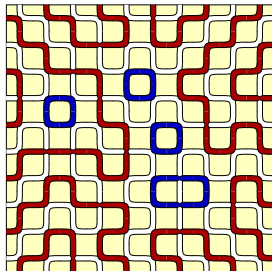
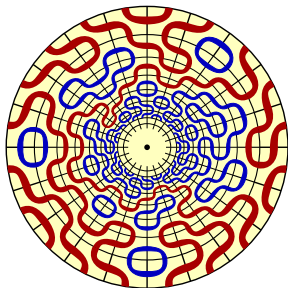


Around the Razumov–Stroganov correspondence: old and new

Andrea Sportiello, based on work with Luigi Cantini (CY Cergy Paris Université)



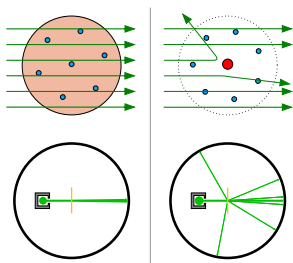
Journées de combinatoire de Bordeaux 2025
LaBRI, 3–5 Février 2025

How Statistical Mechanics met Algebraic Combinatorics

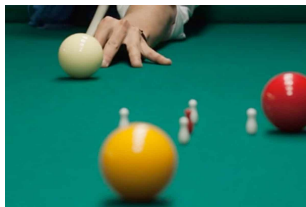
“Newton Physics” is the physics of dynamical systems, i.e. the equations describing the motion of point particles possibly with mutual interactions and in a potential.



Halley comet, photo by Dennis Mammana in northern Mexico, March 13 1986
<https://www.rocketstem.org/2020/03/07/ice-and-stone-comet-of-week-11/>



wikipedia:
Rutherford scattering experiments



How Statistical Mechanics met Algebraic Combinatorics

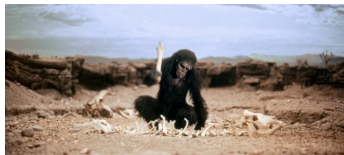
“Newton Physics” is the physics of dynamical systems, i.e. the equations describing the motion of point particles possibly with mutual interactions and in a potential.



Historically, that's **the beginning of all physics**

How Statistical Mechanics met Algebraic Combinatorics

“Newton Physics” is the physics of dynamical systems, i.e. the equations describing the motion of point particles possibly with mutual interactions and in a potential.



By the work of Lagrange, Euler, Hamilton, ... the Hamiltonian has been introduced. It generates the time evolution, is a conserved quantity, and its value describes the total energy of the system.

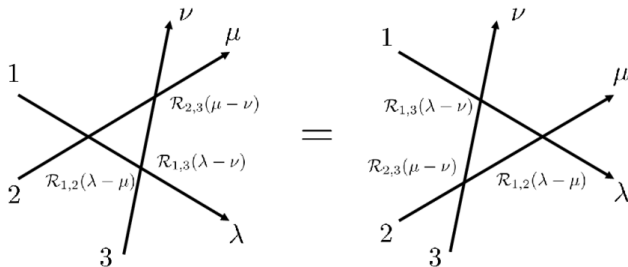
In generic systems, the total energy and momentum are the only conserved quantities, and the motion is typically chaotic. But some “lucky” systems have many more conserved quantities.

Integrable Systems *à la Liouville* possess a full set of conserved charges, that Poisson-commute with each other. The dynamics is so constrained that, essentially, the trajectory in phase space is determined by the conservation laws.

How Statistical Mechanics met Algebraic Combinatorics

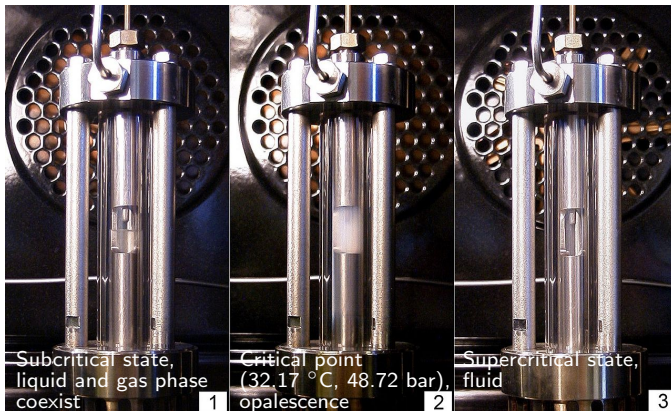
Classical Mechanics evolved into Quantum Mechanics, and then into Quantum Field Theory. **Integrable Quantum Field Theories** then developed as the QFT version of Liouville integrability.

The work of **Yang** (and others!) found that, in a simple class of (1+1-dimensional local relativistic) QFT's, the **scattering matrix factorized** to that of the two-body problem if the (later called) **Yang–Baxter Equation** was satisfied.



How Statistical Mechanics met Algebraic Combinatorics

Statistical Mechanics of phase transitions and critical phenomena is a *different* field of physics, describing matter near to a critical point.



Measurement of the critical temperature of ethane, and illustration of critical opalescence
<https://commons.wikimedia.org/wiki/File:CriticalPointMeasurementEthane.jpg>

How Statistical Mechanics met Algebraic Combinatorics

Statistical Mechanics of phase transitions and critical phenomena is a *different* field of physics, describing matter near to a critical point.

“Solving” a stat-mech problem is, in general, easy in dimension $d = 1$ or in mean-field approximation (corresponding to $d \rightarrow \infty$), but virtually impossible in the “physical” dimensions $d = 2, 3$ or 4 .

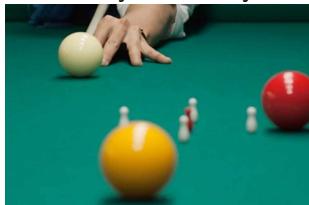
However, by the beginning of the 70’s we had a few **exactly-solved models in $d = 2$** : **Kirchhoff’s** 1847 Matrix–Tree theorem, counting the spanning trees of a graph (for all d); **Onsager’s** 1944 solution of the Ising Model; the **Kac–Ward** 1952 alternate solution of Ising; **Kasteleyn’s** 1961 solution of the Dimer Model; **Lieb and Sutherland’s** 1967 solution of the 6-Vertex Model; . . .

Then, **Baxter** in 1971 had a key idea: in Statistical Mechanics, exactly-solved models in $d = 2$ are solvable because they are, in disguise, models for scattering in 1+1 dimension, which are “integrable” because of the presence of a **Yang–Baxter Equation**.

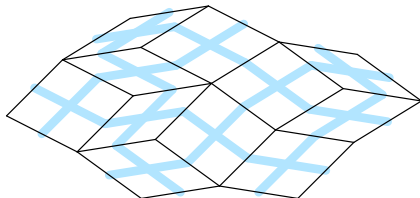
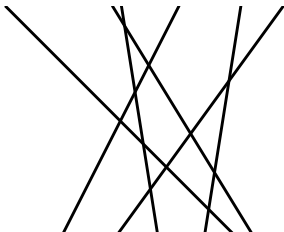
How Statistical Mechanics met Algebraic Combinatorics

This is an incredible change of perspective.

Now, if you suspect that a stat-mech model is specially nice or solvable, you can try to “*Baxterise*” it,



Billiard balls: when they hit each other, they change velocities, but keep their colours. . .

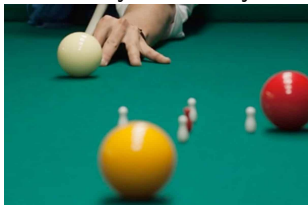


Integrable Baxter models:
“balls” hit at intersections
keep their “rapidities” λ_j ,
but may change their colours x_i

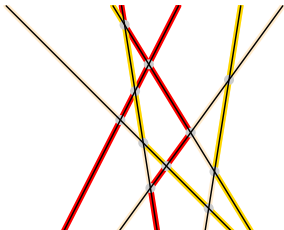
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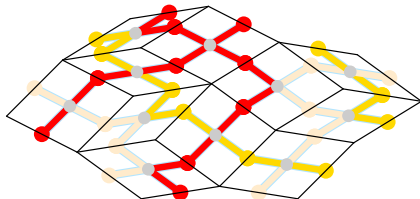
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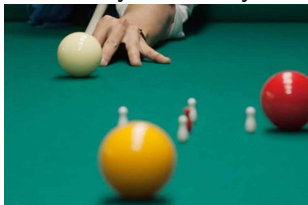
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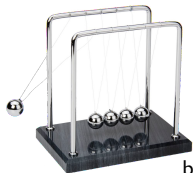
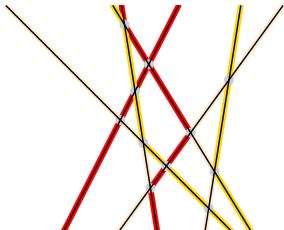
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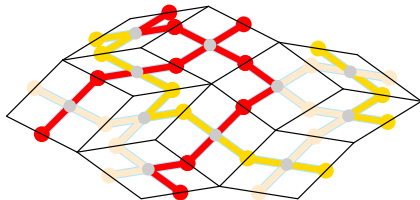
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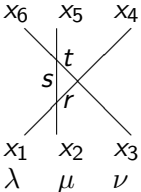


How Statistical Mechanics met Algebraic Combinatorics

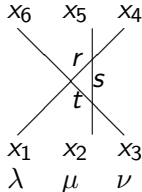
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Now, if you suspect that a stat-mech model is specially nice or solvable, you can try to “*Baxterise*” it, i.e. invent a one-parameter deformation of the weights such that:

1. The graph is described by bundles of lines that intersect each other. A **spectral parameter** λ_j is attached to line j .
2. The variables x_i live on segments between intersections.
3. The measure $\mu(\vec{x})$ factorises into $W_{x_1, x_2}^{x_3, x_4}(\lambda_i - \lambda_j)$ at crossings.
4. The Yang–Baxter equation is an identity for a certain sum over internal states, for a system with three lines.



$$\sum_{r,s,t} W_{x_1, x_2}^{s, r}(\lambda - \mu) W_{r, x_3}^{t, x_4}(\lambda - \nu) W_{s, t}^{x_6, x_5}(\mu - \nu)$$

$$= \sum_{r,s,t} W_{x_2, x_3}^{t, s}(\mu - \nu) W_{x_1, t}^{x_6, r}(\lambda - \nu) W_{r, s}^{x_5, x_4}(\lambda - \mu)$$


$\forall \lambda, \mu, \nu \in \mathbb{C}; x_1, \dots, x_6 \in \mathcal{S}$

How Statistical Mechanics met Algebraic Combinatorics

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Many facts follow from the YBE: in a cylinder geometry **transfer matrices** $T(\lambda)$ **do commute**, which implies that $H = \frac{\partial}{\partial \lambda} T(\lambda)|_{\lambda=0}$ commutes with all charges $Q_j = \frac{\partial^j}{\partial \lambda^j} T(\lambda)|_{\lambda=0}$. All is “clean”, and different notions of integrability come together!

An example: Schur functions and weighted lozenge tilings

Semi-Standard Young Tableaux $\text{SSYT}(\lambda, n)$:

Fillings of λ with the integers $\{1, 2, \dots, n\}$, $\bullet \leq \bullet$
 repetitions allowed, satisfying \bullet

1	1	3	4	4
2	3			
5	6			
6				

Schur polynomials are 'generating functions' of SSYT 's
 and a basis of symmetric polynomials

$$s_{\lambda}(x_1, \dots, x_6) = \dots + x_1^2 x_2 x_3^2 x_4^2 x_5 x_6^2 + \dots$$

$$s_{\lambda}(x_1, \dots, x_n) = \sum_{T \in \text{SSYT}(\lambda, n)} \prod_{i=1}^n x_i^{\#\{i \in T\}} \quad \text{def. SSYT}$$

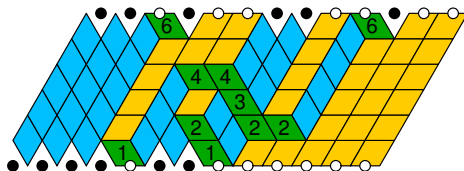
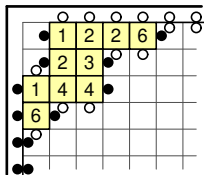
$$[\Delta(\vec{x}) = \prod_{i < j} (x_i - x_j)] = \frac{1}{\Delta(\vec{x})} \det \left((x_i^{\lambda_j + n - j})_{i,j=1, \dots, n} \right) \quad \text{Weyl formula}$$

$$= \det \left((h_{\lambda_i + j - i})_{i,j=1, \dots, \ell(\lambda)} \right) \quad \text{Jacobi-Trudi}$$

$$= \det \left((e_{\lambda'_i + j - i})_{i,j=1, \dots, \lambda_1} \right) \quad \text{dual Jacobi-Trudi}$$

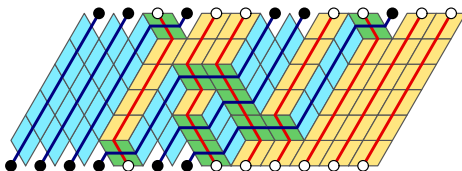
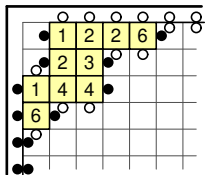
An example: Schur functions and weighted lozenge tilings

(Skew-)Schur polynomials can be represented as partition functions of the (f.f.) **5VM**, which is a $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$ Yang–Baxter integrable model, on a horizontal strip of the square lattice, with homogeneous vertical spectral parameters, the horizontal ones determine the alphabet, and the top/bottom b.c. determine the partitions λ and μ

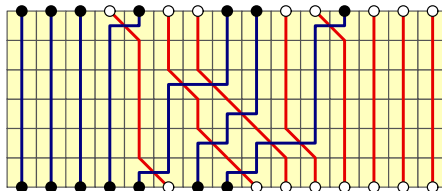
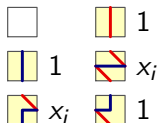


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$T(x_i)$:




Littlewood–Richardson coefficients as a Vertex Model

The structure constants $c_{\mu\nu}^\lambda$ of the algebra $\Lambda = \text{span}_{\mathbb{K}}(s_\lambda(\vec{x}))_\lambda$ are **non-negative integers** known as **Littlewood–Richardson coefficients**

$$s_\mu(\vec{x})s_\nu(\vec{x}) = \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda(\vec{x}); \quad s_\lambda(\vec{x}, \vec{y}) = \sum_{\mu, \nu} c_{\mu\nu}^\lambda s_\mu(\vec{x})s_\nu(\vec{y});$$

Also the Littlewood–Richardson coefficients are described by an **integrable Vertex Model**, now with underlying $\mathcal{U}_q(\widehat{\mathfrak{sl}}_3)$ symmetry.

