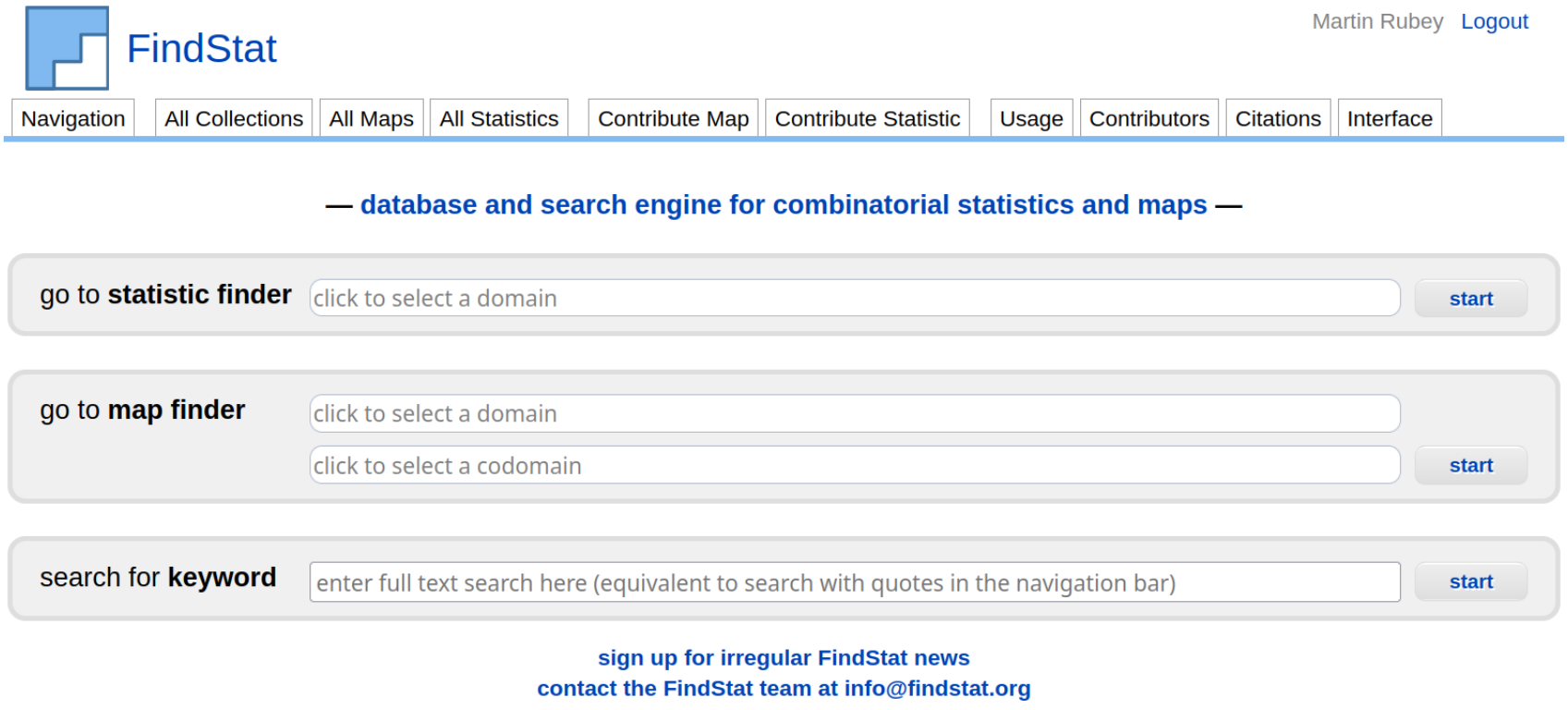
An unexpected symmetry on graphs

Florian Fürnsinn, Moritz Gangl and Martin Rubey

Universität & Technische Universität Wien

Bordeaux 2025

www.findstat.org



www.findstat.org

Let $\mathcal{C}_n$ be a finite collection of objects, e.g.

- permutations in $\mathfrak{S}_n$,
- Dyck paths of semilength n,
- (simple, unlabelled) graphs with n vertices,...

A *statistic* is a function $s : \mathcal{C}_n \to \mathbb{Z}$, its *distribution* is $\sum_{c \in \mathcal{C}_n} q^{s(c)}$.

www.findstat.org

Let $\mathcal{C}_n$ be a finite collection of objects, e.g.

- permutations in $\mathfrak{S}_n$,
- Dyck paths of semilength n,
- (simple, unlabelled) graphs with n vertices,...

A *statistic* is a function $s : \mathcal{C}_n \to \mathbb{Z}$, its *distribution* is $\sum_{c \in \mathcal{C}_n} q^{s(c)}$.

In www.findstat.org, we have

Permutations	$\sim$ 400 statistics	18 bijections
Dyck paths	$\sim$ 300 statistics	24 bijections
...		

www.findstat.org

Let $\mathcal{C}_n$ be a finite collection of objects, e.g.

- permutations in $\mathfrak{S}_n$,
- Dyck paths of semilength n,
- (simple, unlabelled) graphs with n vertices,...

A *statistic* is a function $s : \mathcal{C}_n \to \mathbb{Z}$, its *distribution* is $\sum_{c \in \mathcal{C}_n} q^{s(c)}$.

In www.findstat.org, we have

Permutations	~ 400 statistics	18 bijections
Dyck paths	~ 300 statistics	24 bijections
...		
(simple, unlabelled) graphs	~ 300 statistics	2 bijections

www.findstat.org

A *statistic* is a function $s : \mathcal{C}_n \to \mathbb{Z}$, its *distribution* is $\sum_{c \in \mathcal{C}_n} q^{s(c)}$.

If s_1 and s_2 are equidistributed, $s_1 \sim s_2$, there must be a bijection

$$\phi : \mathcal{C}_n \to \mathcal{C}_n$$

$$s_1\big(\phi(e)\big) = s_2(e).$$

www.findstat.org

A *statistic* is a function $s : \mathcal{C}_n \to \mathbb{Z}$, its *distribution* is $\sum_{c \in \mathcal{C}_n} q^{s(c)}$.
If s_1 and s_2 are equidistributed, $s_1 \sim s_2$, there must be a bijection

$$\phi : \mathcal{C}_n \to \mathcal{C}_n$$
$$s_1\big(\phi(e)\big) = s_2(e).$$

Many statistics on Dyck paths or permutations have the same distribution, e.g.

Permutations	$maj \sim inv$
Dyck paths	$height \sim bounce\ count$

No such (non-trivial) equidistributions were known for graphs.

www.findstat.org

A *statistic* is a function $s : \mathcal{C}_n \to \mathbb{Z}$, its *distribution* is $\sum_{c \in \mathcal{C}_n} q^{s(c)}$.
If s_1 and s_2 are equidistributed, $s_1 \sim s_2$, there must be a bijection

$$\phi : \mathcal{C}_n \to \mathcal{C}_n$$
$$s_1\big(\phi(e)\big) = s_2(e).$$

Many statistics on Dyck paths or permutations have the same distribution, e.g.

Permutations	$maj \sim inv$
Dyck paths	$height \sim bounce\ count$

No such (non-trivial) equidistributions were known for graphs.
However, FindStat found

#graphs without leaves = #graphs with distinct neighbourhoods

mating

Theorem (Kilibarda 2007, Gessel & Li 2011).

#graphs without leaves = #graphs with distinct neighbourhoods

mating

Theorem (Kilibarda 2007, Gessel & Li 2011).

