Stochastic diffusion model on the unit circle

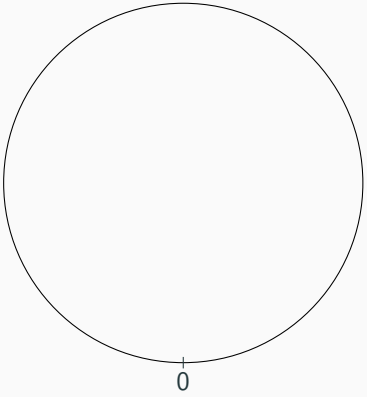
Zoé Varin

February 5th, 2025

Joint work with Jean-François Marckert

Definition of the model

State space $\mathcal{C} = \mathbb{R}/\mathbb{Z}$. $m_0, \cdots, m_n$ with $\sum m_i < 1$.



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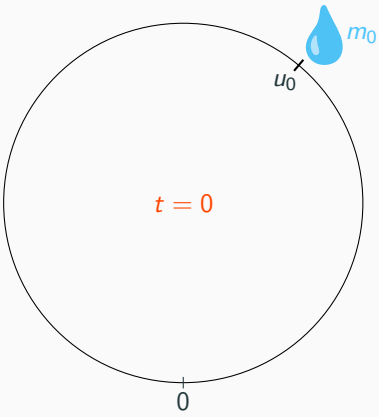

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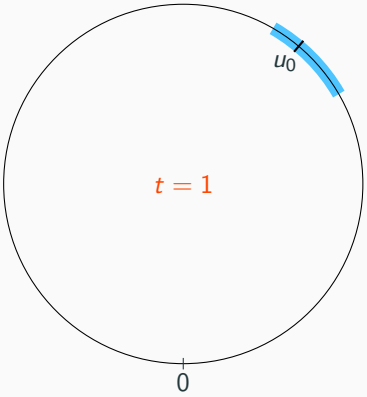

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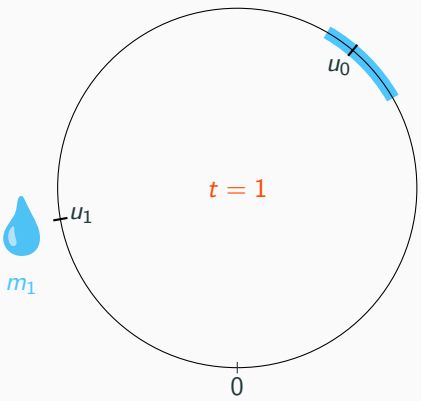

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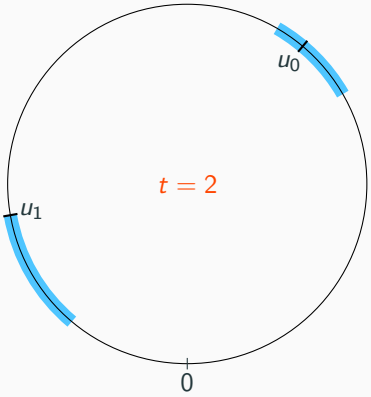

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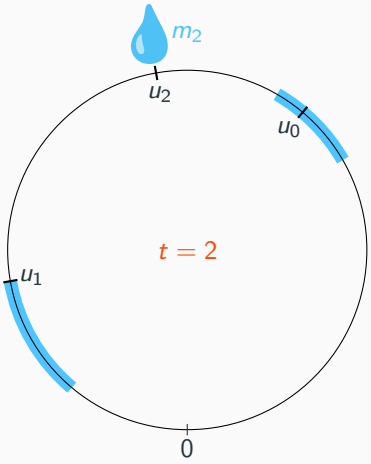