 A. Knutson and T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. **119** (2003); P. Zinn-Justin, *Littlewood–Richardson Coefficients and Integrable Tilings*, EJC **16** (2009)

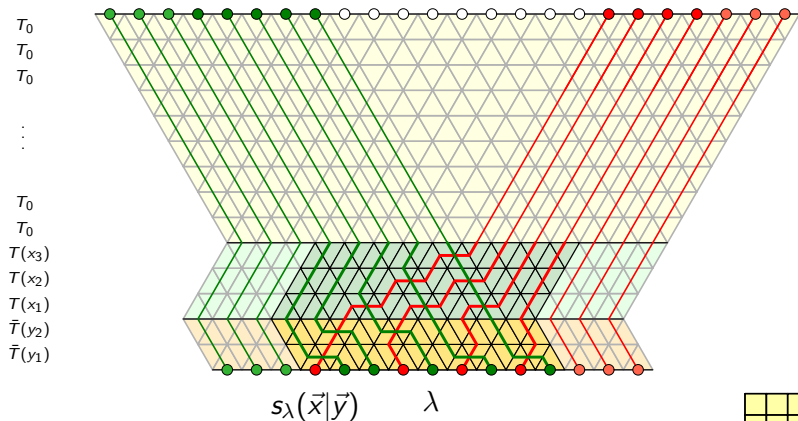
The key idea is to express the two sides of the coproduct identity

$$s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu, \nu} c_{\mu\nu}^\lambda s_\mu(\vec{x})s_{\nu'}(\vec{y})$$

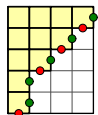
as partition functions in a **rank-2** model (i.e., with particles of three colours)

The identity is a consequence of commutation of transfer matrices, which in turns comes from the **Yang–Baxter Equation** of the rank-2 model

Littlewood–Richardson coefficients as a Vertex Model

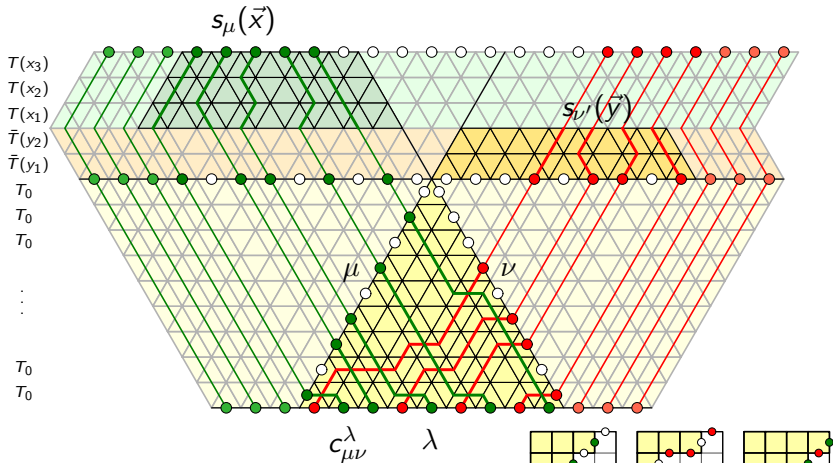


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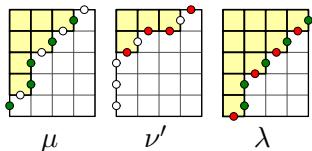


λ

Littlewood–Richardson coefficients as a Vertex Model



$$s_\lambda(\vec{x}|\vec{y}) = \sum_{\mu, \nu} c_{\mu\nu}^\lambda s_\mu(\vec{x}) s_{\nu'}(\vec{y})$$



A partial conclusion from the introduction...

We have seen that the mechanism

$$\text{YBE} \implies [T(\lambda), T(\mu)] = 0 \implies Z_{\lambda/\mu}(\vec{x}) \text{ is symmetric}$$

links Integrable Models *à la* Baxter to the Algebraic Combinatorics of symmetric polynomials, and that, thanks to a further idea by P. Zinn-Justin, also to their structure constants.

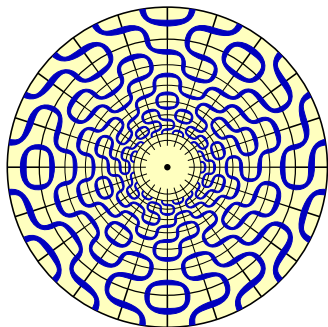
We think that, nowadays, **we essentially understand all this.**

Now we are going to talk about the **Razumov–Stroganov correspondence**, which connects two distinct Integrable Models *à la* Baxter.

We will see symmetric polynomials and structure constants pop out (although this was not quite needed for RS in itself)

We will also see that **we mostly do not understand this...**

Two Problems of (integrable) Random Tilings

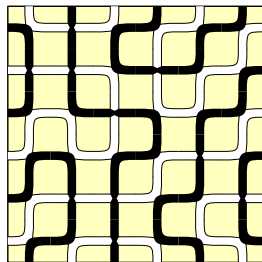


$O(1)$ Dense Loop Model

on a semi-infinite cylinder (or strip)
= XXZ Quantum Spin Chain at $\Delta = -\frac{1}{2}$
= Edge-percolation (Potts Model at $Q = 1$)

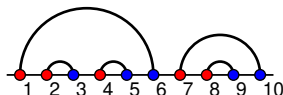
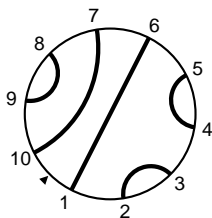
Fully-Packed Loops (FPL) in a square
(or some other domain mostly locally like a square lattice)

= Alternating Sign Matrices (ASM)
= Six-Vertex Model at $\Delta = +\frac{1}{2}$ (Ice Model)
= Non-Intersecting Lattice Paths



Link patterns

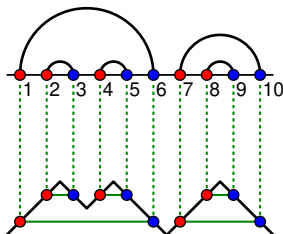
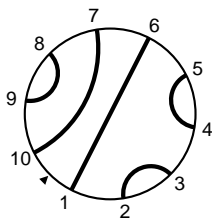
A **link pattern** $\pi \in \mathcal{LP}(2n)$ is a pairing of $\{1, 2, \dots, 2n\}$ having no pairs $(a, c), (b, d)$ such that $a < b < c < d$ (i.e., the drawing consists of n **non-crossing** arcs).



They are $C_n = \frac{1}{n+1} \binom{2n}{n}$ (the n -th *Catalan number*),

Link patterns

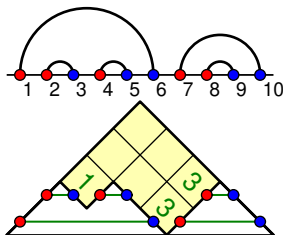
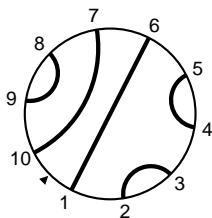
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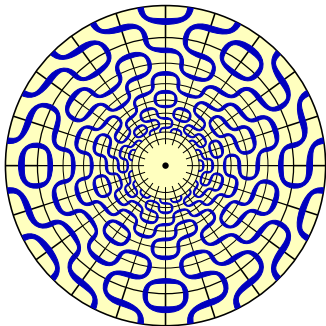
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are in easy bijection with **Dyck Paths** of length $2n$,
and with **integer partitions** boxed in a triangle,...

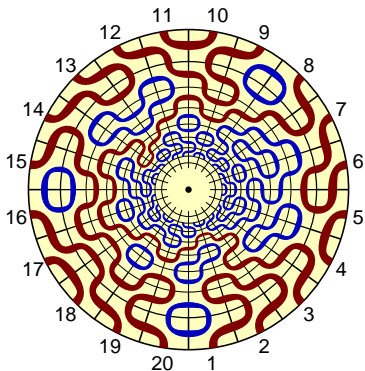
Link patterns in the Dense Loop Model

We can associate a **link pattern** π to any **dense-loop** configuration on a semi-infinite cylinder, as the connectivity pattern among the points on the boundary.



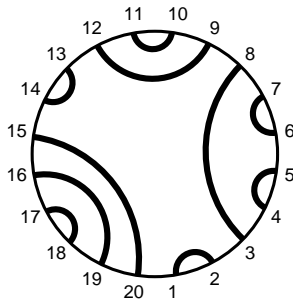
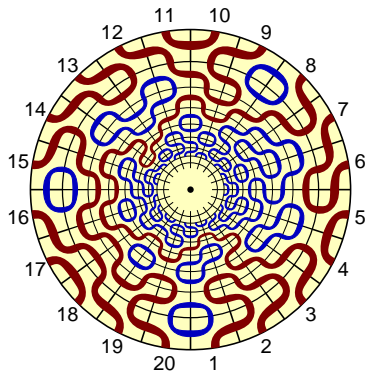
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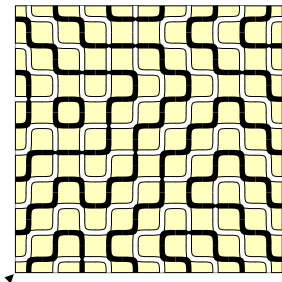
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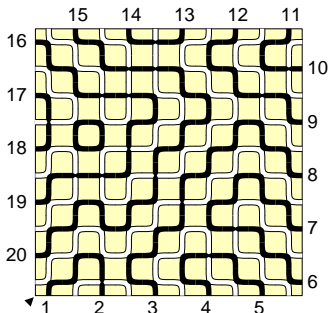
Link patterns in Fully-Packed Loops

We can associate a **link pattern** π also to any **Fully-Packed Loop** configuration, as the connectivity pattern among the black terminations on the boundary.



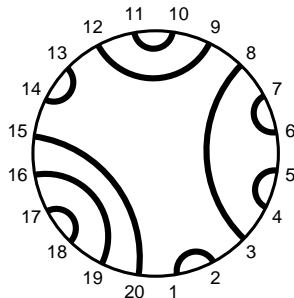
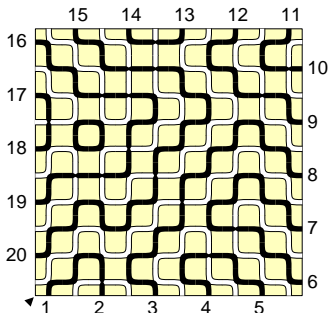
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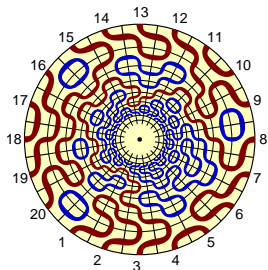


Link patterns in Fully-Packed Loops

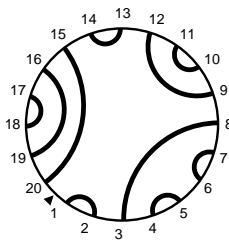
We can associate a **link pattern** π also to any **Fully-Packed Loop** configuration, as the connectivity pattern among the black terminations on the boundary.



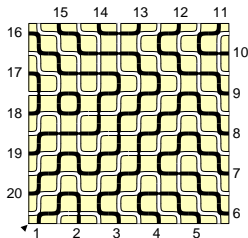
The dihedral Razumov–Stroganov correspondence



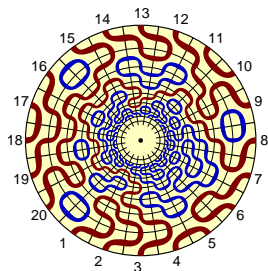
$\tilde{\Psi}_n(\pi)$: probability of π
in the $O(1)$ Dense Loop Model
in the $\{1, \dots, 2n\} \times \mathbb{N}$ cylinder



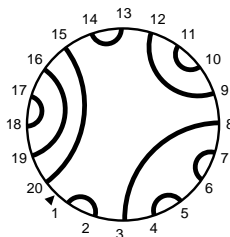
$\Psi_n(\pi)$: probability of π
for FPL with uniform measure
in the $n \times n$ square



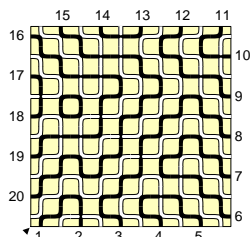
The dihedral Razumov–Stroganov correspondence



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$\Psi_n(\pi)$: probability of π
for FPL with uniform measure
in the $n \times n$ square




Dihedral Razumov–Stroganov correspondence

(conjecture: Razumov Stroganov, 2001; proof: AS Cantini, 2010)

$$\tilde{\Psi}_n(\pi) = \Psi_n(\pi)$$

Many Razumov–Stroganov-type conjectures

In fact, there exists a whole class of Razumov–Stroganov conjectures

 A.V. Razumov and Yu.G. Stroganov, *Combinatorial nature of ground state vector of $O(1)$ loop model*, Theor. Math. Phys. **138** (2004); —, *$O(1)$ loop model with different boundary conditions and symmetry classes of alternating-sign matrices*, Theor. Math. Phys. **142** (2005); J. de Gier, *Loops, matchings and alternating-sign matrices*, Discr. Math. **298** (2005); S. Mitra, B. Nienhuis, J. de Gier and M.T. Batchelor, *Exact expressions for correlations in the ground state of the dense $O(1)$ loop model*, JSTAT(2004); J. de Gier and V. Rittenberg, *Refined Razumov–Stroganov conjectures for open boundaries*, JSTAT(2004); Ph. Duchon, *On the link pattern distribution of quarter-turn symmetric FPL configurations*, FPSAC 2008

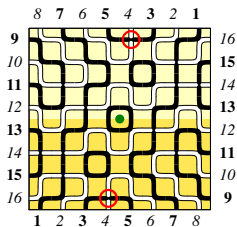
These variants are naturally arranged into two main classes:

dihedral RS: FPL domains with Wieland dihedral symmetry,
 \Leftrightarrow $O(1)$ DLM on the cylinder (the **periodic** quantum spin chain)

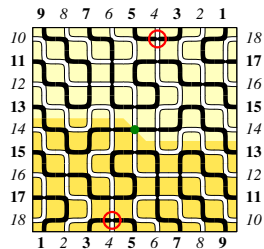
vertical RS: FPL domains with a “reflecting wall” of
U-turn/O-turn \Leftrightarrow versions of the $O(1)$ DLM on the strip
(the **open or closed boundary** quantum spin chain)

Some other dihedral Razumov–Stroganov (ex-)conjectures

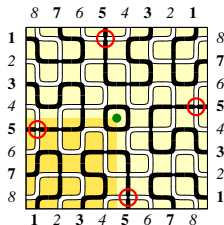
HTASM $L = 2n \dagger$



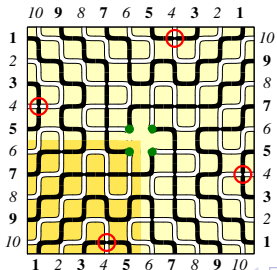
HTASM $L = 2n + 1 \dagger$



QTASM $L = 4n \ddagger$



qQTASM $L = 4n + 2 \ddagger$



\dagger HTASM = Half-turn symmetric ASM's
 \ddagger QTASM = Quarter-turn symmetric ASM's
 qQTASM = quasi-Quarter-turn symmetric ASM's

Dihedral symmetry of FPL

A corollary of the Razumov–Stroganov correspondence. . .