#graphs without leaves = #graphs with distinct neighbourhoods

Theorem (Read 1989). The generating function for graphs with distinct neighbourhoods is

$$\sum_{\lambda} \frac{2^{X_\lambda}}{z_\lambda} x^{|\lambda|} \prod_i (1 - x^{\lambda_i}),$$

where $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \dots$ for $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ and $X_\lambda = \sum_{i<j} \gcd(\lambda_i, \lambda_j) + \sum_i \left\lfloor \frac{\lambda_i}{2} \right\rfloor$.

mating

Definition (Sumner 1973, Bull & Pease 1989).
The *neighbourhood of u* is $\mathrm{N}(u) = \{v \mid (u, v) \in E(G)\}$,
the *closed neighbourhood of u* is $\overline{\mathrm{N}}(u) = \{u\} \cup \mathrm{N}(u)$.

mating

Definition (Sumner 1973, Bull & Pease 1989).
The *neighbourhood of u* is $\mathrm{N}(u) = \{v \mid (u, v) \in E(G)\}$,
the *closed neighbourhood of u* is $\overline{\mathrm{N}}(u) = \{u\} \cup \mathrm{N}(u)$.
A graph is *point determining* or *mating*, if $\mathrm{N}(u) \neq \mathrm{N}(v)$,
it is *point distinguishing* or *co-mating*, if $\overline{\mathrm{N}}(u) \neq \overline{\mathrm{N}}(v)$,
for any pair of vertices $u \neq v$.

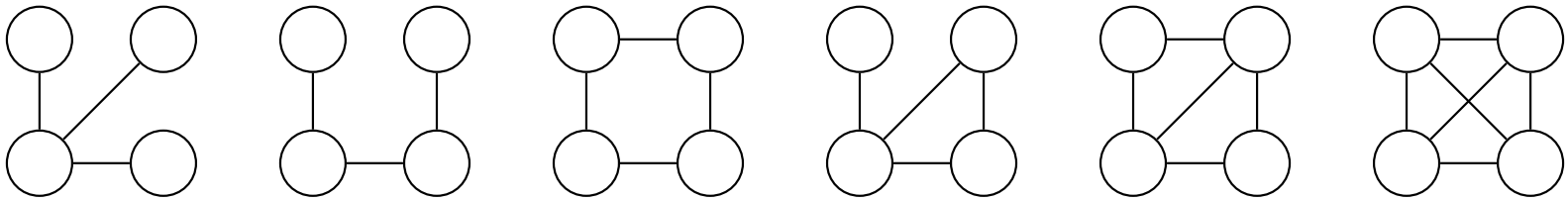
Theorem (Kilibarda 2007, Gessel & Li 2011).

#graphs without leaves = #mating graphs = #co-mating graphs

tufts and siblings

Definition (Alexandersson & R. 2022). The *tuft number* $t(G)$ is the maximal number of leaves adjacent to the same vertex.

a *tuft*:

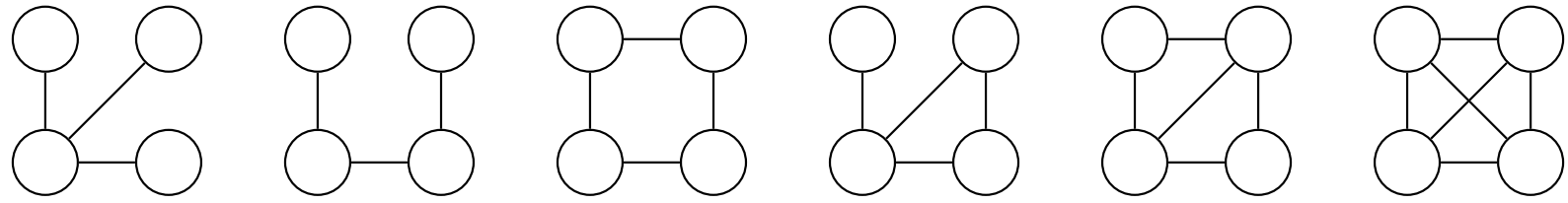


$t(G) =$ $t(G) =$ $t(G) =$ $t(G) =$ $t(G) =$ $t(G) =$

tufts and siblings

Definition (Alexandersson & R. 2022). The *tuft number* $t(G)$ is the maximal number of leaves adjacent to the same vertex.

a *tuft*:

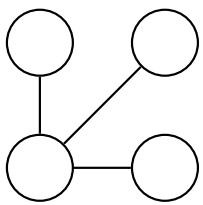


$t(G) = 3$ $t(G) = 1$ $t(G) = 0$ $t(G) = 1$ $t(G) = 0$ $t(G) = 0$

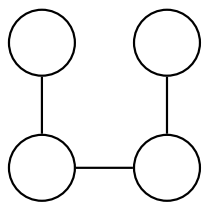
tufts and siblings

Definition (Alexandersson & R. 2022). The *tuft number* $t(G)$ is the maximal number of leaves adjacent to the same vertex.
The *sibling number* is the maximal number of siblings of a vertex:

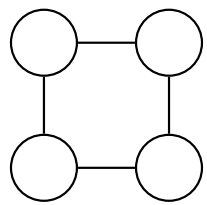
$$s(G) = \max_{v \in V(G)} \left| \{ u \in V(G) \mid \overline{\mathrm{N}}(u) = \overline{\mathrm{N}}(v) \} \right| - 1.$$



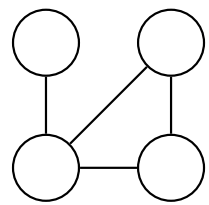
$s(G) =$
$t(G) = 3$



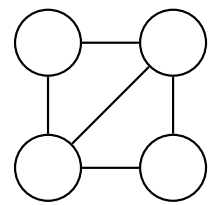
$s(G) =$
$t(G) = 1$



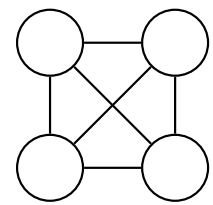
$s(G) =$
$t(G) = 0$



$s(G) =$
$t(G) = 1$



$s(G) =$
$t(G) = 0$

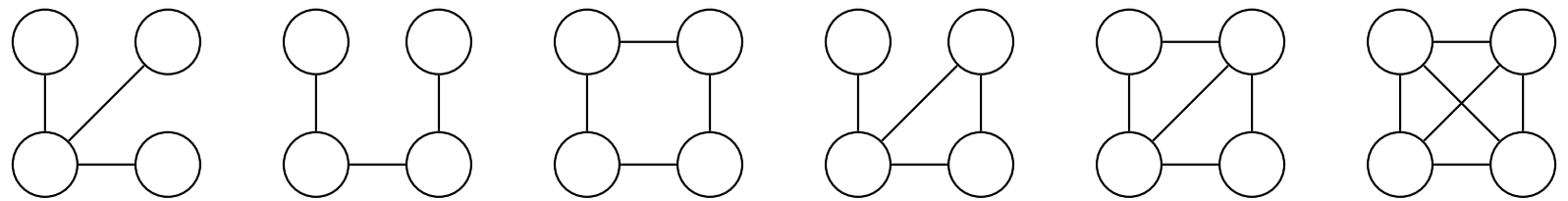


$s(G) =$
$t(G) = 0$

tufts and siblings

Definition (Alexandersson & R. 2022). The *tuft number* $t(G)$ is the maximal number of leaves adjacent to the same vertex.
The *sibling number* is the maximal number of siblings of a vertex:

$$s(G) = \max_{v \in V(G)} \left| \{ u \in V(G) \mid \overline{\mathrm{N}}(u) = \overline{\mathrm{N}}(v) \} \right| - 1.$$



