^{n+1}t^{-1}s^{-n}$  on which we use the Pumping Lemma.

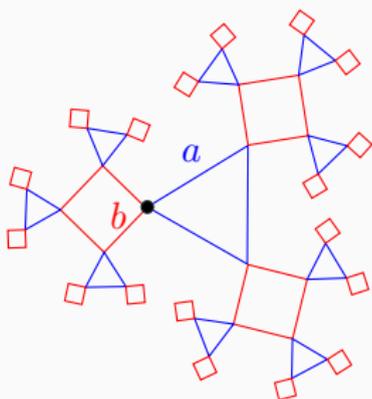
# Skeletons of Finite Type

## Theorem (Aubrun, B. '24)

$G$  admits  $S$  s.t.  $\mathbb{X}_{G,S}$  is an SFT iff  $G$  is a plain group.

A group  $G$  is said to be **plain** if there are finite groups  $\{G_i\}_{i=1}^m$  and  $n \in \mathbb{N}$  such that

$$G \simeq \left( \bigast_{i=1}^m G_i \right) \ast \mathbb{F}_n.$$



$$\begin{aligned} G &= \mathbb{Z}/3\mathbb{Z} \ast \mathbb{Z}/4\mathbb{Z}, \\ &= \langle a, b \mid a^3, b^4 \rangle. \end{aligned}$$

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( $\Leftarrow$ ) Study the skeleton of the corresponding groups

( $\Rightarrow$ ) Understand the geometry of the Cayley graph "at infinity"

To prove this we need the notion of an **end**:

Consider  $\Gamma = (V, E)$  a connected graph.

- A **ray** is an infinite sequence of distinct adjacent vertices  
 $\rho = (v_1, v_2, \dots) \in V^{\mathbb{N}}$ ,
- $\rho_1 \sim \rho_2$  if  $\forall A \subseteq V$ , all but finitely many vertices of  $\rho_1$  and  $\rho_2$  lie on the same connected component of  $\Gamma \setminus A$ ,
- $\text{Ends}(\Gamma) = \Gamma / \sim$ ,
- $e(\Gamma) = |\text{Ends}(\Gamma)|$

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## Theorem (Hopf '43, Freudenthal '44)

For  $\Gamma = \text{Cay}(G, S)$ , we have  $e(\Gamma) \in \{0, 1, 2, \infty\}$ .

## Definition (Thomassen, Woess '93)

The **size**  $m$  of  $\omega \in \text{Ends}(\Gamma)$  is the number of disjoint rays it contains.

We say that  $\omega$  is

- **thin** if  $m < +\infty$ ,
- **thick** if  $m = \infty$ .

## "Lemma" (Lindorfer, Woess '20)

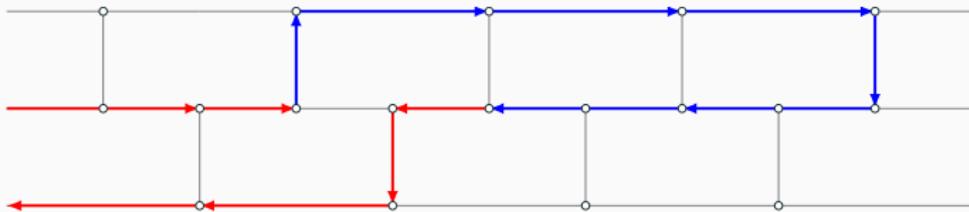
For  $\Gamma = \text{Cay}(G, S)$ , if  $\omega$  is of size  $m$  then  $\Gamma$  contains a "strip" of size  $m$  in the direction of  $\omega$ .





# Proof sketch

- First we show that all ends are **thin** and of size at most 2.



- If all ends are of size 1:

**Theorem (Haring-Smith '83 + Lindorfer, Woess '20)**

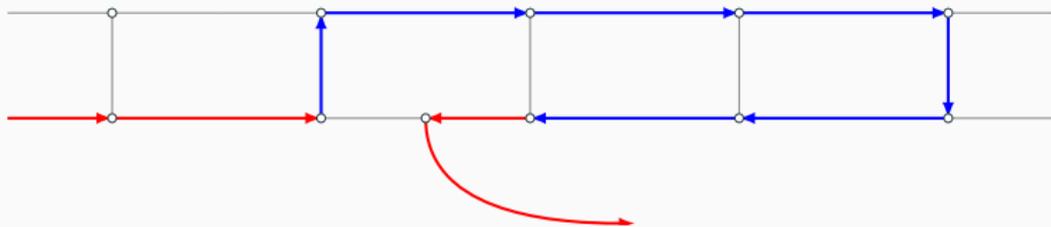
$G$  is plain iff it admits a Cayley graph with ends of size 1.