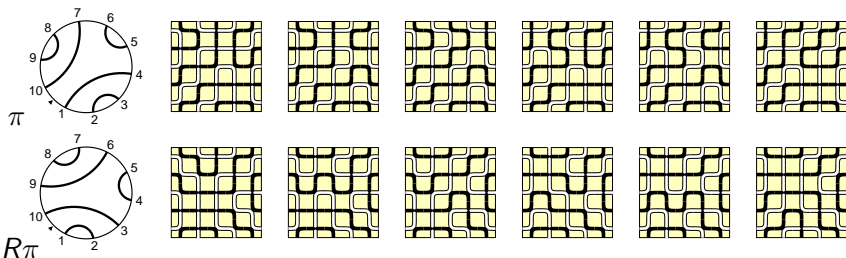
(. . . that was known *before* the Razumov–Stroganov conjecture)

call R the operator that rotates a link pattern by one position

Dihedral symmetry of FPL

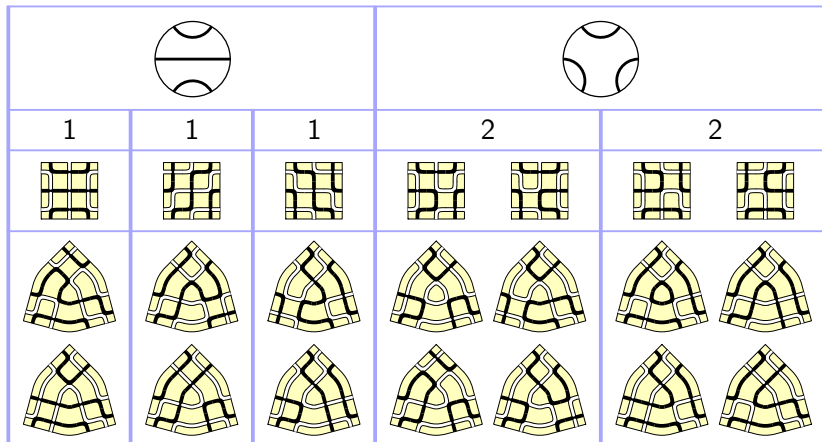
(proof: Wieland, 2000)

$$\Psi_n(\pi) = \Psi_n(R\pi)$$



The domains where dihedral Razumov–Stroganov holds

In the case of the [dihedral Razumov–Stroganov correspondence](#), Wieland gyration (and its generalisations) has been a crucial ingredient and led us to classify the family of domains for which RS holds



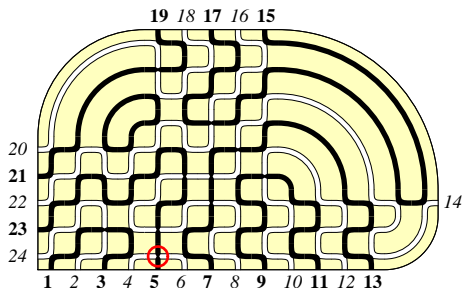
The domains where dihedral Razumov–Stroganov holds

So, in *proving* the various existing (dihedral) Razumov–Stroganov conjectures, we have been led to *generalise* them to a much larger family of domains ($\sim n^3$ different domains for $\mathcal{LP}(2n)$).

There are three subclasses, according to the type of link patterns and Temperley–Lieb algebras: ordinary, punctured even and punctured odd.

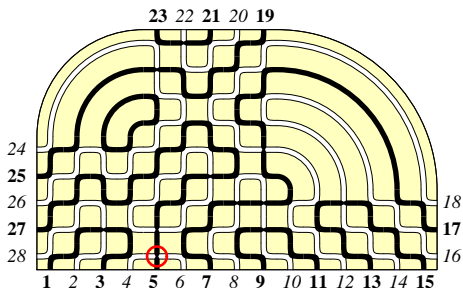
The domains where dihedral Razumov–Stroganov holds

1 corner, 3 triangles:



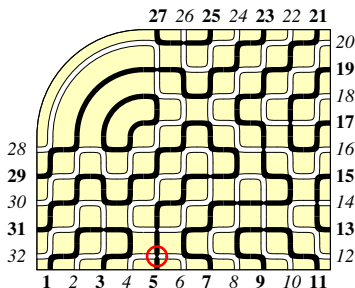
The domains where dihedral Razumov–Stroganov holds

2 corners, 2 triangles:



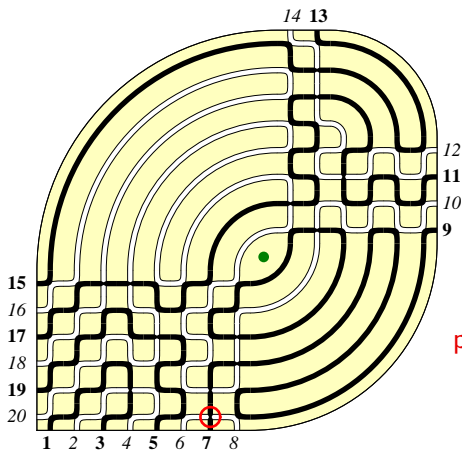
The domains where dihedral Razumov–Stroganov holds

3 corners, 1 triangle:



The domains where dihedral Razumov–Stroganov holds

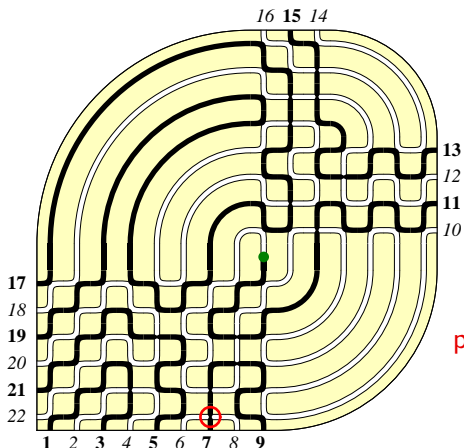
1 corner, 1 face with $\ell = 2$:



(this works with
punctured link patterns
of even size!)

The domains where dihedral Razumov–Stroganov holds

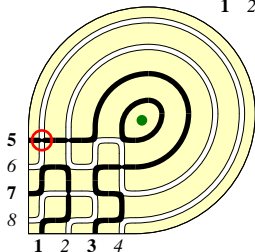
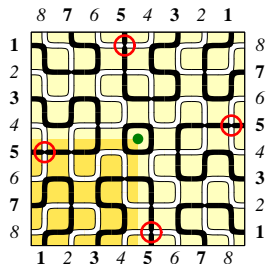
1 corner, 1 vertex of degree 2:



(this works with
punctured link patterns
of odd size!)

The domains where dihedral Razumov–Stroganov holds

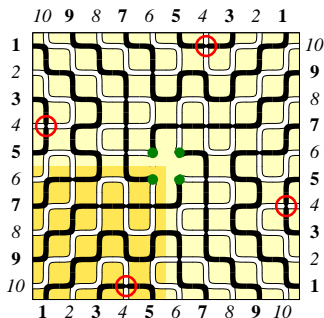
2 corners, 1 face with $\ell = 2$:
(these are QTASM,
quarter-turn symmetric ASM's)
and it works with punctured
link patterns of even size)



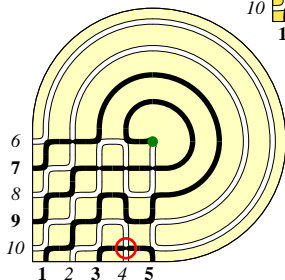
$$L = 4n$$

The domains where dihedral Razumov–Stroganov holds

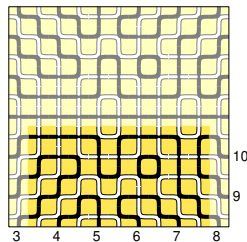
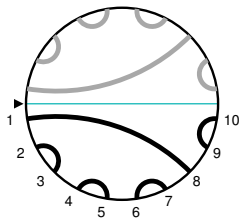
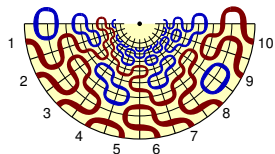
2 corners, 1 face with $\ell = 2$:
(these are q QTASM,
quasi-quarter-turn symmetric ASM's)
and it works with punctured
link patterns of odd size)



$$L = 4n + 2$$



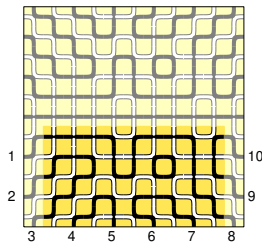
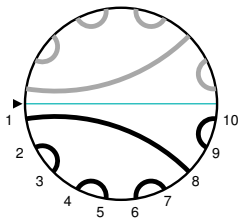
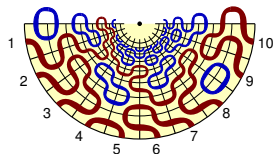
A Vertical Razumov–Stroganov Conjecture



$\tilde{\Psi}_n^V(\pi)$: probability of π
in the $O(1)$ Dense Loop Model
in the $\{1, \dots, 2n\} \times \mathbb{N}$ strip

$\Psi_n^V(\pi)$: probability of π
for vertically-symmetric FPL
with uniform measure in the
 $(2n + 1) \times (2n + 1)$ square

A Vertical Razumov–Stroganov Conjecture



$\tilde{\Psi}_n^V(\pi)$: probability of π
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with uniform measure in the
 $(2n + 1) \times (2n + 1)$ square

Vertical Razumov–Stroganov conjecture

(Razumov and Stroganov, 2001, for the square of side $2n + 1$)

$$\tilde{\Psi}_n^V(\pi) = \Psi_n^V(\pi)$$

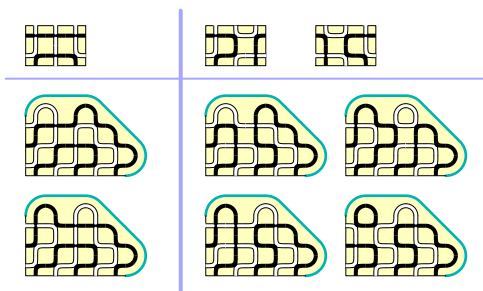
The domains where vertical Razumov–Stroganov holds

The Vertical Razumov–Stroganov conjectures are a whole second family

They involve FPL with some version of **reflecting wall**
and the $O(1)$ Dense Loop Model on a **strip with a boundary**

Our proof methods do **not** seem to work for any of the Vertical Razumov–Stroganov conjectures, **which are all open at present**

But at least we think we know the precise list of domains with Vertical RS



$$3 + x + 7y + 2xy + 4y^2 + xy^2$$

$$6 + 2x + 14y + 4xy + 8y^2 + 2xy^2$$

A personal perspective on Razumov–Stroganov history

2001: Razumov and Stroganov find the first versions of their conjecture (for ASM's and VSASM's)

soon after: Many new conjectures of the same flavours pop up
Many good people work on this. They don't find a proof, but, in passing, they find a lot of other interesting things. . .

2010: Cantini and myself find a **first proof** of **dihedral RS** (and its generalisations)

People starts not caring anymore. But we are still obsessed by the fact that the **vertical RS** is not proven. . .

2012: Cantini and myself find a proof of a conjecture of Di Francesco from 2004, which implies a **second proof** of **dihedral RS** (and its generalisations)

today: Still no proof of **vertical RS** in sight. But Cantini and myself keep on trying, and, in passing, we still find a lot of other interesting things. . .

(Failed) plan of this talk: 1 – first attempts on RS

- ▶ Di Francesco and P. Zinn-Justin solve the “qKZ equation” for the DLM, i.e. “add the spectral parameters” to the DLM side of RS. They find **multi-contour integral formulas** for the $\tilde{\Psi}(\pi)$.
- ▶ For FPL’s, Caselli, Krattenthaler, Lass and Nadeau show that components $\Psi_{n+k}(\pi_n \text{ (diagram)}_k)$ are **polynomials in k** (a bit like Ehrhart polynomials for the number of integer points in a polytope). This leads to the study of “FPL in a triangle”, which form a three-index tensor $a_{\sigma,\pi,\tau}$ of integers.
- ▶ Some statistics d are assigned to the indices, and $a_{\sigma,\pi,\tau} = 0$ unless $d(\sigma) + d(\tau) \leq d(\pi)$. Nadeau shows that, when $d(\sigma) + d(\tau) = d(\pi)$, the $a_{\sigma,\pi,\tau}$ ’s are the **Littlewood–Richardson coefficients $c_{\sigma\tau}^{\pi}$** , as the FPL’s are **Knutson–Tao puzzles** in slight disguise.
- ▶ P. Zinn-Justin conjectures a **three-index multi-contour integral formula** for FPL in a triangle $a_{\sigma,\pi,\tau}$. From that, (dihedral) RS would follow. But, even today that dihedral RS is solved, **this conjecture remains wide open.**

(Failed) plan of this talk: 2 – our work on RS, and new things

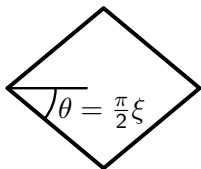
- ▶ We **solve (dihedral) RS** a first time, by promoting Wieland gyration to a “symmetry that leads to Ward identities”. But some parts are a bit tedious.
- ▶ We find a second proof, as a byproduct of the proof of a stronger conjecture by Di Francesco on the eigenstate of the **scattering matrix** $S(x)$. We introduce a “**heretical enumeration**” of FPL, for which the identity is quite simple and only involves R and e_1 . But **still no vertical RS...**
- ▶ ... We look at FPL in UASM, with **black and white LP, and $\#\{\bigcirc\}$ statistics**. We **conjecture** that the enumerations $A_{\pi_\bullet, \pi_\circ}(\tau)$ have leading coeffs coinciding with LR coeffs $c_{\pi_\bullet, \pi_\circ}^{\delta_n}$ **for canonical Grothendieck polys (still no idea for the full polys)**.
- ▶ Also, we search for a (dihedral) RS with **black and white LP statistics**. We find that the “heretical enumeration” of FPL satisfies an identity involving only R_\bullet , R_\circ , e_1^\bullet and e_1° .
The DLM side is a weird new loop model, on finite cylinders.