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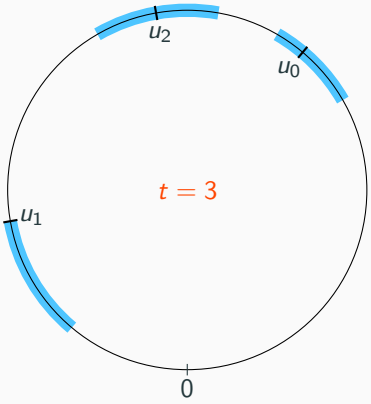

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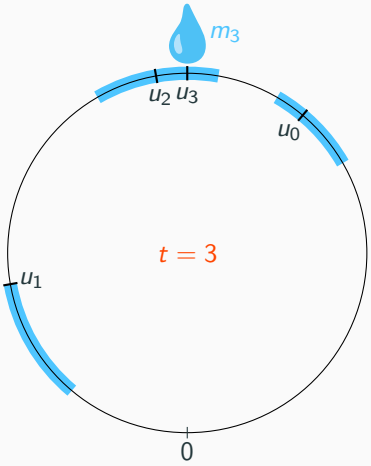

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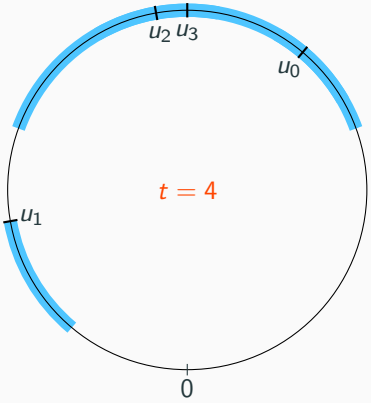

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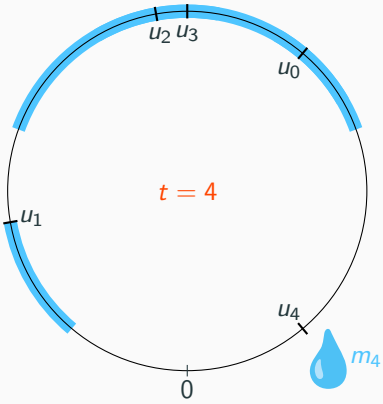

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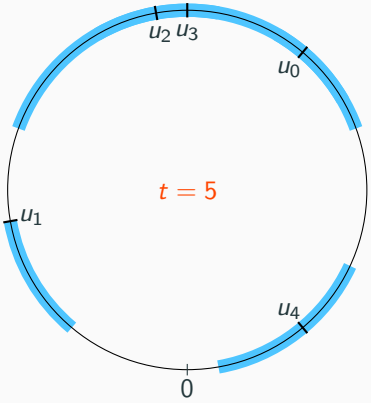

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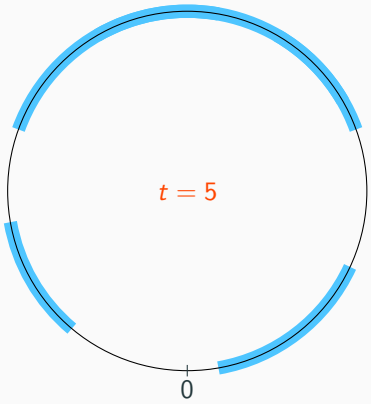

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- **occupied space** $O^{(k)}$ of size $\text{Leb}\left(O^{(k)}\right) = \sum_{i=0}^{k-1} m_i$
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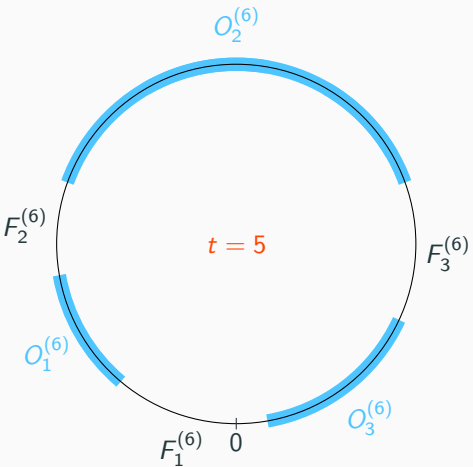