$s(G) = 0$	$s(G) = 0$	$s(G) = 0$	$s(G) = 1$	$s(G) = 1$	$s(G) = 3$
$t(G) = 3$	$t(G) = 1$	$t(G) = 0$	$t(G) = 1$	$t(G) = 0$	$t(G) = 0$

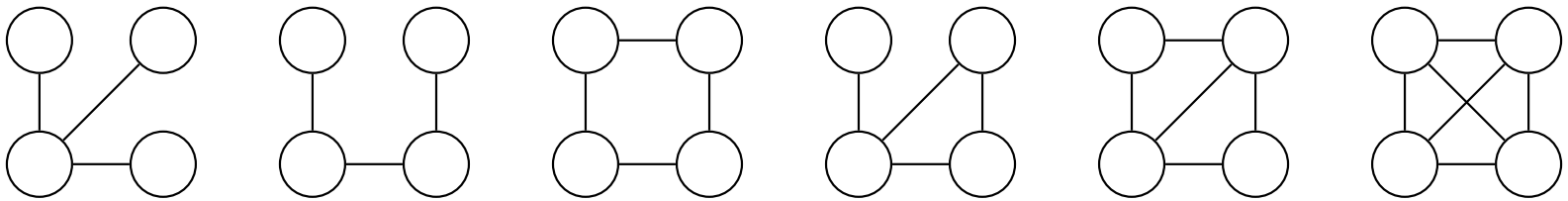
tufts and siblings

Definition (Alexandersson & R. 2022). The *tuft number* $t(G)$ is the maximal number of leaves adjacent to the same vertex.
The *sibling number* is the maximal number of siblings of a vertex:

$$s(G) = \max_{v \in V(G)} |\{u \in V(G) \mid \overline{\mathrm{N}}(u) = \overline{\mathrm{N}}(v)\}| - 1.$$

Conjecture (Alexandersson & R. 2022).

$$(s,t) \sim (t,s)$$



$s(G)=0$	$s(G)=0$	$s(G)=0$	$s(G)=1$	$s(G)=1$	$s(G)=3$
$t(G)=3$	$t(G)=1$	$t(G)=0$	$t(G)=1$	$t(G)=0$	$t(G)=0$

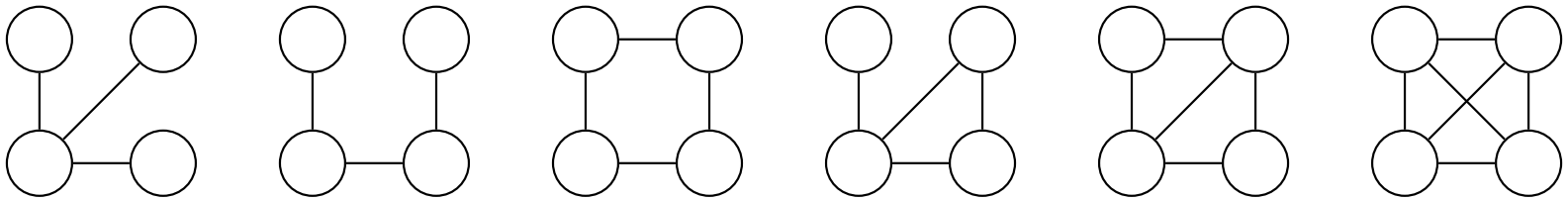
tufts and siblings

Definition (Alexandersson & R. 2022). The *tuft number* $t(G)$ is the maximal number of leaves adjacent to the same vertex.
The *sibling number* is the maximal number of siblings of a vertex:

$$s(G) = \max_{v \in V(G)} |\{u \in V(G) \mid \overline{\mathrm{N}}(u) = \overline{\mathrm{N}}(v)\}| - 1.$$

Theorem (Fürnsinn & Gangl & R. 2024).

$$(s,t) \sim (t,s)$$

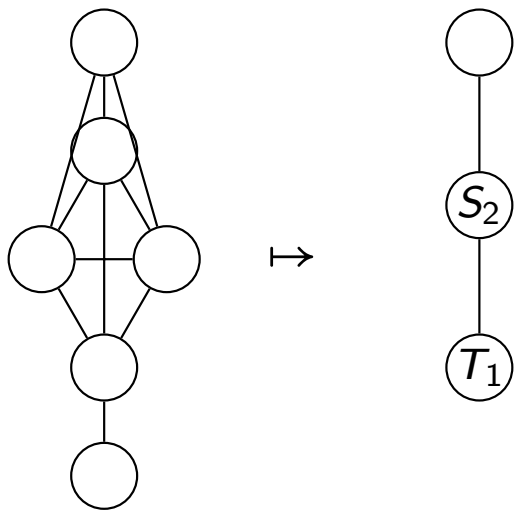


$s(G)=0$	$s(G)=0$	$s(G)=0$	$s(G)=1$	$s(G)=1$	$s(G)=3$
$t(G)=3$	$t(G)=1$	$t(G)=0$	$t(G)=1$	$t(G)=0$	$t(G)=0$

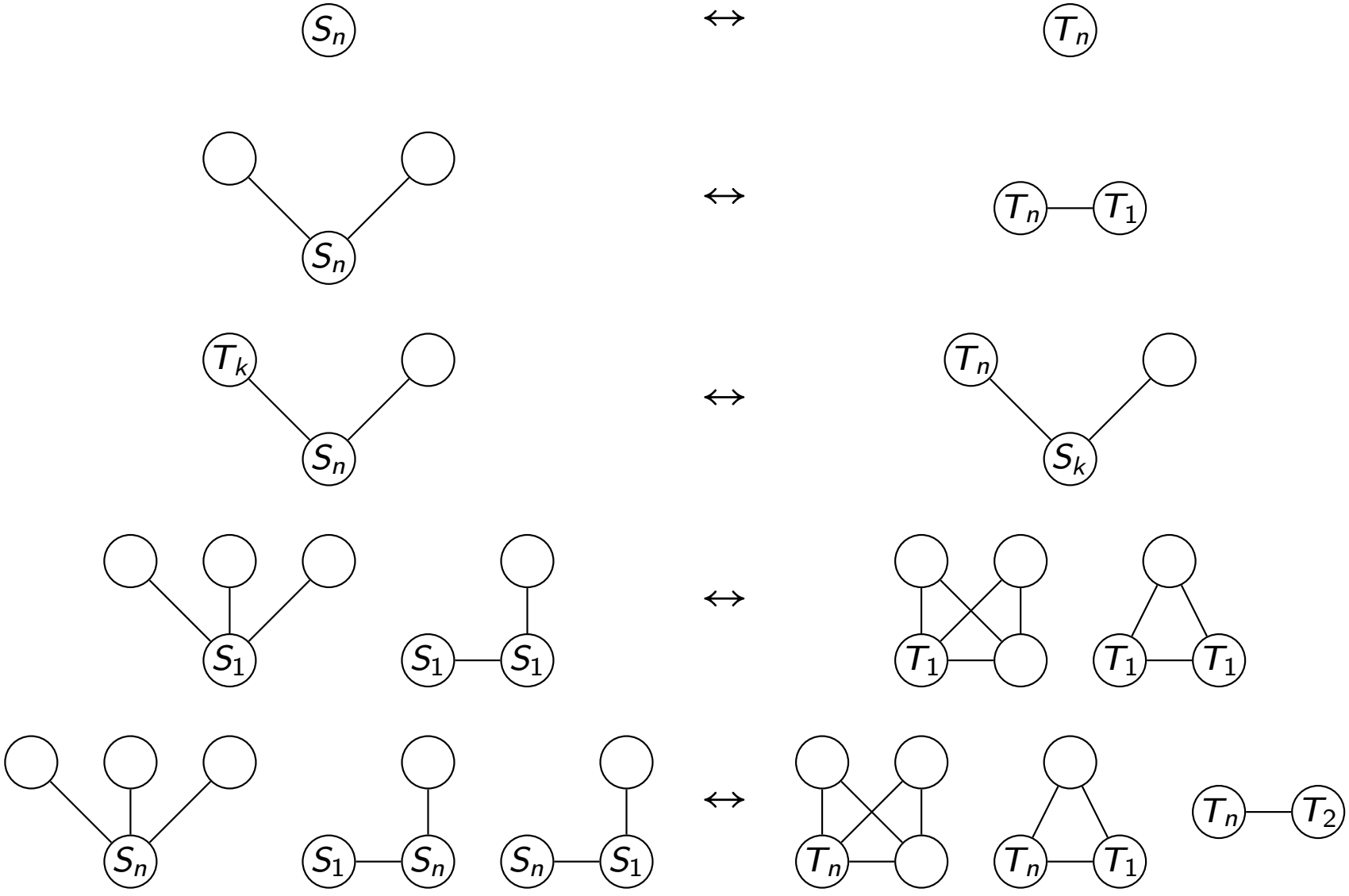
first idea: look at small examples and find a bijection