(Failed) plan of this talk: 2 – our work on RS, and new things

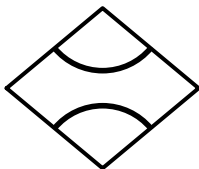
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$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(2n)$

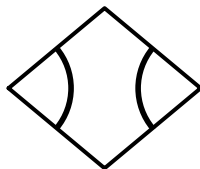
The “integrable weights” of the $O(\tau)$ Dense Loop Model (with $\tau = 2 \cos \gamma$) on isoradial graphs* are



$$\xi \in [0, 1]$$

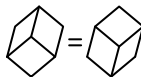


$$\sin \gamma \xi$$



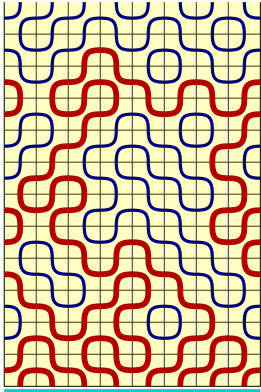
$$\sin \gamma (1 - \xi)$$

* The relation between the angle θ and the integrable weights is natural in two respects: the density of free energy is uniform, and the YBE condition corresponds to “flipping a cube”

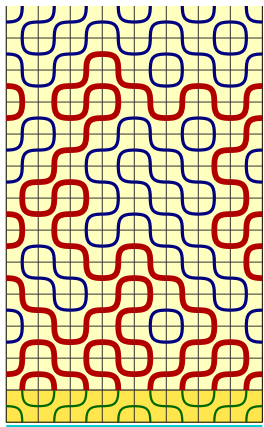


$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(2n)$

A config with $t - 1$ layers, and link pattern π



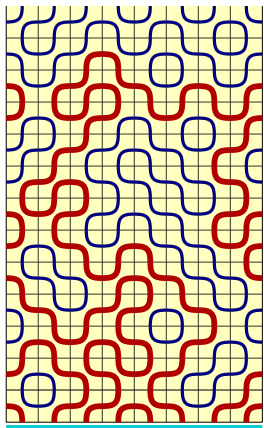
$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(2n)$



A config with $t - 1$ layers, and link pattern π

Add a new layer, of i.i.d. tiles, with probability $p(\diamond)/p(\diamond)$
 $= p/(1 - p) = \sin(\frac{\pi}{3}\xi)/\sin(\frac{\pi}{3}(1 - \xi))$
(say, $p = 1/2$, i.e. $\xi = 1/2$)...

$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(2n)$



A config with $t - 1$ layers, and link pattern π

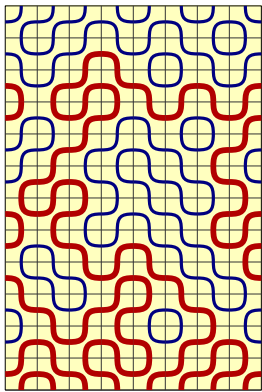
Add a new layer, of i.i.d. tiles, with probability $p(\diamondsuit)/p(\blacklozenge)$
 $= p/(1 - p) = \sin(\frac{\pi}{3}\xi)/\sin(\frac{\pi}{3}(1 - \xi))$
(say, $p = 1/2$, i.e. $\xi = 1/2$)...

Some loops get detached from the boundary. You have a config with t layers, and a new link pattern π' .

The rates $W_p(\pi, \pi')$ are encoded by a big polynomial T_p in $R^{\pm 1}$ and the e_j 's:
 $T_p|\pi\rangle = \sum_{\pi'} W_p(\pi, \pi')|\pi'\rangle$

$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(2n)$

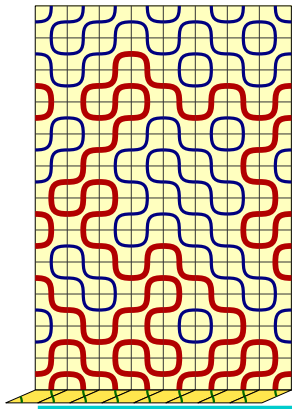
Now repeat the game,



$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(2n)$

Now repeat the game,

but add i.i.d. tiles, with prob. $p \rightarrow 0$

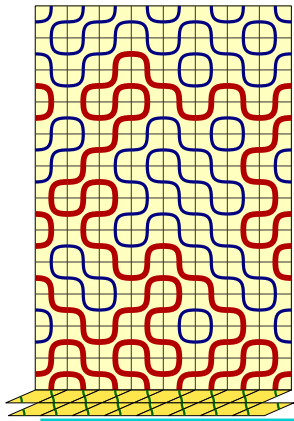


$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(2n)$

Now repeat the game,

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For most of the layers you just rotate



$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(2n)$

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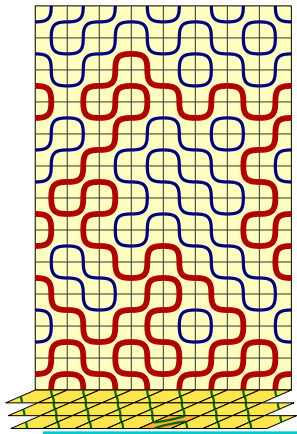
For most of the layers you just rotate
From time to time, you have a single non-trivial tile. In the limit, the probability of having two non-trivial tiles in the same row vanishes.

The rates are

$$W_{p \rightarrow 0}(\pi, \pi') = \delta(\pi', R\pi) + \mathcal{O}(p).$$

More precisely, the operator T_p has the form

$$T_p = R(I + p \sum_j (e_j - 1) + \mathcal{O}(p^2))$$



$O(1)$ Dense Loop Model: the Markov Chain over $\mathcal{LP}(2n)$

Now repeat the game,

but add i.i.d. tiles, with prob. $p \rightarrow 0$

For most of the layers you just rotate
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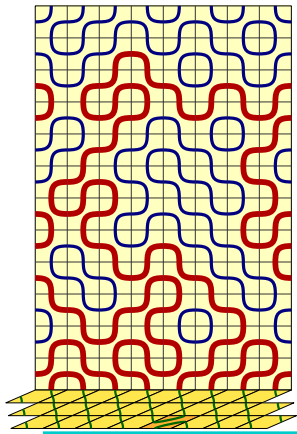
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Hamiltonian H



Integrability: commutation of Transfer Matrices

The 1-parameter family of matrices for the transition rates, $W_p(\pi, \pi')$, acting on $\mathbb{C}^{\mathcal{L}^{\mathcal{P}}(2n)}$ by filling one layer of lozenges with angle θ , form a commuting family. I.e. the family of polynomials T_p in the Temperley–Lieb Algebra form a commuting family.

Trivial: $\tilde{\Psi}_p(\pi)$, the steady state, is the **unique** eigenstate of $T_p(\pi, \pi')$ with all positive entries

The Yang–Baxter relation implies:

$$\sum_{\pi'} W_{p_1}(\pi, \pi') W_{p_2}(\pi', \pi'') = \sum_{\pi'} W_{p_2}(\pi, \pi') W_{p_1}(\pi', \pi''),$$

or also, in TL alg., $[T_p, T_{p'}] = 0$

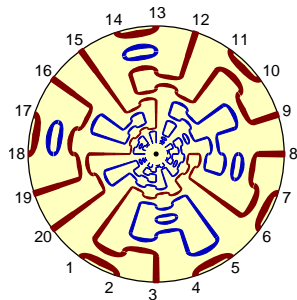
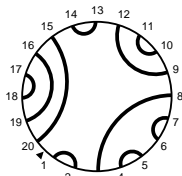
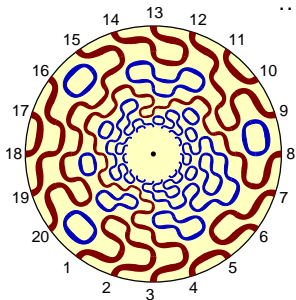
Consequence: $\tilde{\Psi}_p(\pi) \equiv \tilde{\Psi}_{p'}(\pi)$ and we can get $\tilde{\Psi}(\pi) := \tilde{\Psi}_{1/2}(\pi)$ from the study of $T_{p \rightarrow 0}$. Namely, calling $|\tilde{\Psi}_n\rangle = \sum_{\pi} \tilde{\Psi}(\pi) |\pi\rangle$ and $H_n = \frac{\partial}{\partial p}(R^{-1} T_p)|_{p=0} = \sum_{i=1}^{2n} (e_i - 1)$, we have

$$H_n |\tilde{\Psi}_n\rangle = 0$$

linear-algebra characterization of $\tilde{\Psi}(\pi)$

Integrability: commutation of Transfer Matrices

...said with a picture...



$$|\tilde{\Psi}_n\rangle := \sum_{\pi \in \mathcal{LP}(2n)} \tilde{\Psi}_n(\pi) |\pi\rangle$$

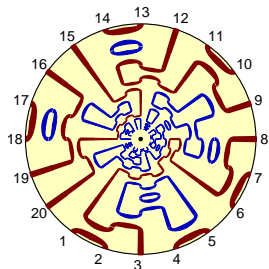
$$(T_n - 1)|\tilde{\Psi}_n\rangle = 0$$

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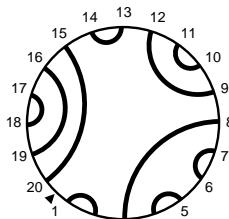
the two linear equations for $|\tilde{\Psi}_n\rangle$ are equivalent!

The Razumov–Stroganov correspondence: reloaded



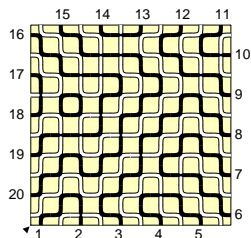
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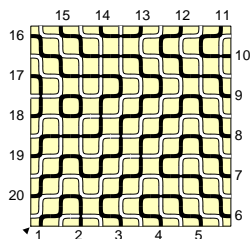
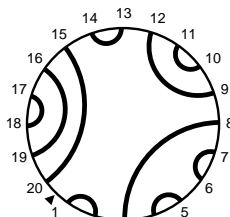
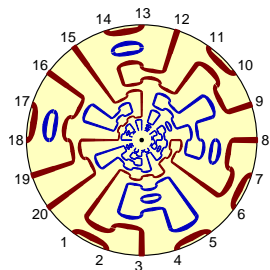


$$|\Psi_n\rangle = \sum_{\phi \in \mathcal{Fpl}(n)} |\pi(\phi)\rangle$$

$$\mathcal{Fpl}(n) = \{ \text{FPL in } n \times n \text{ square} \}$$



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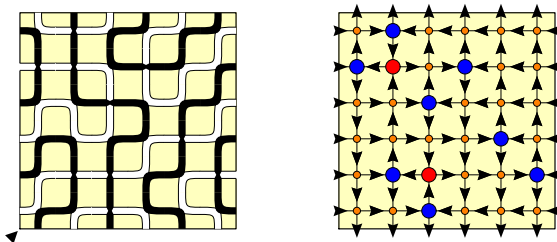
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(conjecture: Razumov Stroganov, 2001; proof: AS Cantini, 2010)

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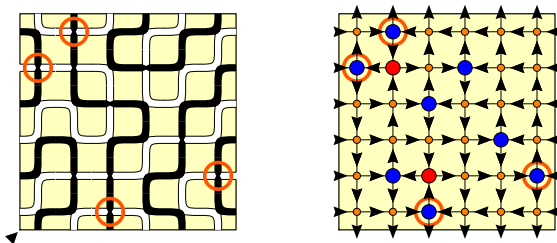
Refinement position in Fully-Packed Loops

Fully-Packed Loops have a **unique** straight tile on any external line
(and Alternating Sign Matrices have a **unique** +1 on any external line)



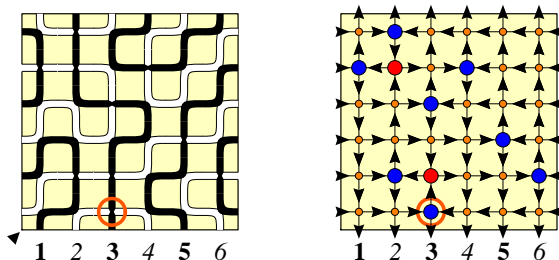
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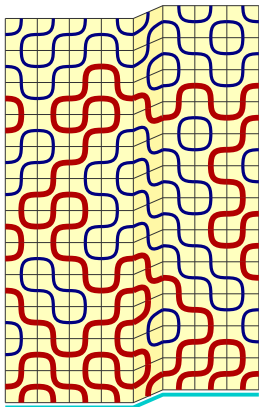
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Concentrate on the bottom row, and call **refinement position** the corresponding column index.

The Izergin–Korepin determinant gives us the **total number of FPL configurations**, possibly refined according to these 4 statistics, but **not** the numbers refined according also to the **link patterns**...

Repeat the game once more...

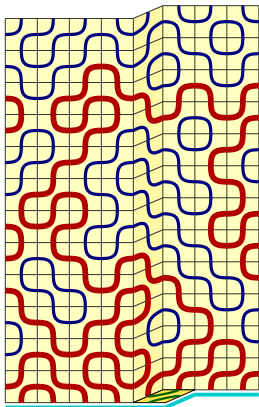


Repeat the game once more...

...but this time keep all tiles frozen, except for the one in column $i + 1$

$$RX_i(t) = R(t + (1 - t)e_j)$$

These simple operators seem to have nothing to do with the original problem.



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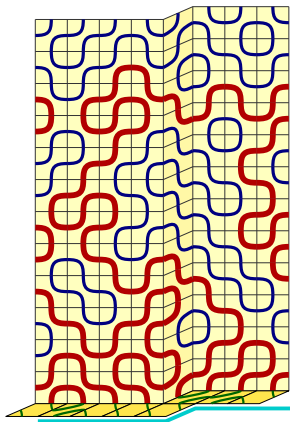
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Nonetheless, calling $S_i(t) = (RX_i(t))^N$ the **Scattering Matrix** on column i , we have $S_i(1 - t) = 1 + tH + \mathcal{O}(t^2)$

So, if we understand the Frobenius vector $\Psi(t)$ of $S_i(1 - t)$, (i.e. the Frobenius vector of $RX_i(1 - t)$), we also understand the RS vector Ψ .



Dihedral covariance of the eigenvectors $|\tilde{\Psi}_n^{(i)}(t)\rangle$

In the original formulation of the Razumov–Stroganov conjecture we have $|\tilde{\Psi}_n\rangle = \sum_{\pi} \tilde{\Psi}(\pi)|\pi\rangle$, satisfying $H_n|\tilde{\Psi}_n\rangle = 0$

The operators $RX_i(t)$, and the scattering matrices $S_i(t)$, induce the deformation

$$|\tilde{\Psi}_n^{(i)}(t)\rangle = \sum_{\pi} \tilde{\Psi}^{(i)}(t; \pi)|\pi\rangle, \text{ satisfying } (RX_i(t) - 1)|\tilde{\Psi}_n^{(i)}(t)\rangle = 0.$$

Because of a **dihedral covariance** of these equations, (and unicity of the Frobenius vector) it suffices to study $RX_1(t)$ and $|\tilde{\Psi}_n^{(1)}(t)\rangle$

$$\text{i.e., } 0 = (X_i(t) - R^{-1})|\tilde{\Psi}_n^{(i)}(t)\rangle = R(X_{i+1}(t) - R^{-1})R^{-1}|\tilde{\Psi}_n^{(i)}(t)\rangle \\ \text{implies } |\tilde{\Psi}_n^{(i+1)}(t)\rangle \propto R^{-1}|\tilde{\Psi}_n^{(i)}(t)\rangle$$

Call $\text{Sym} = N^{-1} \sum_{i=0}^{N-1} R^i$, the operator that projects on the rotationally-invariant subspace of $\mathbb{C}^{\text{LP}(N)}$.