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$\Longrightarrow$ $(O_i^{(k)}, F_i^{(k)})_{1 \leq i \leq N^{(k)}}$ ordered around the circle with $0 \in O_1^{(k)} \cup F_1^{(k)}$

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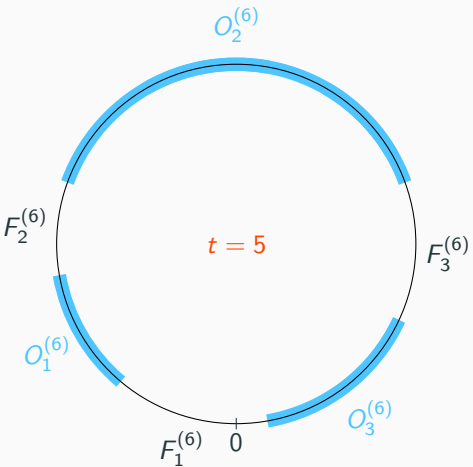

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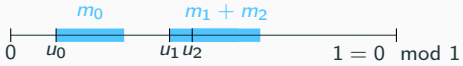

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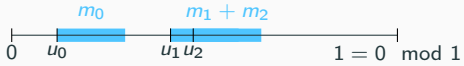

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- Diffusion to the closest side (with or without constant reevaluation)
- Range of a brownian path
- short-sighted jam spreader
- for any ball, pick at random some spreading policy
- ...

What is a valid spreading policy ?

As a continuous process: local and continuous diffusion

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Hypotheses:

- dl and dr only depend on what is inside the current component of u_k (one of the $O_i^{(k+\varepsilon)}$)
- invariance by translation of the process

A universality result

We consider $\frac{\sigma.|F^{(k)}|}{R} = \left(\frac{|F^{(k)}_{\sigma_i}|}{R}\right)_{1\leq i\leq N^{(k)}}$, for $\boldsymbol{\sigma} \sim \mathcal{U}(\mathfrak{S}_{N^{(k)}})$. Let $R = 1 - \sum_{i=0}^{k-1} m_i$.

Theorem

Independently of the diffusion policy,

- ***Number of blocks:*** $N^{(k)} \overset{(d)}{=} 1 + \text{Binomial}(k-1, R)$
- ***Lengths of the free blocks:*** $\frac{\boldsymbol{\sigma}.|F^{(k)}|}{R} \sim \text{Dirichlet}(N^{(k)}; 1, \dots, 1)$
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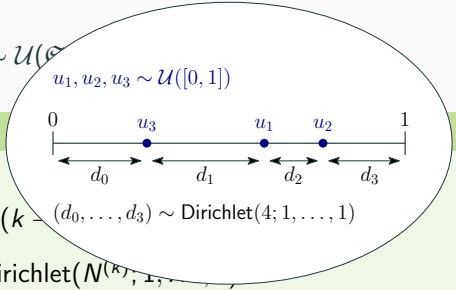

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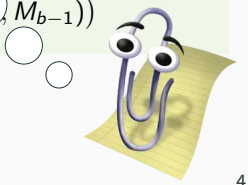

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Some background on the continuous and discrete parking models

Discrete parking

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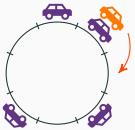

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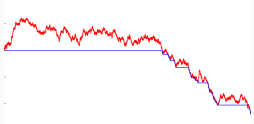

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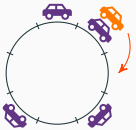

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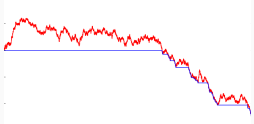

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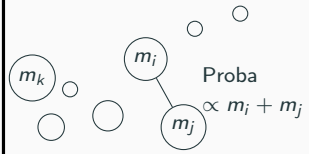

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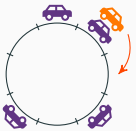

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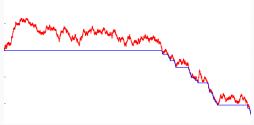