Fact. In graphs other than K_2, siblings cannot be adjacent to leaves.

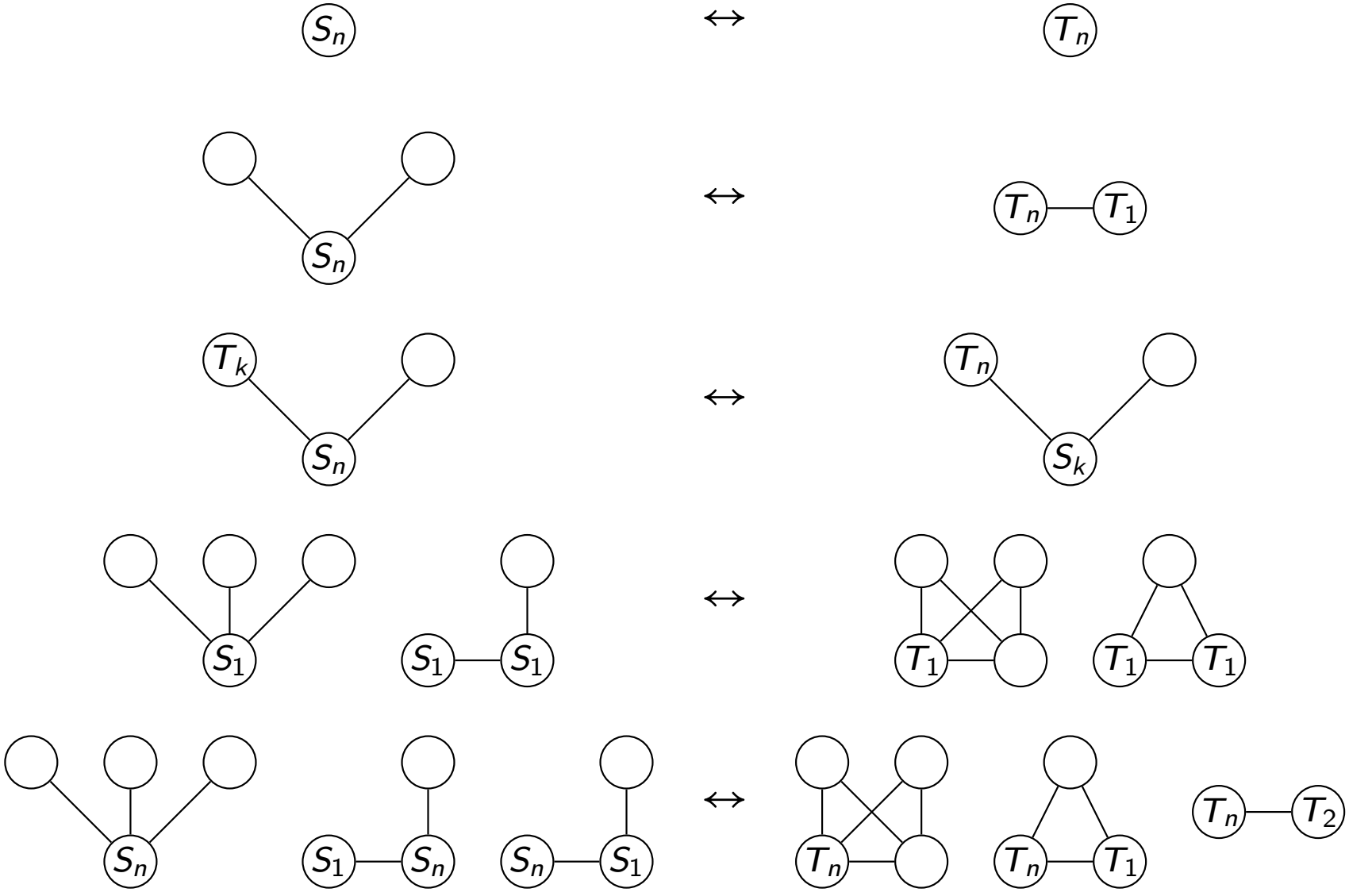
Contract tufts with n leaves to a vertex T_n
and groups of $n+1$ siblings to a vertex S_n.



first idea: look at small examples and find a bijection



first idea: look at small examples and find a bijection



second idea: follow Read, Gessel & Li and count

combinatorial species (Joyal 1981)

a combinatorial species is . . .

Definition. . . . a gadget to enumerate symmetry classes of labelled objects. Given a set of *labels*, a species 'produces' the set of *(labelled) structures*.

- *Par*, set partitions

$$Par[a, 1, \heartsuit] = \{a1\heartsuit,\ a1|\heartsuit,\ a\heartsuit|1,\ 1\heartsuit|a,\ a|1|\heartsuit\}$$

a combinatorial species is . . .

Definition. . . . a gadget to enumerate symmetry classes of labelled objects. Given a set of *labels*, a species 'produces' the set of *(labelled) structures*.

- *Par*, set partitions

$$Par[a, 1, \heartsuit] = \{a1\heartsuit,\ a1|\heartsuit,\ a\heartsuit|1,\ 1\heartsuit|a,\ a|1|\heartsuit\}$$

- $\mathcal{G}$, (vertex labelled simple) graphs

a combinatorial species is . . .

Definition. . . . a gadget to enumerate symmetry classes of labelled objects. Given a set of *labels*, a species 'produces' the set of *(labelled) structures*.

- Par, set partitions

$$Par[a, 1, \heartsuit] = \{a1\heartsuit,\ a1|\heartsuit,\ a\heartsuit|1,\ 1\heartsuit|a,\ a|1|\heartsuit\}$$

- $\mathcal{G}$, (vertex labelled simple) graphs
- E, sets, and E_n, sets of size n

$$E[a, 1, \heartsuit] = \{\{a, 1, \heartsuit\}\} \quad E_2[a, 1, \heartsuit] = \{\}$$

a combinatorial species is . . .

Definition. . . . a functor F from the category of finite sets with bijections to the category of finite sets with maps:

- for a finite set of *labels* U,
  $F[U]$ is the finite set of *F-structures on U*,

a combinatorial species is . . .