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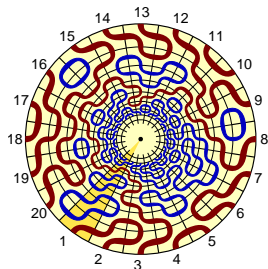
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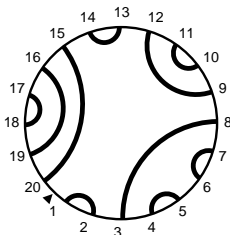
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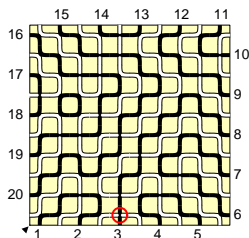
The refined Razumov–Stroganov correspondence



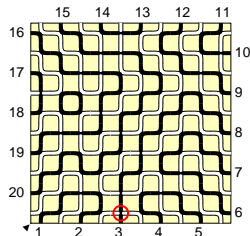
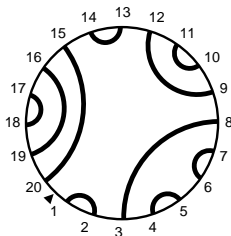
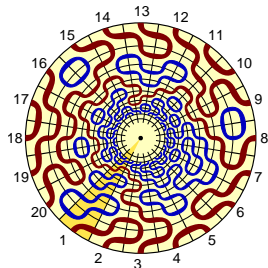
$\tilde{\Psi}_n(t; \pi)$: probability of π
in the $O(1)$ Dense Loop Model
with dynamics given by $RX_1(t)$



$\Psi_n(t; \pi)$: count FPL's ϕ
having link pattern π
give $t^{h(\phi)-1}$ weight



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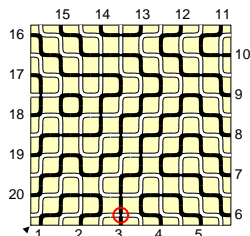
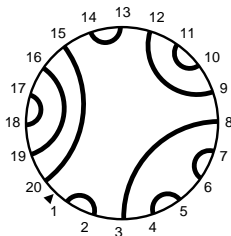
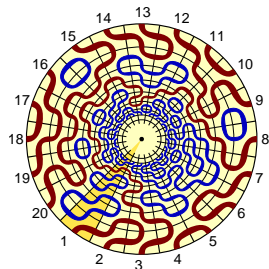
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(conjecture: Di Francesco, 2004; proof: AS Cantini, 2012)

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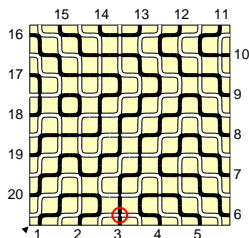
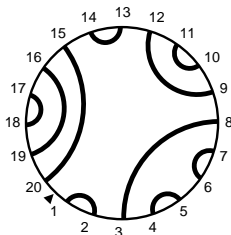
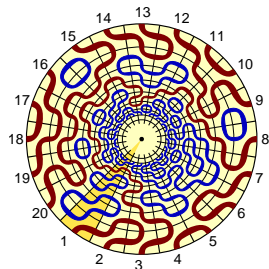
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A quest for a new strategy

In 2010, Cantini and myself gave a first proof of the (unrefined) Razumov–Stroganov conjecture. Later on, in 2012 we gave a proof for the refined Di Francesco conjecture, which also provides a *(more illuminating?)* proof of the original RS

- 2010:
- Realize that $H|\tilde{\Psi}\rangle = 0$ fixes $|\tilde{\Psi}\rangle$ univocally;
 - Prove combinatorially that also $|\Psi\rangle$ satisfies $H|\Psi\rangle = 0\dots$

...But the $|\tilde{\Psi}^{(i)}\rangle$'s **differ** (they are only **dihedrally covariant**), and satisfy **different** linear equations (with $RX_i(t)$)...

...and $\text{Sym } |\tilde{\Psi}^{(i)}\rangle$ does **not** satisfy any simple linear equation that fixes it univocally!

- 2012:
- Find a **new way** $\pi'(\phi)$ of associating link patterns to FPL;
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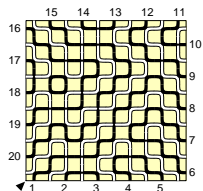
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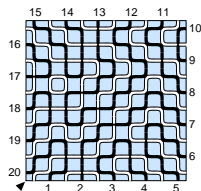
Bonus: The new enumeration is interesting by itself

The heretical enumeration



The role of black and white is symmetrical...

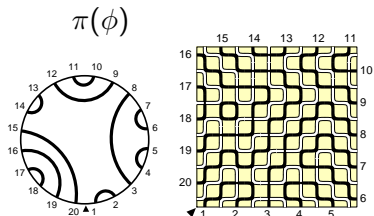
The heretical enumeration



...who's who is a matter of convention.

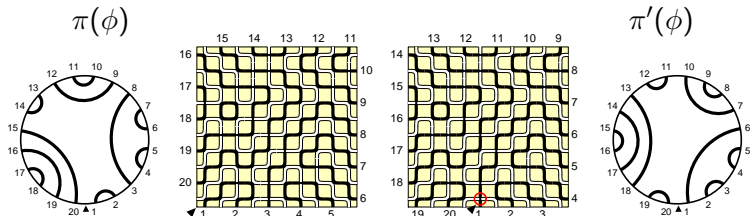
Swapping coloration in **all** FPL's leads to an equivalent conjecture

The heretical enumeration



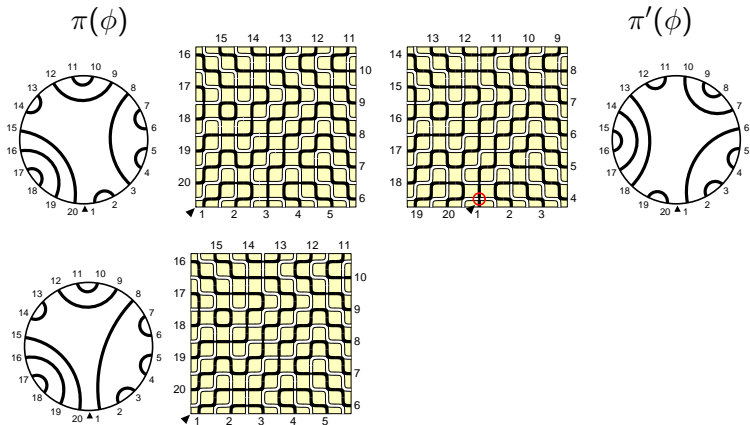
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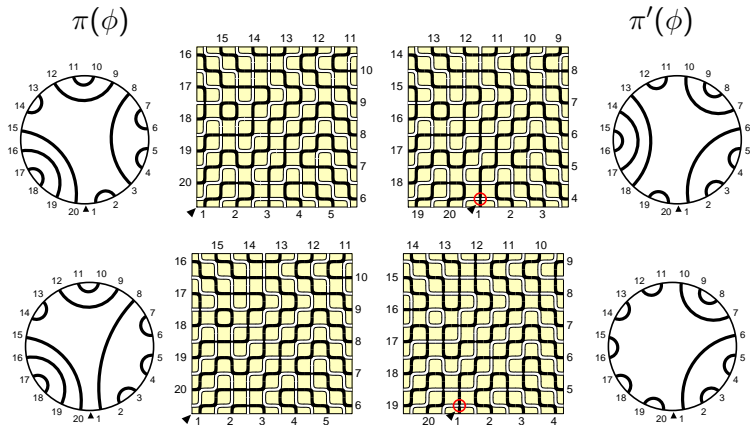
Here's the **new** rule: if the refinement position is **odd**...
...you just **rotate** the starting point to the refinement position

The heretical enumeration



if the refinement position is **even**...

The heretical enumeration



if the refinement position is **even**...

...you **swap** black and white, and **rotate** the starting point

Use projectors to get 2 simple equations (instead of 1 difficult eq.)

We wanted to prove Di Francesco 2004 conjecture:

$$\text{Sym} |\tilde{\Psi}(t)\rangle = \text{Sym} |\Psi(t)\rangle$$

with $|\tilde{\Psi}(t)\rangle$ solving $(X_1(t) - R^{-1})|\tilde{\Psi}(t)\rangle = 0$

$$\text{and } |\Psi(t)\rangle = \sum_{\phi} t^{h(\phi)-1} |\pi(\phi)\rangle$$

We have been led to split this in two parts:

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$$|\tilde{\Psi}(t)\rangle = |\Psi'(t)\rangle \quad \text{and} \quad \text{Sym} |\Psi'(t)\rangle = \text{Sym} |\Psi(t)\rangle$$

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The first relation is proven if you show that

$$(X_1(t) - R^{-1})|\Psi'(t)\rangle \equiv (t - R^{-1} - (t - 1)e_1)|\Psi'(t)\rangle = 0$$

recalling that $e_1^2 = e_1$, and $(1 - e_1)^2 = (1 - e_1)$:

Use projectors to get 2 simple equations (instead of 1 difficult eq.)

We wanted to prove Di Francesco 2004 conjecture:

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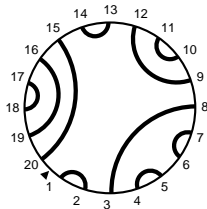
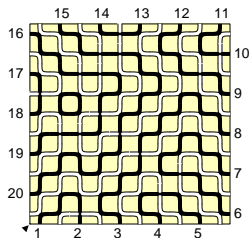
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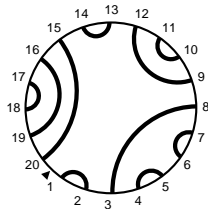
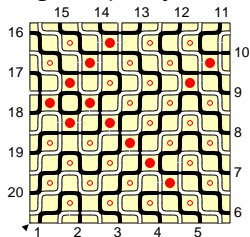
Wieland gyration: how it works

FPL config







Wieland gyration: how it works

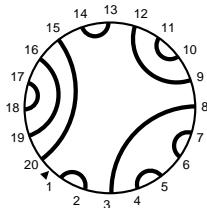
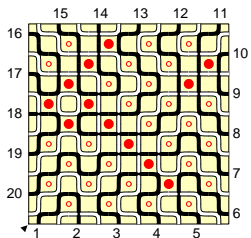
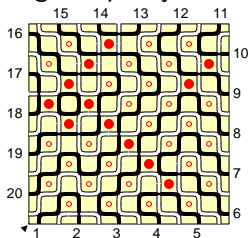
Mark faces  and ,
of given parity





Wieland gyration: how it works

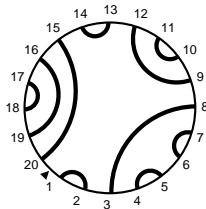
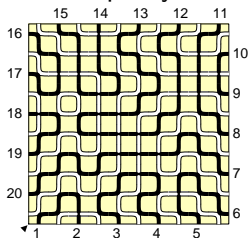
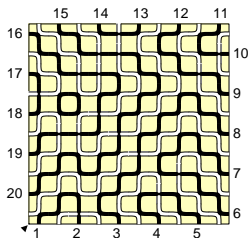
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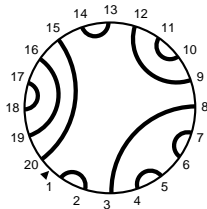
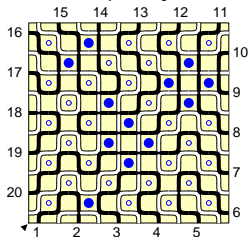
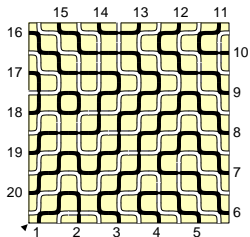
Wieland gyration: how it works

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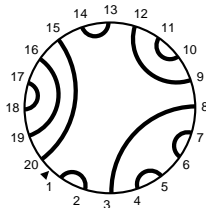
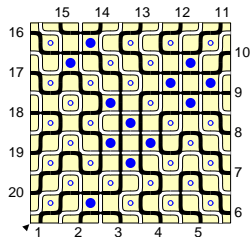
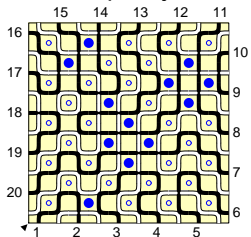
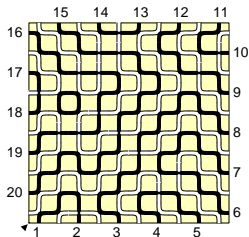
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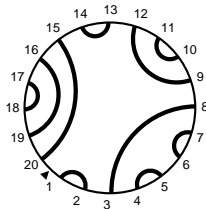
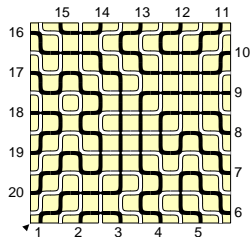
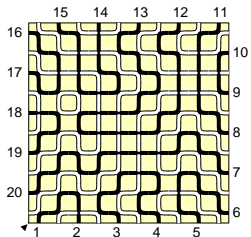
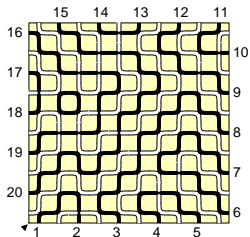


Wieland gyration: how it works

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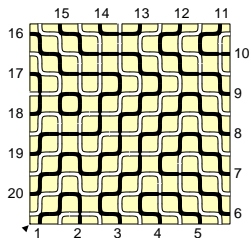


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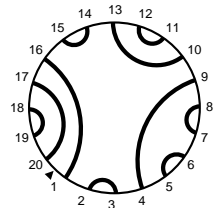
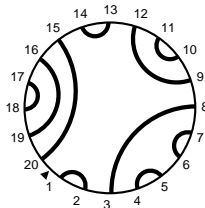
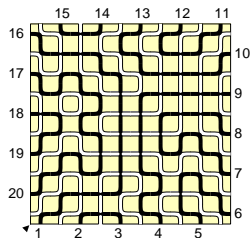
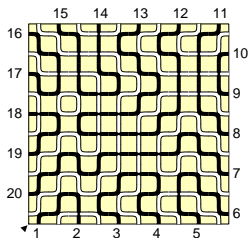


Wieland gyration: how it works

Link pattern $\pi\dots$

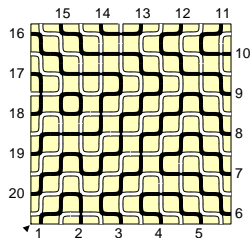


\dots and $R\pi\dots$

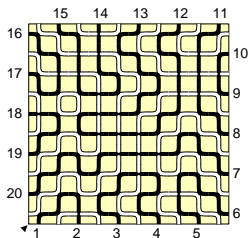


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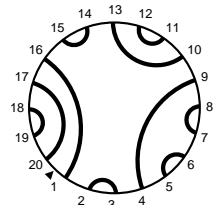
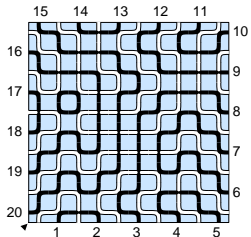
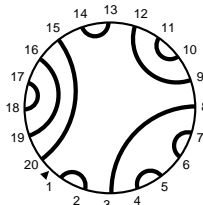
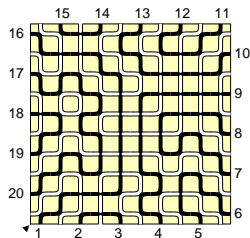
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...and, on the conjugate
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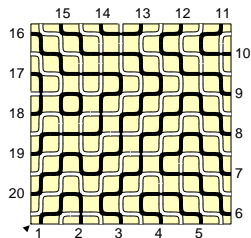