Generalized parking

- Parking on $\mathbb{Z}$ (Przykucki, Roberts, Scott [PRS23])
- Parking on (random) trees (Contat et. al.)
- Bilateral parking procedures (Nadeau), Golf model [Var24]

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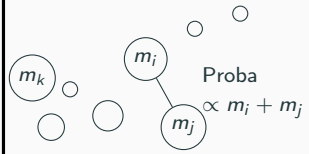

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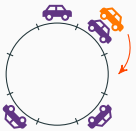

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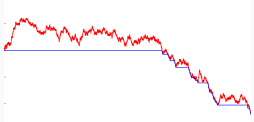

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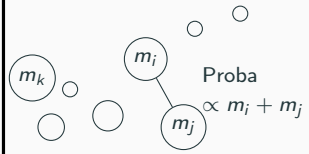

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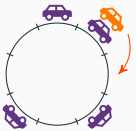

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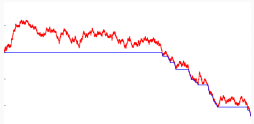

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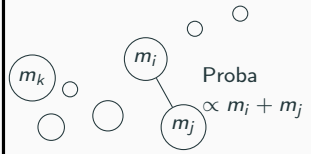

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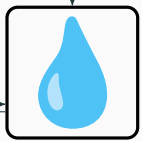

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Main principle

Consider 4 uniform points on $[0, 1]$.

Conditional on  :

Then:

4 uniform points in

3 uniform points in

Intuition on the proof

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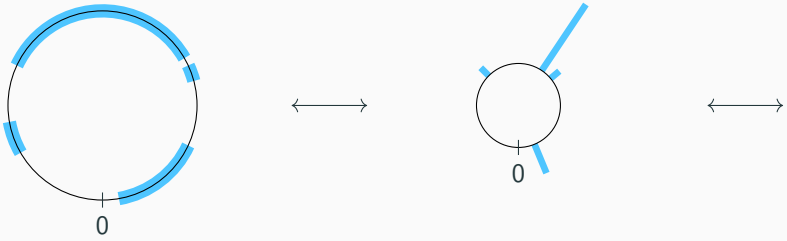

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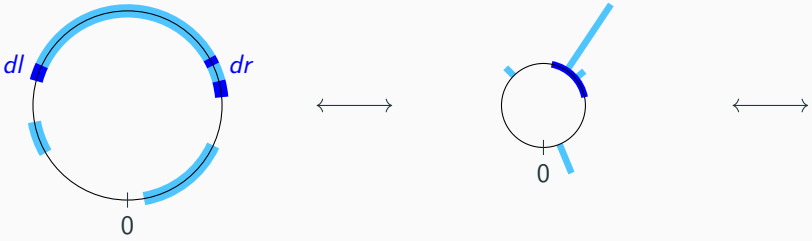

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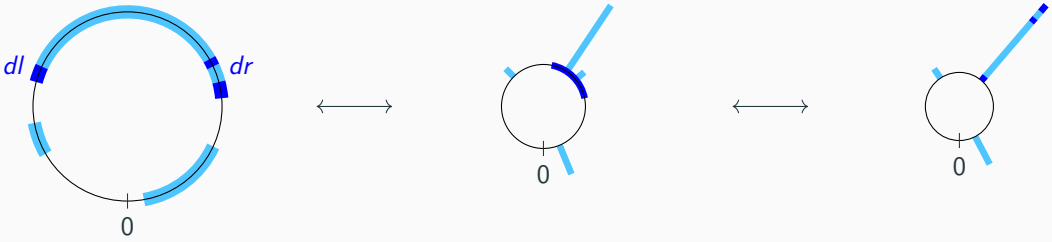