Definition. . . . a functor F from the category of finite sets with bijections to the category of finite sets with maps:

- for a finite set of *labels* U,
  $F[U]$ is the finite set of *F-structures on U*,
- for a bijection $\sigma : U \to V$,
  $F[\sigma] : F[U] \to F[V]$ is the *relabelling* of the structures,

such that

$$F[id_U] = id_{F[U]}$$

and, for any two bijections $\sigma : U \to V$ and $\tau : V \to W$,

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma].$$

a combinatorial species is . . .

Definition. . . . a functor F from the category of finite sets with bijections to the category of finite sets with maps:

- for a finite set of *labels* U,
  $F[U]$ is the finite set of *F-structures on U*,
- for a bijection $\sigma : U \to V$,
  $F[\sigma] : F[U] \to F[V]$ is the *relabelling* of the structures,

such that

$$F[id_U] = id_{F[U]}$$

and, for any two bijections $\sigma : U \to V$ and $\tau : V \to W$,

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma].$$

usually, the definition of relabelling is clear from context!

a combinatorial species is . . .

Definition. . . . a functor F from the category of finite sets with bijections to the category of finite sets with maps:

- for a finite set of *labels* U,
  $F[U]$ is the finite set of *F-structures on U*,
- for a bijection $\sigma : U \to V$,
  $F[\sigma] : F[U] \to F[V]$ is the *relabelling* of the structures,

such that

$$F[id_U] = id_{F[U]}$$

and, for any two bijections $\sigma : U \to V$ and $\tau : V \to W$,

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma].$$

$$Par[a, 1, \heartsuit] = \{a1\heartsuit,\ a1|\heartsuit,\ a\heartsuit|1,\ 1\heartsuit|a,\ a|1|\heartsuit\}$$

$$Par[\left(\begin{smallmatrix} a & 1 & \heartsuit \\ x & y & z \end{smallmatrix}\right)](a1|\heartsuit) = xy|z$$

isomorphism of structures

Definition. Structures $s \in F[U]$ and $t \in F[V]$ are *isomorphic* if t is obtained from s by relabelling:

$$t = F[\sigma](s) \quad \text{for a bijection } \sigma : U \to V$$

$(a1|\heartsuit) \in Par[a, 1, \heartsuit]$ and $(xy|z) \in Par[x, y, z]$ are isomorphic, because

$$Par[\left(\begin{smallmatrix} a & 1 & \heartsuit \\ x & y & z \end{smallmatrix}\right)](a1|\heartsuit) = xy|z.$$

computing with species

Definition.

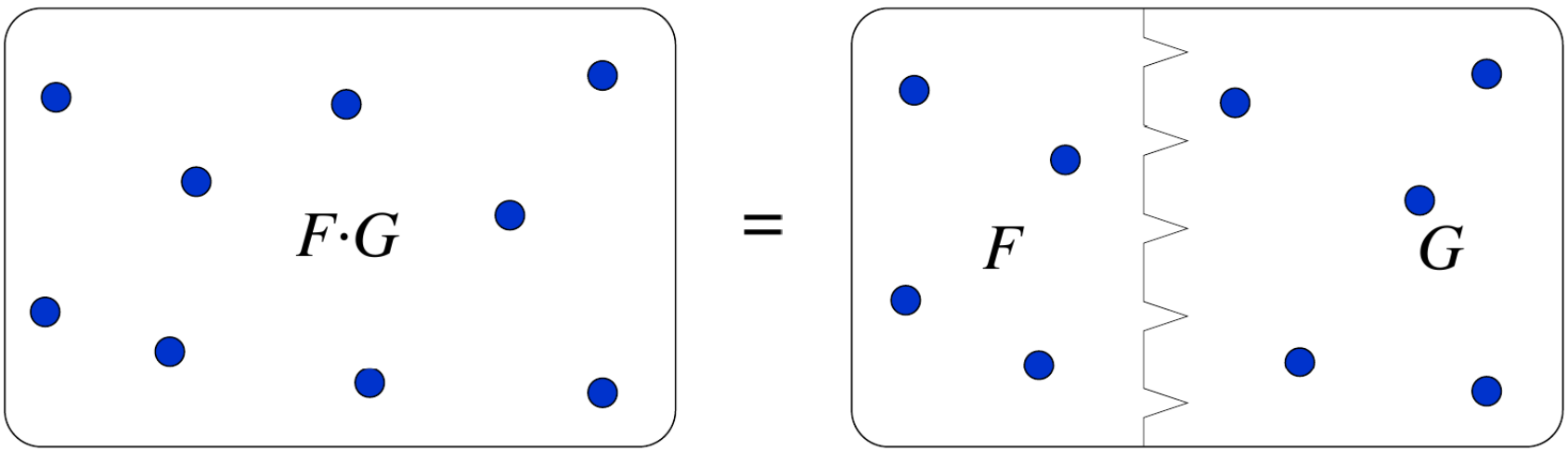
$$(F + G)[U] = F[U] + G[U] \qquad \textit{sum}$$

computing with species

Definition.

$$(F+G)[U] = F[U] + G[U] \qquad \textit{sum}$$

$$(F \cdot G)[U] = \sum_{V+W=U} F[V] \times G[W] \qquad \textit{product}$$



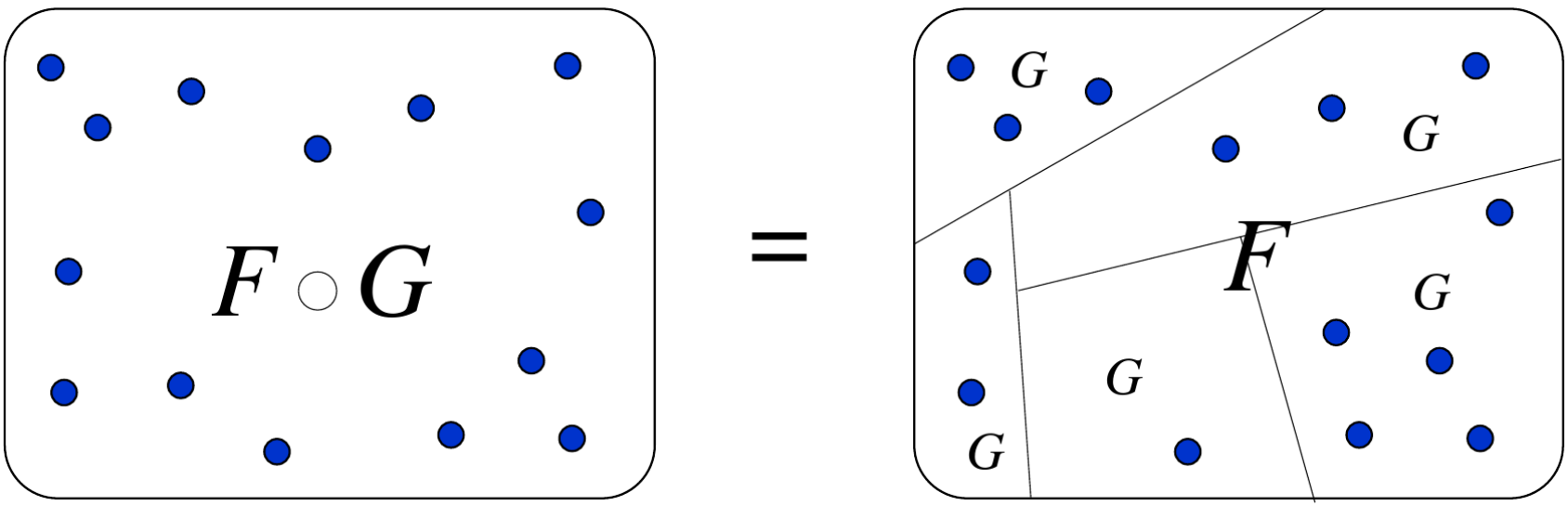
computing with species

Definition.