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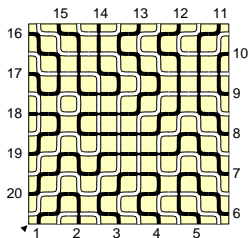


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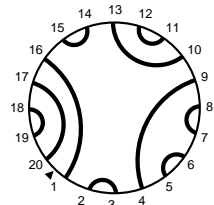
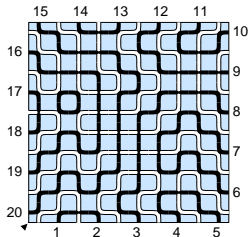
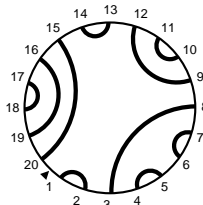
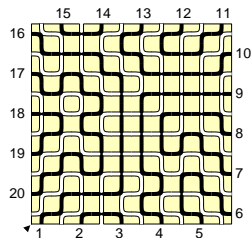
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$\dots R^{\frac{1}{2}} \pi \dots$

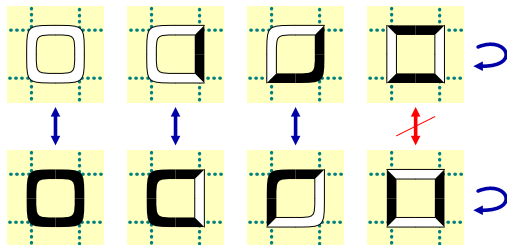


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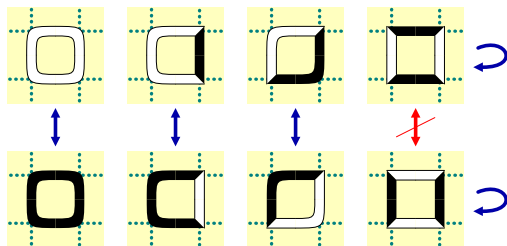
Wieland gyration: why it works

Easier to visualize the $\square \leftrightarrow \square$ exchange on the few \square , \square faces...
...but better use the conjugate config at intermediate step,
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Wieland gyration: why it works

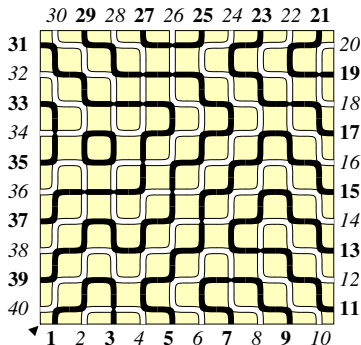
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This rule **inverts** $\deg_{\text{black}}(v) \leftrightarrow \deg_{\text{white}}(v)$,
and **preserves** connectivity of open-path endpoints

Wieland gyration: where it works

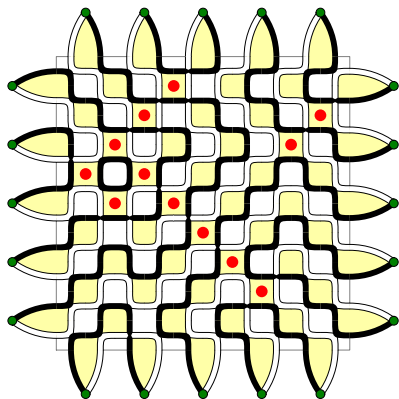
...in the original square domain for FPL we have “external legs” (i.e., vertices of degree 1)... if we **pair** them, to produce triangles, we solve this annoyance...



A configuration on (Λ, τ_+)
(i.e., first leg is black)

Wieland gyration: where it works

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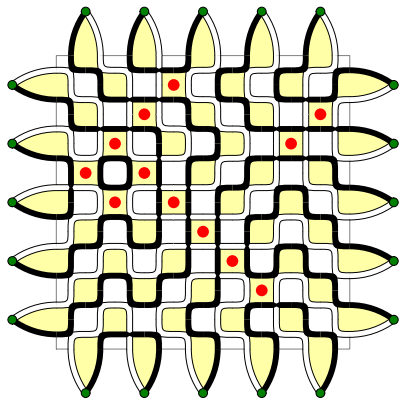


The construction of \mathcal{G}_+ ,
pairing $(2j - 1, 2j)$ legs
(plaquettes are in yellow)

mark in red  and 

Wieland gyration: where it works

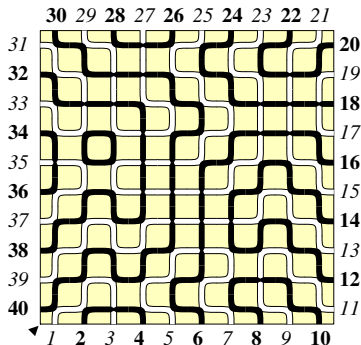
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The result of map H_+

Wieland gyration: where it works

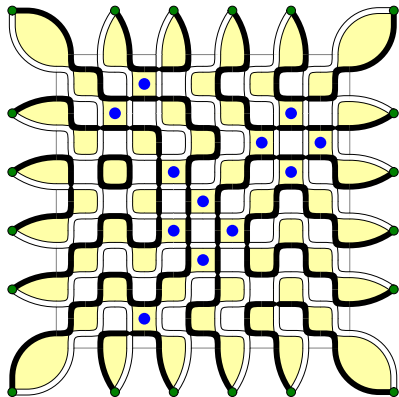
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Split auxiliary vertices
to recover the (Λ, τ_-)
geometry
(i.e., first leg is white)

Wieland gyration: where it works

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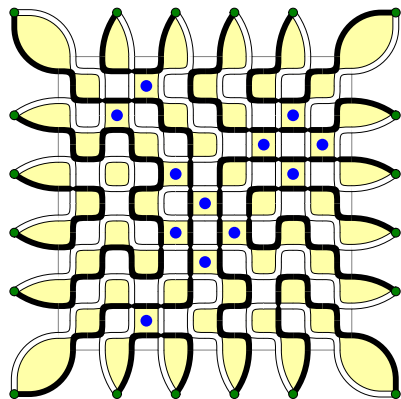


The construction of \mathcal{G}_- ,
pairing $(2j, 2j + 1)$ legs

mark in blue  and 

Wieland gyration: where it works

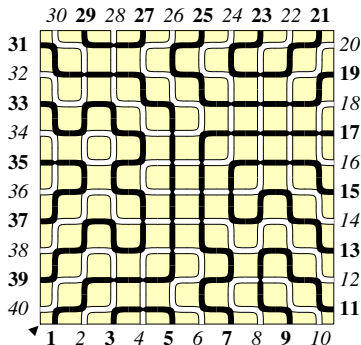
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Wieland gyration: where it works

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Split auxiliary vertices to recover the (Λ, τ_+) original geometry (with a rotated link pattern)...

Wieland gyration: where it works

So, the trick is:

- invert $\deg_{\text{black}}(v) \leftrightarrow \deg_{\text{white}}(v)$
- preserve connectivity of open paths

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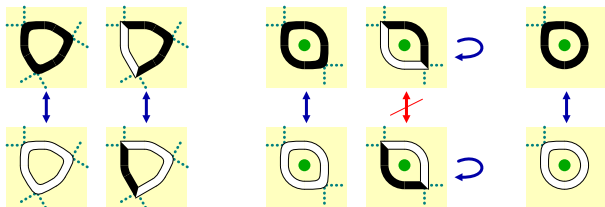
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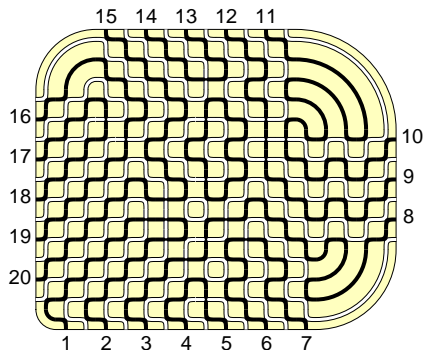
A **single** move exists on plenty of graphs...

then, **rotation** comes from **two** moves

...many more domains than just $n \times n$ squares have this property!

Wieland gyration: where it works

We can trade corners for points of curvature (i.e., faces with less than 4 sides). But we need at least one corner, because closed spectral lines have a trivial behaviour ($1 + R$)



(bottom line: an **elementary** generalization of Wieland strategy gives **rotational symmetry** for FPL enumerations above)

Yet one word on gyration... the boundary conditions

We have seen how to generalise the **domain**,
using black/white alternating boundary conditions

What does it happen if we generalise on **boundary conditions**?

Pairing consecutive legs with the same colour produces arcs,
and “**loses link-pattern information**”: gyration holds for
linear combinations of $\Psi(\pi)$, instead of component-wise.

These linear combinations, induced by arcs, are well-described by
Temperley-Lieb operators.

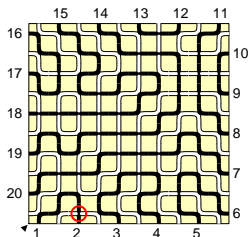
We will not need this in full generality. . .

the study of **a single defect** is sufficient at our purposes.

Alternating boundary conditions, with one defect

Example: the state $|\Psi^{[j]}\rangle = \sum_{\phi: h(\phi)=j} |\pi'(\phi)\rangle$ satisfies

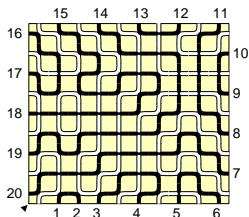
$$(R e_{j-1} - e_j) |\Psi^{[j]}\rangle = 0$$



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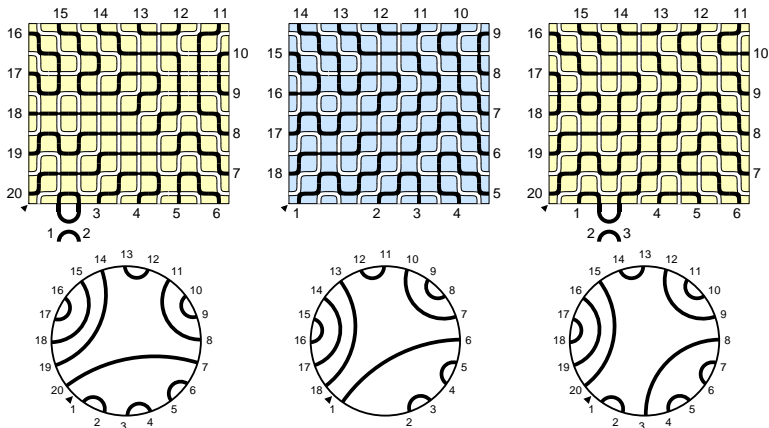
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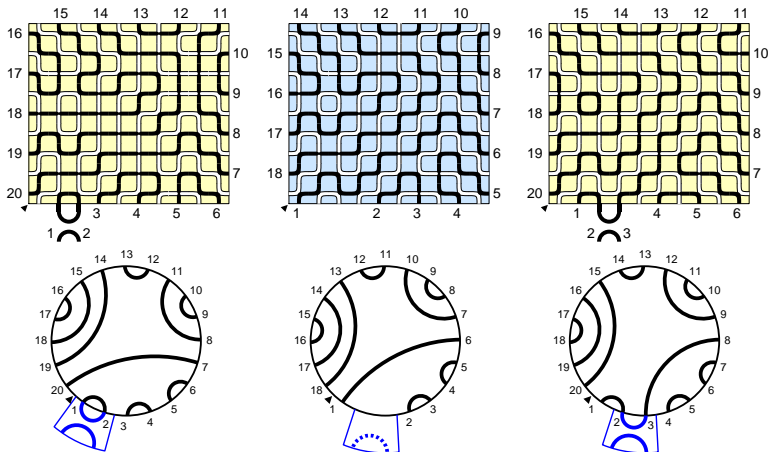
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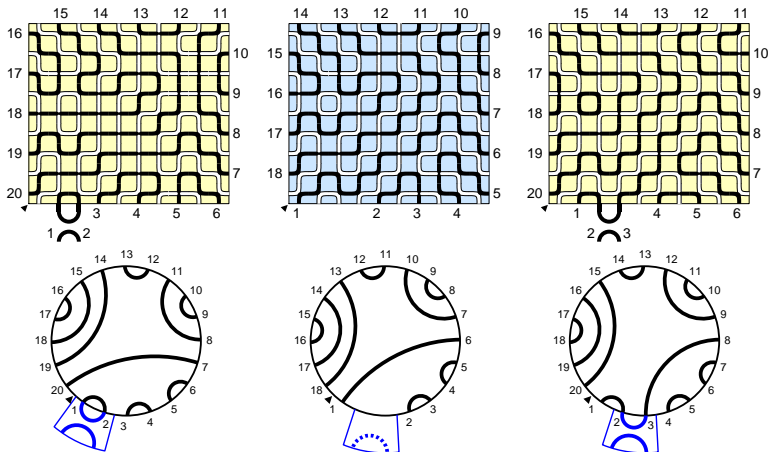
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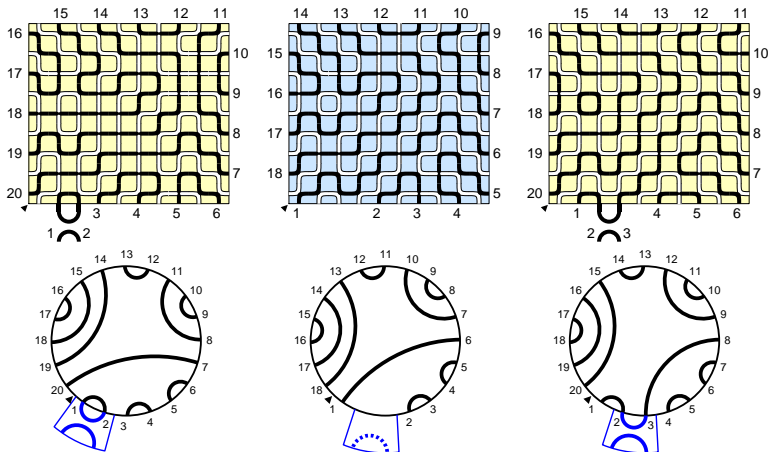
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 $(e_j R - e_j) |\Psi^{[j]}\rangle = 0$



Alternating boundary conditions, with one defect

Example: the state $|\Psi^{[l]}\rangle = \sum_{\phi: h(\phi)=j} |\pi'(\phi)\rangle$ satisfies

$$Re_j(1 - R^{-1})|\Psi^{[l]}\rangle = 0$$



A first consequence

Recall our checklist of identities:

$$(1) : e_1 (1 - R^{-1})|\Psi'(t)\rangle = 0$$

$$(2) : (1 - e_1) (t - R^{-1})|\Psi'(t)\rangle = 0$$

$$(3) : \text{Sym } |\Psi'(t)\rangle = \text{Sym } |\Psi(t)\rangle$$

(2) is equivalent to ask that $t\Psi(t; \pi) = \Psi(t; R^{-1}\pi)$,
for all π such that $1 \approx 2$...

but this is easily seen: $1 \approx 2$ forces a small region,
that in turns implies a simple behaviour of the refinement position
under Wieland gyration

A first consequence

Recall our checklist of identities:

(1) : $e_1 (1 - R^{-1})|\Psi'(t)\rangle = 0$ ✓ We have just proven this!