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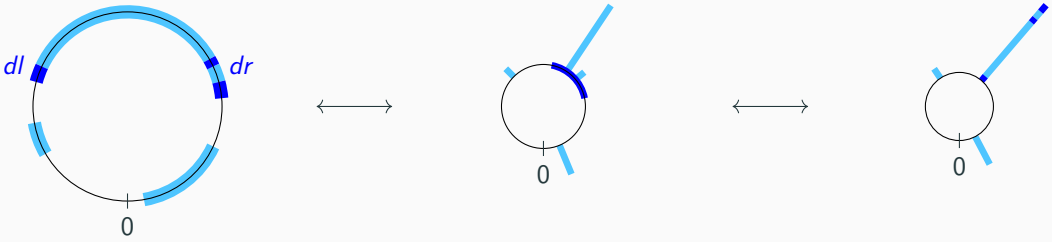

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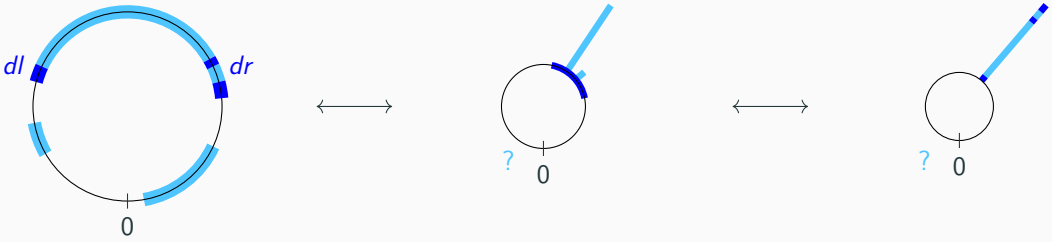

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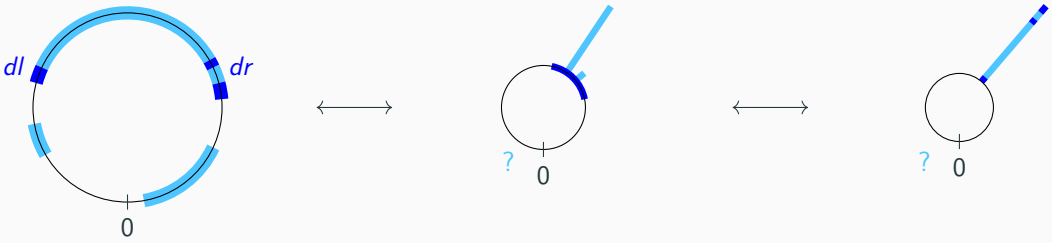

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Distribution of the lengths of the free blocks

Reminder: $|F| = (|F_i|)_{1 \leq i \leq N^{(k)}}$ and $\frac{\sigma.|F|}{R} = \left(\frac{|F_{\sigma_i}|}{R}\right)_{1 \leq i \leq N^{(k)}}$, for any $\sigma \in \mathfrak{S}_{N^{(k)}}$.

Theorem (Distribution of $|F|$ conditional on $N^{(k)}$)

Recall that $R = 1 - \sum m_i$.

- ***unbiased version:*** *if* $\boldsymbol{\sigma} \sim \mathcal{U}(\mathfrak{S}_{N^{(k)}})$, $\frac{\boldsymbol{\sigma}.|F|}{R} \sim \text{Dirichlet}(N^{(k)}; 1, \cdots, 1)$

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- ***biased version:***

$$\frac{(|F_1|, \ldots, |F_{N^{(k)}}|)}{R} \sim \begin{cases} \text{Dirichlet}(N^{(k)}; 1, \cdots, 1) & \textit{with probability } 1 - R \\ \text{Dirichlet}(N^{(k)}; 2, 1, \cdots, 1) & \textit{with probability } R \end{cases}$$

Intuition on the proof (bis)