$$(F+G)[U] = F[U] + G[U] \qquad \textit{sum}$$

$$(F \cdot G)[U] = \sum_{V+W=U} F[V] \times G[W] \qquad \textit{product}$$

$$(F \circ G)[U] = \sum_{\pi \in Par[U]} F[\pi] \times \prod_{B \in \pi} G[B] \qquad \textit{composition}$$



computing with species

Definition.

$$(F+G)[U] = F[U] + G[U] \qquad \textit{sum}$$

$$(F \cdot G)[U] = \sum_{V+W=U} F[V] \times G[W] \qquad \textit{product}$$

$$(F \circ G)[U] = \sum_{\pi \in Par[U]} F[\pi] \times \prod_{B \in \pi} G[B] \qquad \textit{composition}$$

Definition.

$$F(x) = \sum_{n \geq 0} |F[n]| \frac{x^n}{n!}$$

$$\widetilde{F}(x) = \sum_{n \geq 0} \#\text{isomorphism types of } F[n]\, x^n$$

computing with species

Definition.

$$(F+G)[U] = F[U] + G[U] \qquad \textit{sum}$$

$$(F\cdot G)[U] = \sum_{V+W=U} F[V]\times G[W] \qquad \textit{product}$$

$$(F\circ G)[U] = \sum_{\pi\in Par[U]} F[\pi]\times \prod_{B\in\pi} G[B] \qquad \textit{composition}$$

Definition.

$$F(x) = \sum_{n\geq 0} |F[n]| \frac{x^n}{n!}$$

$$\widetilde{F}(x) = \sum_{n\geq 0} \#\text{isomorphism types of } F[n]\, x^n$$

Proposition.

$$(F+G)(x) = F(x)+G(x), \quad (F\cdot G)(x) = F(x)G(x), \quad (F\circ G)(x) = F\big(G(x)\big)$$

$$\widetilde{(F+G)}(x) = \widetilde{F}(x)+\widetilde{G}(x), \quad \widetilde{(F\cdot G)}(x) = \widetilde{F}(x)\widetilde{G}(x), \quad \widetilde{(F\circ G)}(x) \neq \widetilde{F}\big(\widetilde{G}(x)\big)$$

computing with species

Definition. $Z_F(x_1, x_2, \dots) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \operatorname{fix} F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} \dots,$

where

$\operatorname{fix} F[\sigma] = \#\text{fixed points of } F[\sigma],$

$\sigma_k = \#k\text{-cycles of } \sigma,$

is the *cycle index series* or *Frobenius character* of F.

Example. $E(x) = \exp(x), \quad \widetilde{E}(x) = \frac{1}{1-x}, \quad Z_E = \exp \sum_{n>0} \frac{x_n}{n}$

computing with species

Definition. $Z_F(x_1, x_2, \dots) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \operatorname{fix} F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} \dots,$

where

$\operatorname{fix} F[\sigma] = \#\text{fixed points of } F[\sigma],$

$\sigma_k = \#k\text{-cycles of } \sigma,$

is the *cycle index series* or *Frobenius character* of F.

Example. $E(x) = \exp(x), \quad \widetilde{E}(x) = \frac{1}{1-x}, \quad Z_E = \exp \sum_{n>0} \frac{x_n}{n}$

Fact. $F(x) = Z_F(x, 0, \dots), \quad \widetilde{F}(x) = Z_F(x, x^2, \dots)$

Theorem. $\widetilde{F \circ G}(x) = Z_F\big(\widetilde{G}(x), \widetilde{G}(x^2), \dots\big)$

molecular decompositions and virtual species

Definition. F is *molecular* if it is not a non-trivial sum.

Fact. F is molecular if any two F structures are isomorphic.
$F \cdot G$ is molecular if F and G are molecular.
The molecular decomposition is unique.

$$\mathcal{G} = 1 + X + 2E_2 + (2E_3 + 2XE_2) + \left(2E_4 + 2E_2^2 + 2XE_3 + 2P_4 + 2X^2E_2 + E_2(X^2)\right) + \ldots$$

molecular decompositions and virtual species

Definition. F is *molecular* if it is not a non-trivial sum.

Fact. F is molecular if any two F structures are isomorphic.
$F \cdot G$ is molecular if F and G are molecular.
The molecular decomposition is unique.

$$\mathcal{G} = 1 + X + 2E_2 + (2E_3 + 2XE_2) + \left(2E_4 + 2E_2^2 + 2XE_3 + 2P_4 + 2X^2E_2 + E_2(X^2)\right) + \dots$$

Definition. A *virtual species* is a formal sum of molecular species.

Fact. F has a compositional inverse if $F_0 = 0$ and $F_1 \neq 0$.

$$E_{\geq 1}^{(-1)} = X + (-E_2) + (-E_3 + XE_2) + (-E_4 + P_4 + XE_3 - X^2E_2) + \dots$$

Read's idea

- $\mathcal{G}$ . . . graphs
- $\mathcal{M}$ . . . co-mating graphs

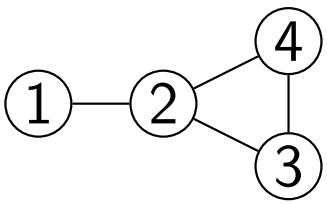
Theorem (Read 1989). $\mathcal{M} \circ E_{\geq 1} = \mathcal{G}$.

Read's idea

- $\mathcal{G}$ . . . graphs
- $\mathcal{M}$ . . . co-mating graphs

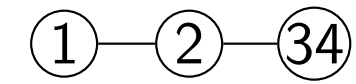
Theorem (Read 1989). $\mathcal{M} \circ E_{\geq 1} = \mathcal{G}$.

Proof. $$(F \circ G)[U] = \sum_{\pi \in Par[U]} F[\pi] \times \prod_{B \in \pi} G[B].$$



$\in \mathcal{G}[1,2,3,4]$

$$= \Big(\underbrace{\big\{\{1\},\{2\},\{3,4\}\big\}}_{\in Par[1,2,3,4]}, \underbrace{\quad}_{\in \mathcal{M}[\{1\},\{2\},\{3,4\}]} , \big(\{1\},\{2\},\{3,4\}\big) \Big)$$



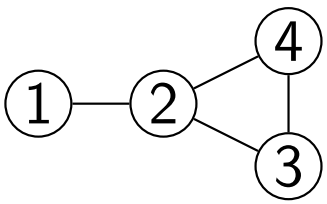
□

Read's idea

- $\mathcal{G}$ . . . graphs
- $\mathcal{M}$ . . . co-mating graphs

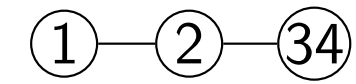
Theorem (Read 1989). $\mathcal{M} \circ E_{\geq 1} = \mathcal{G}$.

Proof. $$(F \circ G)[U] = \sum_{\pi \in Par[U]} F[\pi] \times \prod_{B \in \pi} G[B].$$



$\in \mathcal{G}[1,2,3,4]$

$$= \Big(\underbrace{\big\{\{1\},\{2\},\{3,4\}\big\}}_{\in Par[1,2,3,4]}, \underbrace{\quad}_{\in \mathcal{M}[\{1\},\{2\},\{3,4\}]} , \big(\{1\},\{2\},\{3,4\}\big) \Big)$$



□

Corollary (Read 1989). $\mathcal{M} = \mathcal{G} \circ E_{\geq 1}^{(-1)}$.