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The final orbit lemma

Consider the **orbits** under Wieland half-gyration

As FPL in the same orbit have the same link pattern up to rotation,

$\text{Sym} |\Psi'(t)\rangle = \text{Sym} |\Psi(t)\rangle$ follows if, for every j , and every orbit, there are as many contributions t^{j-1} to $|\Psi'(t)\rangle$ as to $|\Psi(t)\rangle$.

Study the behavior of the **trajectory** $h(x)$ of the refinement position:

- ▶ $h(x+1) - h(x) \in \{0, \pm 1\}$
- ▶ In a periodic function, any height value is attained alternately on ascending and descending portions (or maxima/minima)
- ▶ All maxima/minima plateaux have length 2, the slope is ± 1 elsewhere
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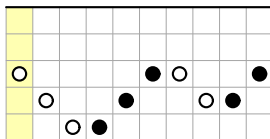
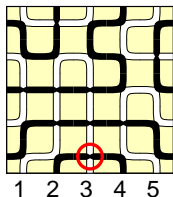
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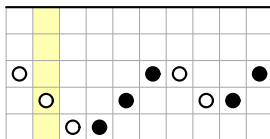
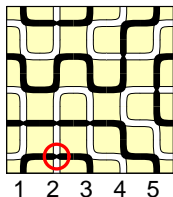


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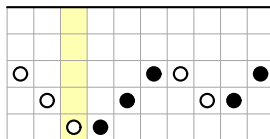
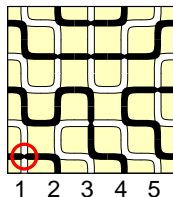


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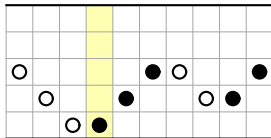
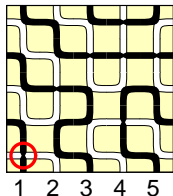


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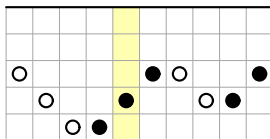
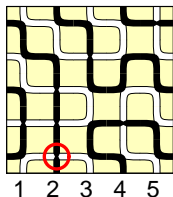


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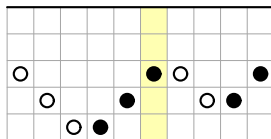
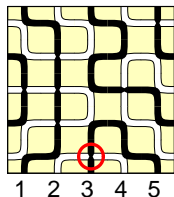


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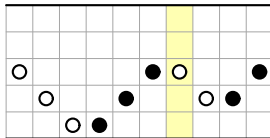
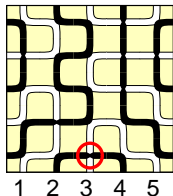


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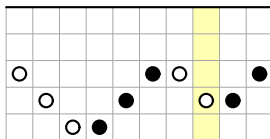
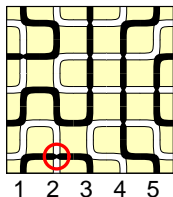


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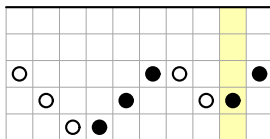
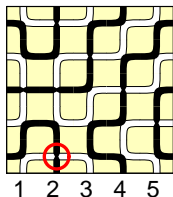


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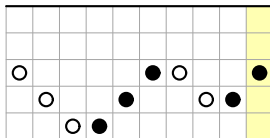
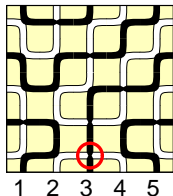


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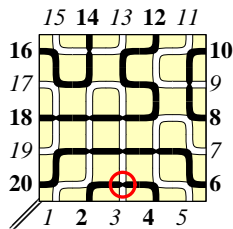
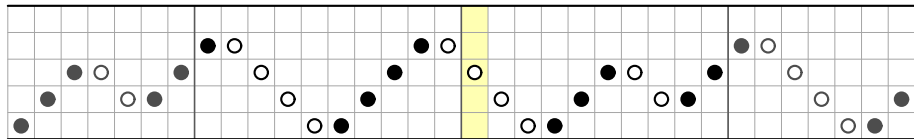
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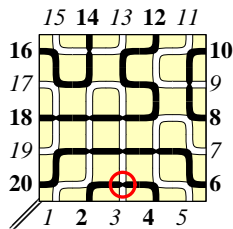
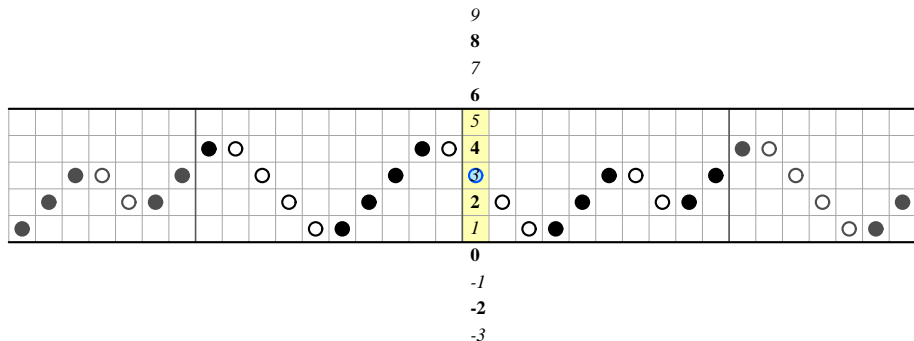


A bijective version of the last lemma



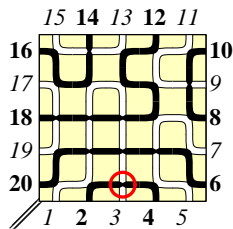
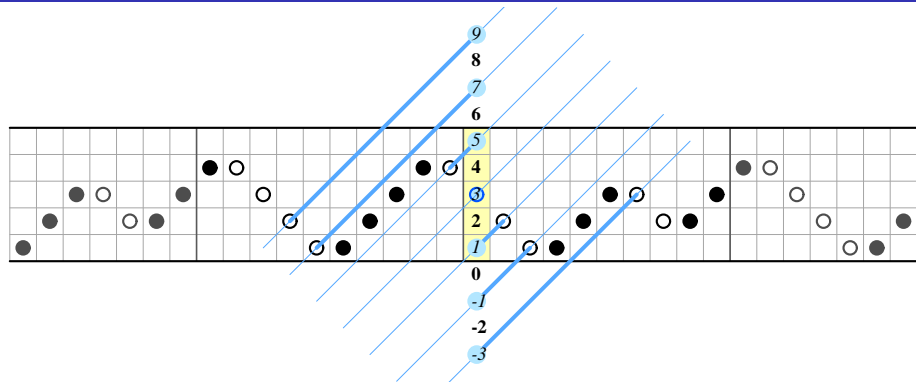
The structure of the orbits gives
a **bijection factory**...

A bijective version of the last lemma



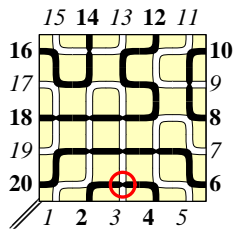
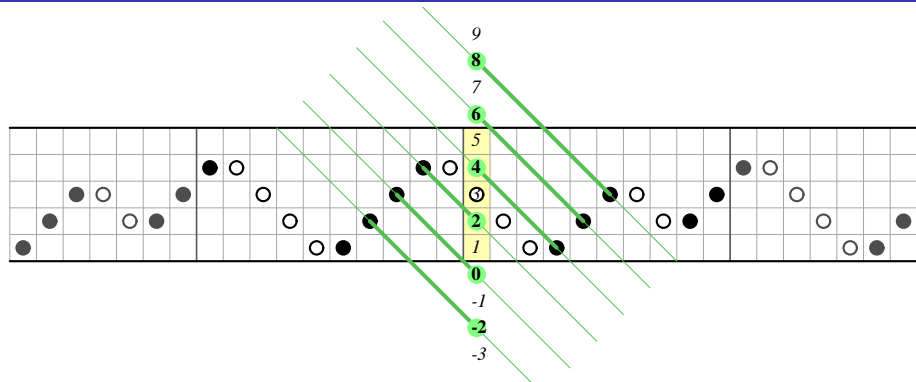
Consider the infinite orbit, and extend ordinates from $\{1, \dots, L\}$ to $\mathbb{Z} \dots$

A bijective version of the last lemma



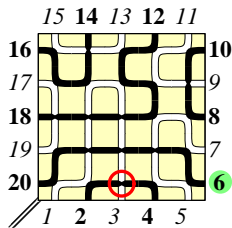
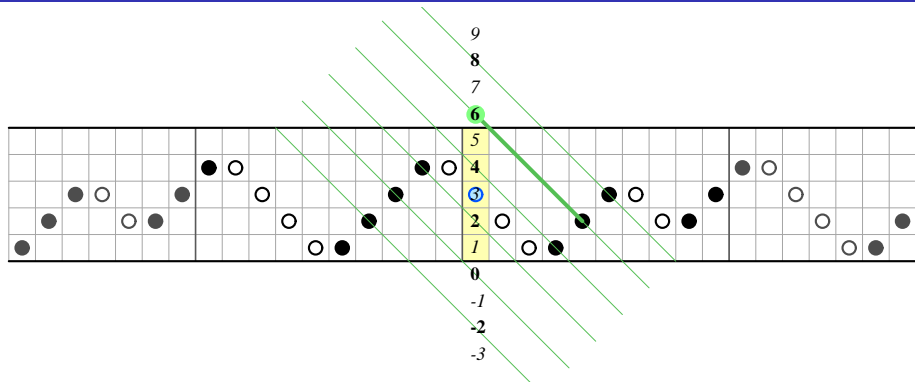
All 45-degree diagonals with odd ordinate have a **unique** intersection with the orbit, and it is **white**

A bijective version of the last lemma



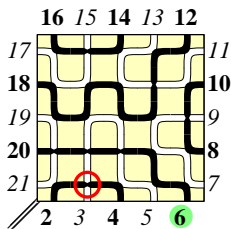
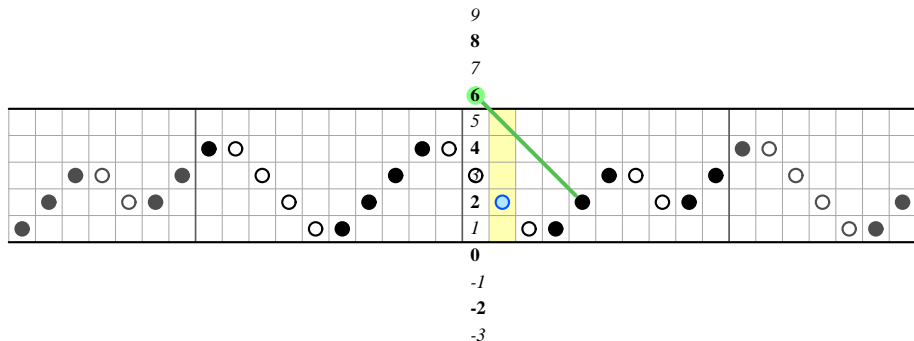
All 135-degree diagonals with even ordinate have a **unique** intersection with the orbit, and it is **black**

A bijective version of the last lemma



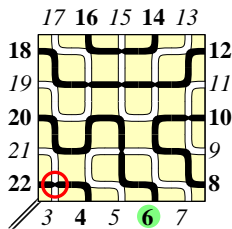
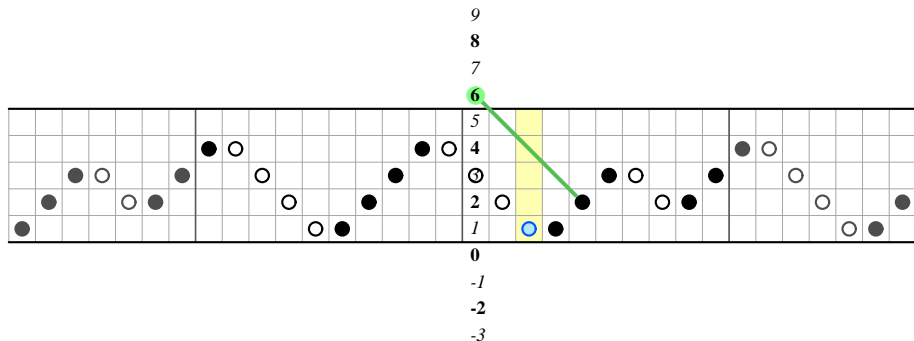
This implies that, for any integer $c \in \mathbb{Z}$, there is a **unique** config in the infinite orbit with refinement position on leg c !