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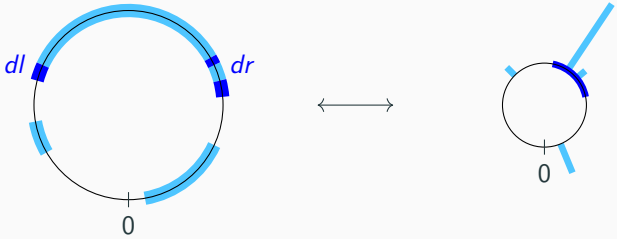

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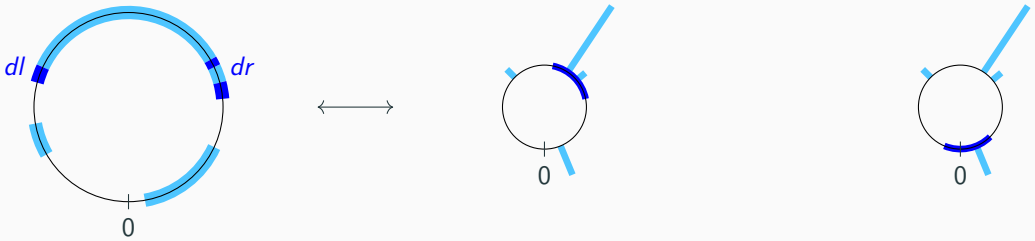

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- even more surprisingly, for the peaks **number** and **positions**: do not depend on which peak is extended by the diffusion

Distribution of the number of blocks $N^{(k)}$

Theorem

$$\mathcal{L}\left(|F^{(k)}| \;\middle|\; (m_0, \cdots, m_{k-1})\right) = \mathcal{L}\left(|\overrightarrow{F^{(k)}}| \;\middle|\; (m_0, \cdots, m_{k-1})\right)$$

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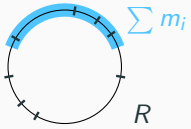
Theorem (Distribution of $N^{(k)}$)

Let $B(k-1, R) \sim \text{Binomial}(k-1, R)$, *then*

$$N^{(k)} \overset{(d)}{=} 1 + B(k-1, R)$$

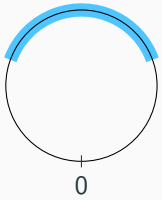


$$N^{(k)} = 1 + \sum_{j=1}^{k-1} \mathbb{1}_{u_k \notin}$$

Distribution of the occupied blocks

One block case:

$$\mathbb{P}\left(N^{(k)} = 1\right) = \left(\sum m_i\right)^{k-1} =: Q\left(\sum m_i, k\right)$$



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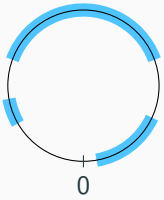

General case:

Theorem

$$\mathbb{P}\left(|O^{(k)}|=(M_0,\dots,M_{b-1})\right)=\mathsf{T}(M_0,\dots,M_{b-1})\sum_{P\in\mathcal{P}(k,b)}\left[\prod_{\ell=0}^{b-1}Q(M_j,|P_j|)\ \mathbb{1}_{\sum_{i\in P_\ell}m_i=M_\ell}\right]$$

where

- $\mathcal{P}(k,b)$ *is the set of partitions* $P=(P_0,\dots,P_{b-1})$ *of* $\{1,\dots,k-1\}$ *into* b *non empty parts,*
- $\mathsf{T}(M_0,\dots,M_{b-1})=M_0\frac{(1-\Sigma M_\ell)^{b-1}}{(b-1)!}+\frac{(1-\Sigma M_\ell)^b}{b!}$.

Summary of universality results

Theorem

*For any continuous model with valid spreading policy, the following distributions are **explicit** and **independent of the spreading policy**:*

- *With k fixed:*
  - $\mathcal{L}(O^{(k)}, F^{(k)})$
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Corollary: results on $O^{(k)}, F^{(k)}$ for one spreading policy are valid for any spreading policy !