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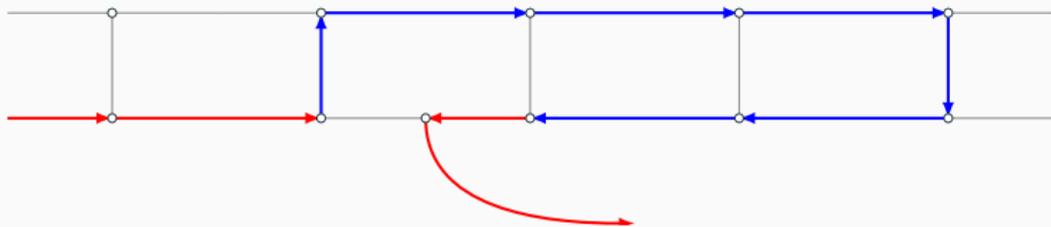
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## Theorem (Scott, Wall '79)

$e(\Gamma) = 2$  iff  $G$  is virtually  $\mathbb{Z}$

## Lemma (Aubrun, B. '24)

If  $G \notin \{\mathbb{Z}, \mathcal{D}_\infty, \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathcal{D}_\infty \times \mathbb{Z}/2\mathbb{Z}\}$  and virtually  $\mathbb{Z}$ , then every Cayley graph of  $G$  has ends of size  $\geq 3$ .

# What now?

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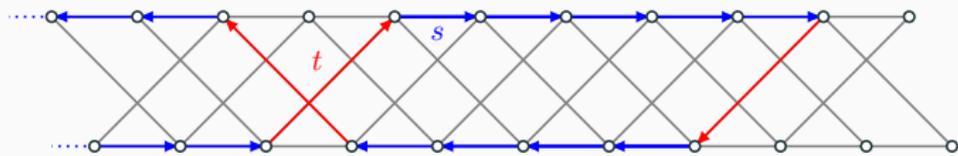
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## Proposition (Folk.)

If  $X$  is an irreducible sofic shift,

$$h_{\text{top}}(X) = \lim_{n \rightarrow \infty} \frac{\log(q_X(n))}{n},$$

where  $q_X(n)$  = number of periodic points of period  $n$  in  $X$ .

## Proposition (Clisby '13)

For  $\mathbb{Z}^2$  with standard gen. set,  $\mu(\mathbb{Z}^2) = \lim_{n \rightarrow \infty} c_e(n)^{\frac{1}{n}}$ , where  $c_e(n)$  is the number of *endless SAWs* of length  $n$ .

## Theorem (Aubrun, B. '24)

If  $\Gamma = \text{Cay}(G, S)$  admits a graph height function  $(h, H)$  such that  $H$  acts transitively on  $\Gamma$ , then

$$h_{\text{top}}(X) = \lim_{n \rightarrow \infty} \frac{\log(q(n))}{n},$$

where  $q(n) =$  number of periodic points of period  $n$  in  $X_{G,S}$ .

A **graph height function**  $(h, H)$  is composed of a function  $h : V \rightarrow \mathbb{Z}$  and a subgroup  $H \leq \text{Aut}(\Gamma)$  acting quasi-transitively on  $\Gamma$  such that

- for all  $u, v \in V_\Gamma$  and  $g \in H$

$$h(g \cdot v) - h(g \cdot u) = h(v) - h(u),$$

- for all  $u \in V_\Gamma$ , there exists  $v, v' \in V_\Gamma$  adjacent to  $u$  such that  $h(v) < h(u) < h(v')$ .

## Example:

For  $G = \mathbb{Z}^2$  with standard gen. set,  $h(i, j) = i$  is a height function along with  $H = \mathbb{Z}^2$

## Proposition (Aubrun, B. '24)

If  $\text{Cay}(G, S)$  admits a graph height function, then  $\mathbb{X}_{G,S}$  contains a periodic point.

## Corollary

Infinite torsion f.g. groups do not admit graph height functions.

This generalizes a result by Grimmet and Li from '17.

## Other properties

$X \subseteq A^{\mathbb{Z}}$  is effective if  $\exists \mathcal{F}$  computable s.t.  $X = \mathcal{X}_{\mathcal{F}}$ .

- $G$  recursively presented  $\implies \mathcal{X}_{G,S}$  is effective  $\forall S$ ,
- $\exists G$  s.t.  $\forall S$ ,  $\mathcal{X}_{G,S}$  is effective and has no computable points,

$X \subseteq A^{\mathbb{Z}}$  is minimal if it does not contain non-empty subshifts.

- $\mathcal{X}_{G,S}$  minimal  $\implies$  proper quotients are finite,  $G$  is not torsion-free.

### Conjecture

$\mathcal{X}_{G,S}$  minimal iff  $G = \mathcal{D}_{\infty}$  with  $S = \{a, b\}$  where  $\mathcal{D}_{\infty} = \langle a, b \mid a^2, b^2 \rangle$ .

## Other properties and questions

- What are the invariant measures of  $\mathbb{X}_{G,S}$ ? What do they represent?
- Describe  $\mathcal{M}(G) = \{\mu(G, S) \mid S \text{ gen. set.}\}$  and  $\mathcal{M} = \bigcup_G \mathcal{M}(G)$ .

### Conjecture (Grimmet, Li '17)

Let  $\Gamma$  be a Cayley graph of degree 3. Then,  $\mu(\Gamma) \geq \frac{1+\sqrt{5}}{2}$ .

### Definition

The skeleton of  $(G, S)$  wr to  $(H, T)$  is

$$X(G, H, S, T) = \{df \in (T^S)^H \mid f : H \rightarrow G \text{ injective, 1-Lipschitz}\},$$

where  $df : H \rightarrow T^S$  is  $df(h)(t) = f(h)^{-1}f(ht)$ .

Thank you for listening!