Gessel & Li's idea: a tuft decomposition

- $X + \mathcal{T} := X \cdot E(Y)$ . . . a singleton vertex or a tuft
- $\mathcal{G}_{\neq 2}(X, Y)$ . . . graphs with leaves Y

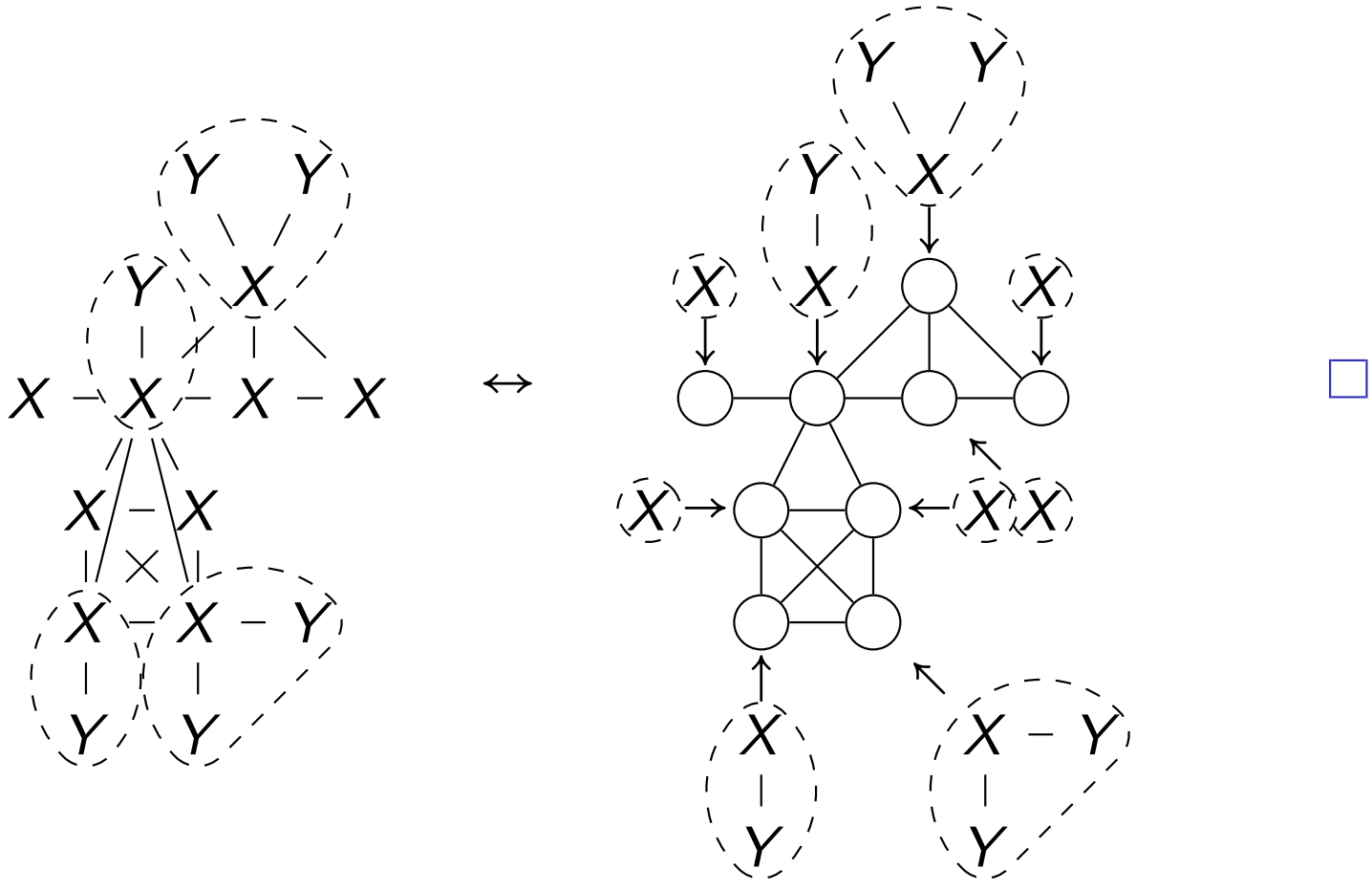
Theorem (Gessel & Li 2011). $\mathcal{G}_{\neq 2}(X, X + Y) = \mathcal{G}\big(X + \mathcal{T}\big) - XY - E_2$.

Gessel & Li's idea: a tuft decomposition

- $X + \mathcal{T} := X \cdot E(Y)$ . . . a singleton vertex or a tuft
- $\mathcal{G}_{\neq 2}(X, Y)$ . . . graphs with leaves Y

Theorem (Gessel & Li 2011). $\mathcal{G}_{\neq 2}(X, X + Y) = \mathcal{G}\big(X + \mathcal{T}\big) - XY - E_2$.

Proof.



□

new idea: a patch decomposition

- $\mathcal{P}(X, Y, Z) := \big(1 + X + E_{\geq 2}(Z)\big) \cdot \big(E \circ \mathcal{T}\big) - 1$ . . . a *patch*
- $\mathcal{G}_{\neq 2}(X, Y, Z)$ . . . graphs with leaves Y, siblings Z