Asymptotic results

With n (random) masses, $n \to \infty$, for example if

- $\forall i, m_i = 1/n$ and consider the process at time n
- $\forall i, m_i = \ell_i/n$ (where ℓ_i are i.i.d. with $\mathbb{E}[\ell_i] < \infty$ and satisfy some regularity assumption), at time $t = \sup\{k : \sum_{i=0}^{k} m_i < 1\}$.

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Corollary (Bertoin, Miermont [BM06]; Marckert, V.)

There exists a limit process S such that

$$\left(\frac{\mathsf{LargestBlock}^{(\mathsf{i})}}{n}, 1 \le i \le j\right) \xrightarrow[n\to\infty]{(d)} (\mathsf{SortedExc}(S)_i, 1 \le i \le j)\,.$$

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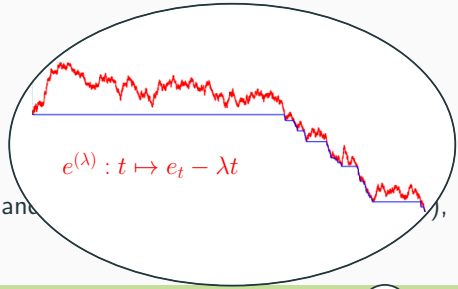

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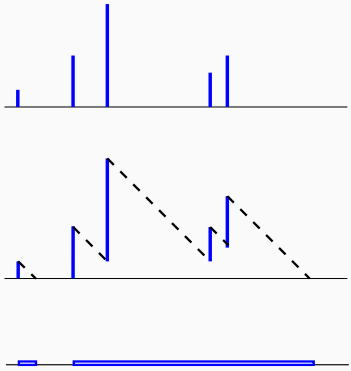
Key tools : the “collecting paths”

Illustration:



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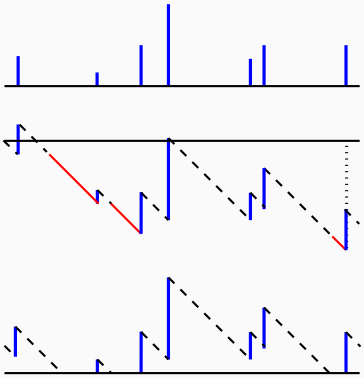

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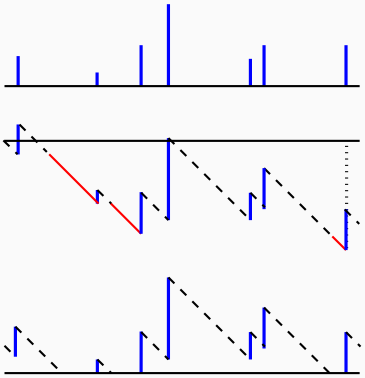

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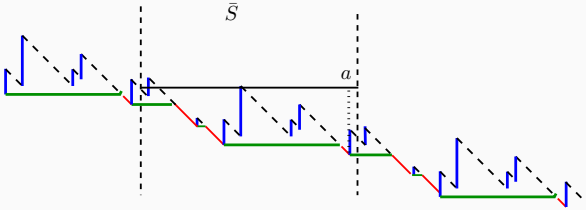

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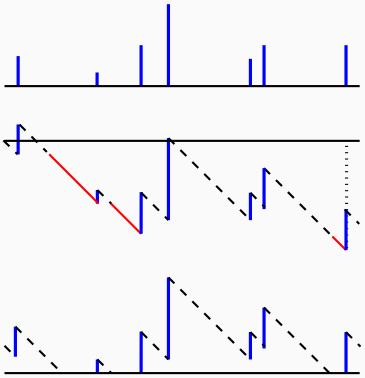

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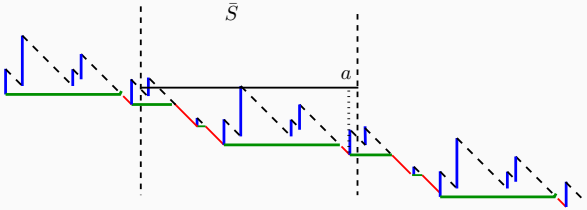

Convergence: when $\bar{S}$ converges to $e^{(\lambda)}$,
$\left(\text{SortedExc}(\bar{S})_i\right)_{1\leq i\leq j} \xrightarrow[n\to\infty]{(d)} \left(\text{SortedExc}(e^{(\lambda)})_i\right)_{1\leq i\leq j}$.