A bijective version of the last lemma



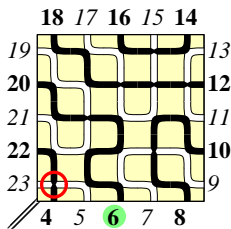
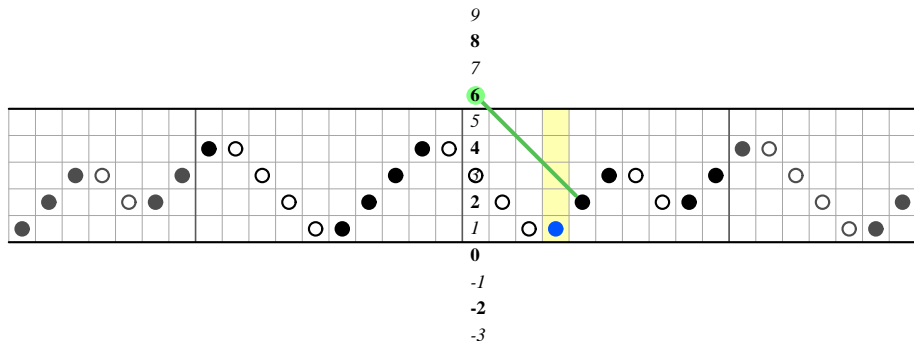
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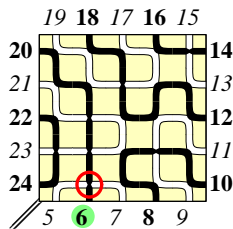
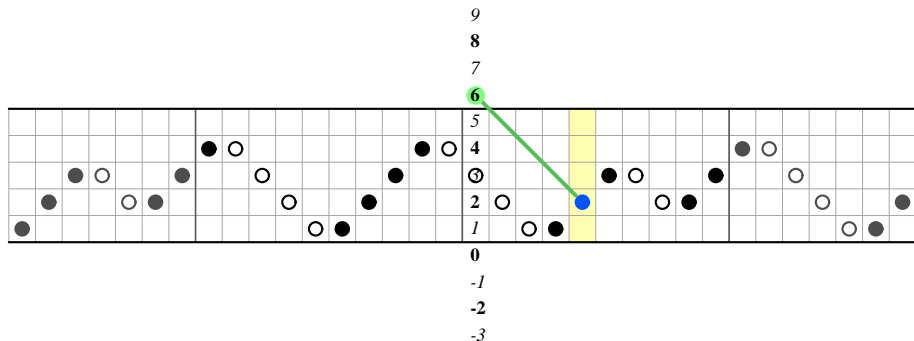
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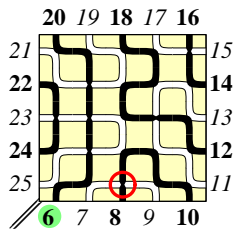
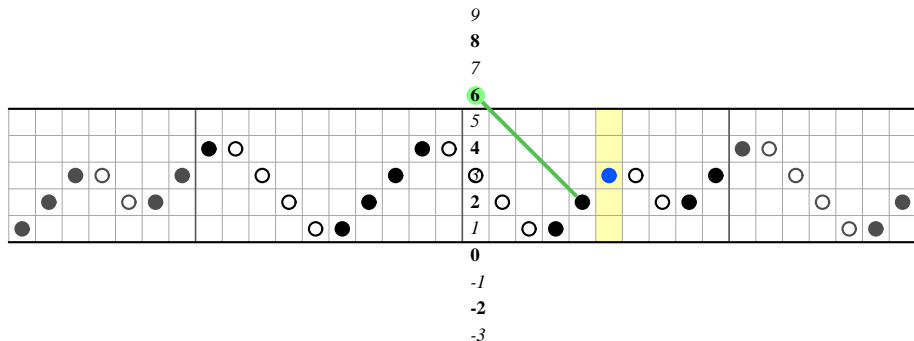
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The RS obsession: why no black+white RS?

Last June there was the “*At the crossroads of physics and mathematics: the joy of integrable combinatorics — A conference in the honor of Philippe Di Francesco’s 60th birthday*” in IPTTh I was presenting my *other* results, on the structure constants of the canonical Grothendieck poly’s popping out in the $(\pi_{\bullet}, \pi_{\circ}, \#\{\circ\})$ statistics of FPL’s in the VSASM’s.

A point of my talk was the disappointment for the fact that we only have a (dihedral) RS correspondence for the π_{\bullet} statistics, despite the fact that Wieland gyration is a statement on the triple $(\pi_{\bullet}, \pi_{\circ}, \#\{\circ\})$.

Since then, I have been thinking back to the question. And I found a sensible recipe, *and* a further generalisation...

...at this aim, we need to introduce a **gauge theory for FPL’s**

Colour-changing cuts in FPL's

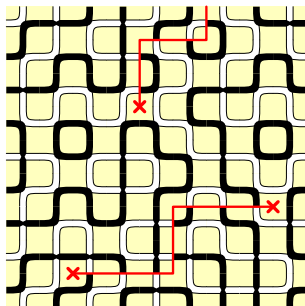
Let G be a 4-valent planar graph, and consider FPL on G

Let \mathcal{C} be a subset of edges of the dual G^* (the *cuts*)

Our FPL paths **change colour** when they cross \mathcal{C}

The set of FPL's is defined up to a gauge transformation of \mathcal{C} , that is, the lattice version of local deformation of cuts, while keeping the endpoint fixed.

In particular, the enumeration according to the link pattern is invariant under cut deformations.



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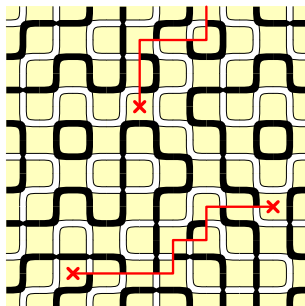
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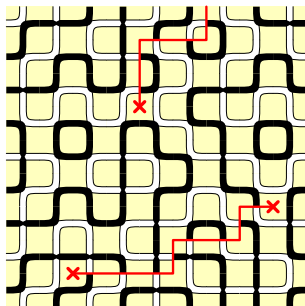
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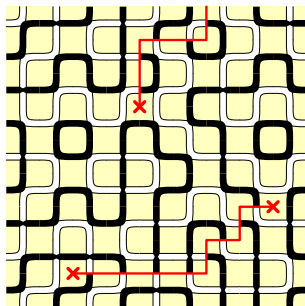
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The forgotten dihedral domains

Above, we classified the domains allowing for the (dihedral, π_{\bullet}) Razumov–Stroganov correspondence. None of them involved cuts (as on such graphs we find paths that “start black and finish white”, and we wouldn’t know what to do with them. . .)

If we had an idea on how to construct a bicoloured ($\pi_{\bullet}, \pi_{\circ}$) Razumov–Stroganov correspondence, nothing would prevent in principle to consider also domains with cuts.

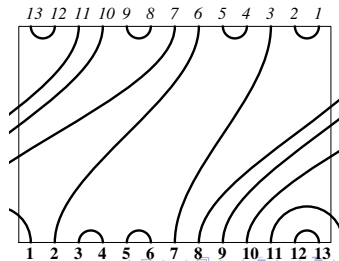
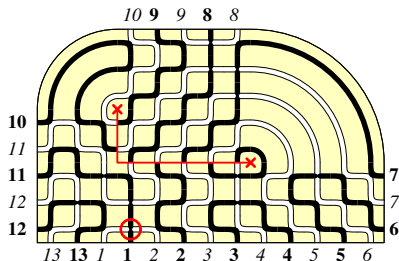
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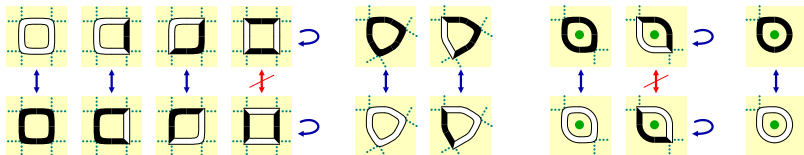
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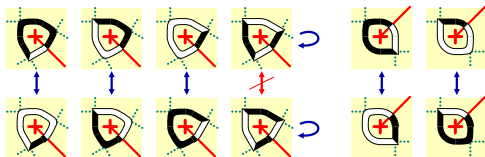
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The forgotten plaquettes



We said that: **①** squares barely work (must not swap if B/W/B/W) **②** triangles and lower work easily **③** 2-gons and 1-gons work so well that you can even put a puncture in them.



Now, with cut endpoints: **①** triangles barely work (must not swap if B/W/B/W) **②** 2-gons and lower work easily **③** no room for a puncture anymore.

Reverse-engineering RS from FPL towards a loop model

We want to “invent” a black+white Razumov–Stroganov correspondence

But are disappointed to find no known good candidate on the DLM side (the “rotor model” would have been promising, if it weren't for certain incompatibility issues on known enumerations)

So, we try to build on what we already know, from the FPL side, and hope to interpret *a posteriori* what we find as a loop model.

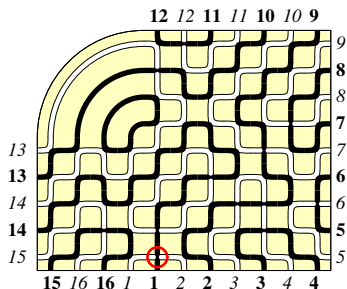
Our best starting point is the Di Francesco 2004 (ex-)conjecture, i.e. our proof that the [heretical-enumeration-vector for FPL](#) $|\Psi'(t)\rangle$ satisfies

$$\begin{aligned}e_1^\bullet (1 - R_\bullet^{-1})|\Psi'(t)\rangle &= 0 \\(1 - e_1^\bullet) (t - R_\bullet^{-1})|\Psi'(t)\rangle &= 0\end{aligned}$$

The black+white equations

So, we consider the black+white heretical enumeration $|\Psi_{\text{bw}}(t)\rangle$, e.g. with the convention that:

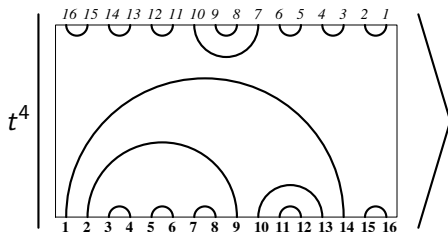
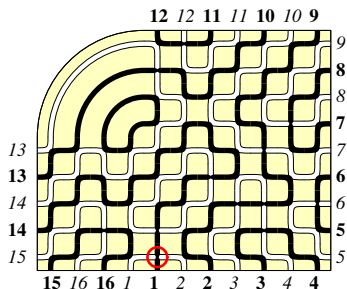
- ▶ the refinement is always on a black leg
- ▶ the refinement leg has black index **1**
- ▶ the black indices grow CCW
- ▶ the white leg left of **1** has white index **1**
- ▶ the white indices grow CCW



The black+white equations

So, we consider the black+white heretical enumeration $|\Psi_{\text{bw}}(t)\rangle$, e.g. with the convention that:

- ▶ the refinement is always on a black leg
- ▶ the refinement leg has black index **①**
- ▶ the black indices grow CCW
- ▶ the white leg left of **①** has white index **①**
- ▶ the white indices grow CCW



The black+white equations

We can work out lemmas as in our previous proof, but now keeping track of both black and white legs. It is more complicated but similar in spirit. We find equations depending only on e_1^\bullet , e_1° , R_\bullet and R_\circ , namely: (note: these equations do not have a unique solution!)

$$(1 - e_1^\circ) |\Psi_{\text{bw}}(t)\rangle = 0$$

$$(1 - e_1^\bullet) (t - R_\bullet^{-1}) |\Psi_{\text{bw}}(t)\rangle = 0$$

$$e_1^\bullet e_1^\circ \left(1 - \frac{1}{1 - tR_\circ(1 - e_1^\circ)} R_\circ R_\bullet^{-1} \right) |\Psi_{\text{bw}}(t)\rangle = 0$$

These are consistent with our previous equations. Indeed, the first one states that the refinement leg is black, and the other two, under the identification $R_\circ \rightarrow 1$, $e_1^\circ \rightarrow 1$, become

$$e_1^\bullet (1 - R_\bullet^{-1}) |\Psi'(t)\rangle = 0$$

$$(1 - e_1^\bullet) (t - R_\bullet^{-1}) |\Psi'(t)\rangle = 0$$

First nice surprise: a hidden symmetry

The heretical enumeration breaks the symmetry between black and white.

Define the combination

$$|\Phi(t)\rangle = \frac{R_{\bullet}^{-1}}{1 - t(1 - e_1^{\circ})R_{\circ}} |\Psi_{\text{bw}}(t)\rangle$$

Our equations above read

$$(1 - e_1^{\circ})(1 - tR_{\circ})|\Phi(t)\rangle = 0$$

$$(1 - e_1^{\bullet})(1 - tR_{\bullet})|\Phi(t)\rangle = 0$$

$$e_1^{\bullet}e_1^{\circ}(R_{\circ} - R_{\bullet})|\Phi(t)\rangle = 0$$

which are both very compact, and symmetric in black and white.

Second nice surprise: separation of TL_{\bullet} and TL_{\circ}

In terms of the vector $|\Psi'_{\text{bw}}(t)\rangle = R_{\bullet}^{-1}|\Psi_{\text{bw}}(t)\rangle$, we can write our equations as

$$\begin{aligned}(1 - e_1^{\circ})|\Psi'_{\text{bw}}(t)\rangle &= 0 \\(1 - e_1^{\bullet})(1 - tR_{\bullet})|\Psi'_{\text{bw}}(t)\rangle &= 0 \\ \left(e_1^{\circ} \frac{1}{1 - tR_{\circ}} - \frac{1}{1 - tR_{\bullet}} e_1^{\bullet} \right) |\Psi'_{\text{bw}}(t)\rangle &= 0\end{aligned}$$

These equations can be interpreted as if TL_{\bullet} and TL_{\circ} act on the bottom and top sides of a cylinder, with the stochastic operator:

- ▶ apply e_1
- ▶ apply R^{ℓ} , with probability $p(\ell) = (1 - t)t^{\ell}$

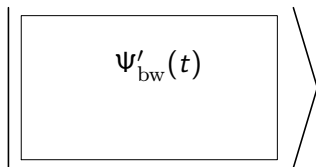
and its adjoint (that is, the two steps are performed in reverse order)

Second nice surprise: separation of TL_{\bullet} and TL_{\circ}

$$(1 - e_1^{\circ})|\Psi'_{\text{bw}}(t)\rangle = 0$$

$$(1 - e_1^{\bullet})(1 - tR_{\bullet})|\Psi'_{\text{bw}}(t)\rangle = 0$$

$$\left(e_1^{\circ} \frac{1}{1 - tR_{\circ}} - \frac{1}{1 - tR_{\bullet}} e_1^{\bullet} \right) |\Psi'_{\text{bw}}(t)\rangle = 0$$

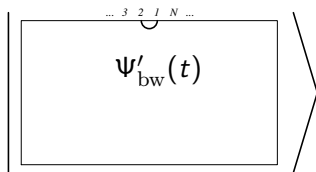


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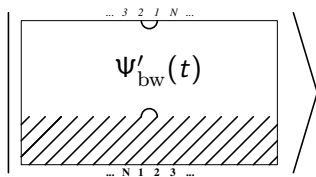


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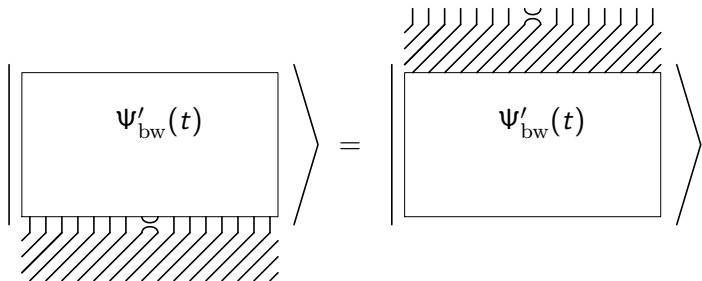


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Third nice surprise: FPL domains with a cut as $T(t)$

Let us try to invent a loop model with properties above

It must contain loop diagrams, for the Temperley–Lieb action

It must also contain a “mark” from where to start counting

We will interpret the mark as a particle like in a 6VM in NILP representation, **in the sector in which there is a single particle**

A set of tiles with these properties is a 5VM mixed to a $O(1)$ DLM



Nice surprise: **the Transfer Matrix of this model is the partition function for FPL in the simplest cut domain** (two rows)

More general cut domains can also be interpreted as Transfer Matrices, but with more complicated (horizontal) auxiliary space, and more complicated tiles (as for a “higher spin” line in the 6VM)

Third nice surprise: FPL domains with a cut as $T(t)$