Discrete space and discrete masses

New definition on $\mathcal{C}_n := \{0/n, \cdots, (n-1)/n\} \subset \mathcal{C}$:

- masses arrive on $\mathcal{C}_n$: $\forall i, u_i \sim \mathcal{U}(\mathcal{C}_n)$
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Theorem (Similar universality result)

For any discrete model with valid spreading policy, the following distributions are ***explicit*** *and* ***independent of the spreading policy****:*

- *With k fixed:*
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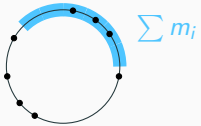
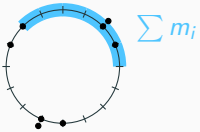
Discrete VS Continuous model

Continuous process:



$N^{(k)} \overset{(d)}{=} 1 + \text{Binomial}(k-1, R)$

Discrete process:



$N^{(k)} \overset{(d)}{=} 1 + \dots$

Discrete VS Continuous model

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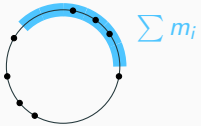

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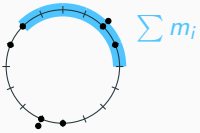

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Discrete VS Continuous model

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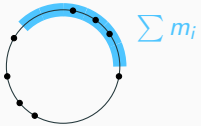

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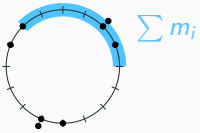

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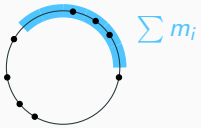

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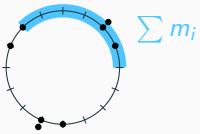

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Asymtotic behavior when $k = n - \lambda\sqrt{n}$ and $\forall i, m_i = 1/n$

Number of blocks: $\frac{N_k}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \lambda$

Number of blocks: $\frac{N_k^{(n)}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \lambda(1-e^{-1})$

Discrete VS Continuous model

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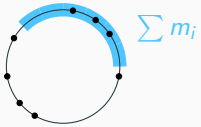
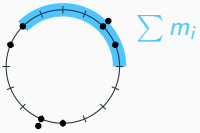


$N^{(k)} \overset{(d)}{=} 1 + \text{Binomial}(k-1, R)$

Discrete process:



$N^{(k)} \overset{(d)}{=} 1 + \text{NbDiff}(\text{Binomial}(k-1, R-1/n))$

Asymtotic behavior when $k = n - \lambda\sqrt{n}$ and $\forall i, m_i = 1/n$

Number of blocks: $\frac{N_k}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \lambda$

Number of blocks: $\frac{N_k^{(n)}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \lambda(1-e^{-1})$

Large block sizes: $\left(\frac{\text{LargestBlock}^{(i)}}{n}, 1 \le i \le j\right) \xrightarrow[n\to\infty]{(d)} \left(\text{SortedExc}(e^{(\lambda)})_i, 1 \le i \le j\right)$

Conclusion and perspectives

Cost of parking procedures:

- Standard parking : if car k falls in a block of size B_k, it costs $C_k = \lfloor U.B_k \rfloor$, with $U \sim \mathcal{U}([0,1])$. Chassaing-Louchard [CL02]:

$$\frac{1}{n^{3/2}} \sum_{k=1}^{\lfloor n-\lambda\sqrt{n} \rfloor} C_k \xrightarrow[n\to\infty]{(d)} \int_0^1 e_t^{(\lambda)} dt$$

- Parking with different parking policies : see Marckert-V. for some of them.

Conclusion and perspectives

Thank you !

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