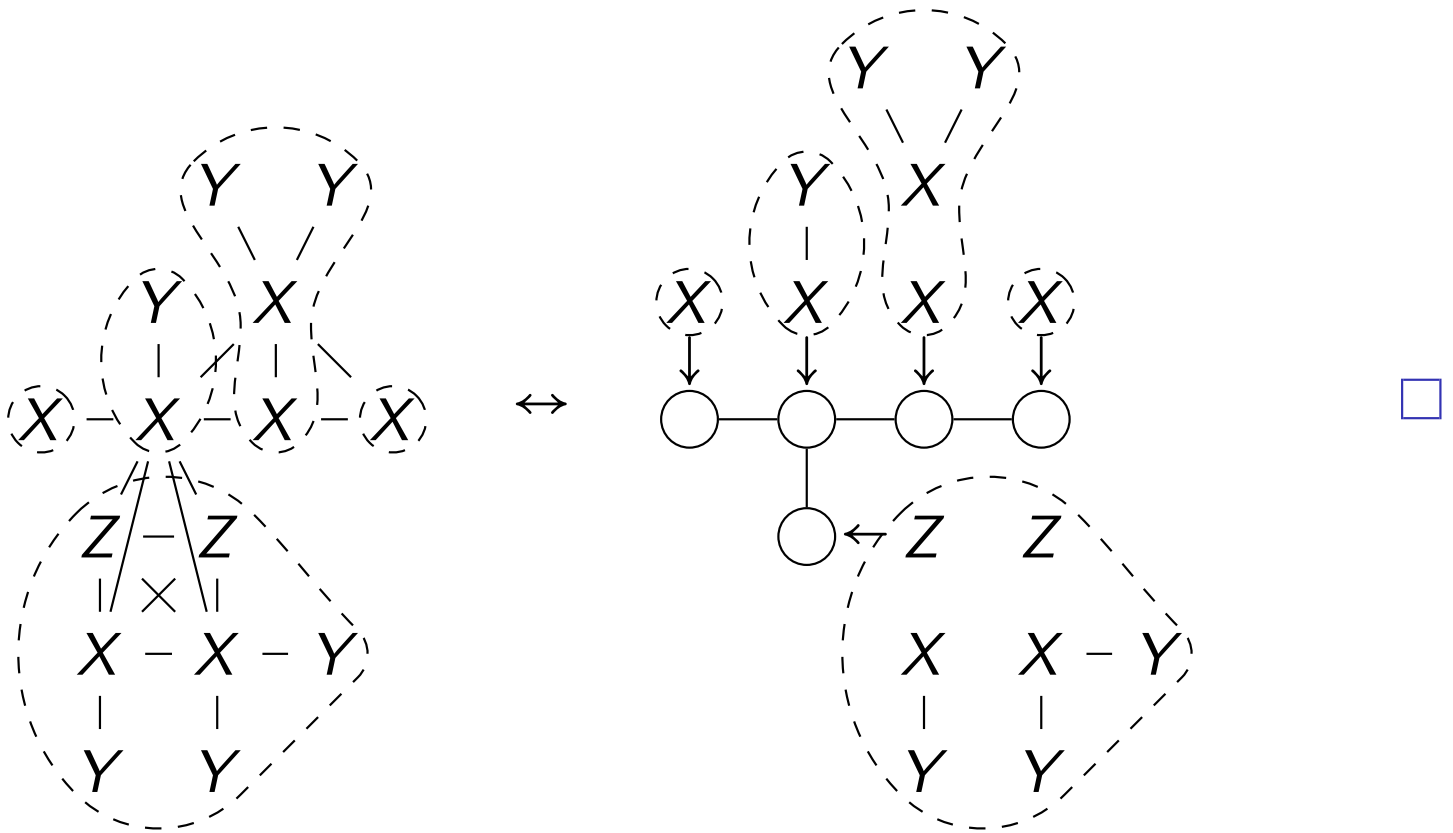
Theorem (Fürnsinn & Gangl & R. 2024).
$\mathcal{G}_{\neq 2}(X, X + Y, Z) = \mathcal{M} \circ \mathcal{P} - XY - E_2(Z)$.

new idea: a patch decomposition

- $\mathcal{P}(X, Y, Z) := \big(1 + X + E_{\geq 2}(Z)\big) \cdot \big(E \circ \mathcal{T}\big) - 1$ . . . a *patch*
- $\mathcal{G}_{\neq 2}(X, Y, Z)$ . . . graphs with leaves Y, siblings Z

Theorem (Fürnsinn & Gangl & R. 2024).
$\mathcal{G}_{\neq 2}(X, X + Y, Z) = \mathcal{M} \circ \mathcal{P} - XY - E_2(Z)$.

Proof.



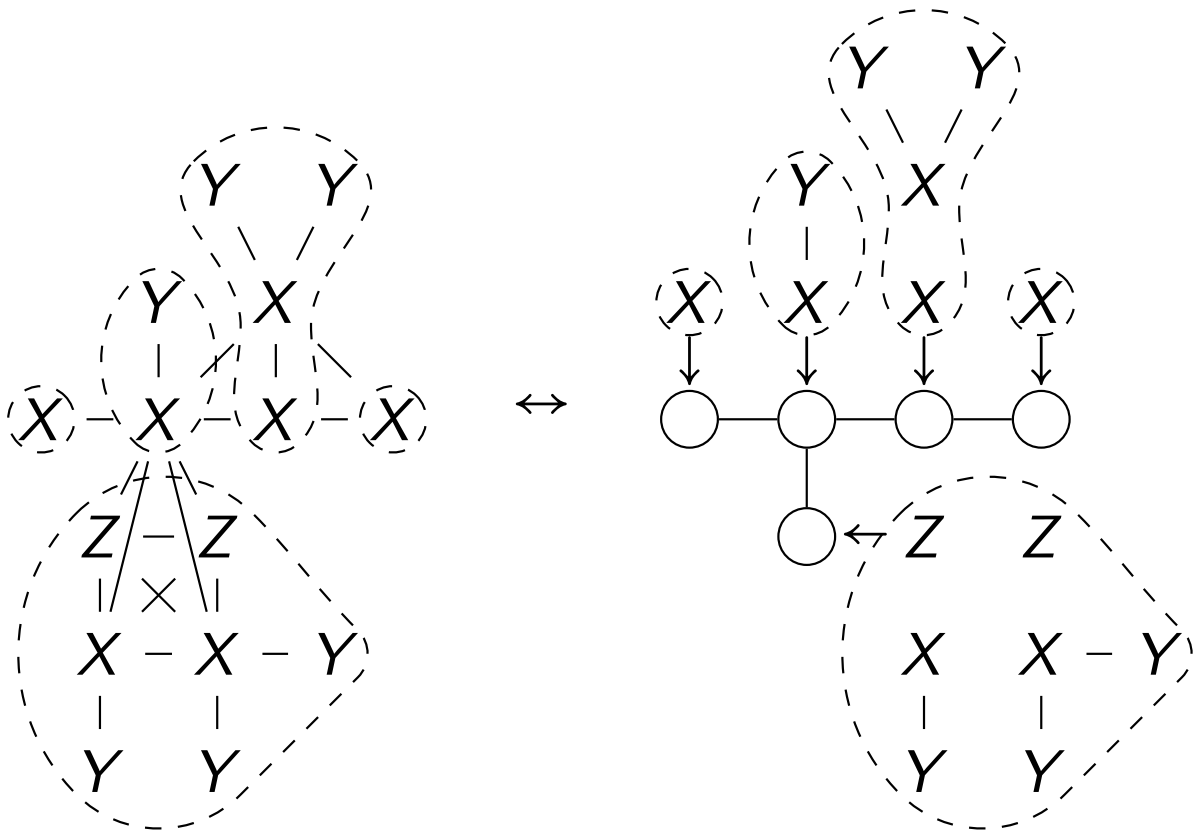
□

new idea: a patch decomposition

- $\mathcal{P}^s(X,Y,Z) := \big(1 + X + E_{2\le\bullet<2+s}(Z)\big)\cdot\big(E\circ\mathcal{T}\big) - 1$ ... a *patch*
- $\mathcal{G}^s_{\neq 2}(X,Y,Z)$ ... graphs with leaves Y, siblings Z, sibling number $\le s$

Theorem (Fürnsinn & Gangl & R. 2024).
$\mathcal{G}^s_{\neq 2}(X, X+Y, Z) = \mathcal{M}\circ\mathcal{P}^s - XY - [s>0]E_2(Z)$.

Proof.



□

harvest

Lemma. Let

$$\mathcal{V}^{s,t} := \big(1 + X + E_{2\le\bullet<2+s}(Z)\big) \cdot E\big(X \cdot E_{<1+t}(Y)E(-X) - 1)\big) - 1.$$

Then $\mathcal{G}^{s,t}_{\neq 2}(X, Y, Z) = \mathcal{M} \circ \mathcal{V}^{s,t} + X^2 - [t > 0]XY - [s > 0]E_2(Z)$.

Proof.
$Y := Y - X$ in the patch decomposition, and similar tricks. □

harvest

Lemma. Let

$$\mathcal{V}^{s,t} := \big(1 + X + E_{2\le\bullet<2+s}(Z)\big) \cdot E\big(X \cdot E_{<1+t}(Y)E(-X) - 1)\big) - 1.$$

Then $\mathcal{G}^{s,t}_{\neq 2}(X, Y, Z) = \mathcal{M} \circ \mathcal{V}^{s,t} + X^2 - [t > 0]XY - [s > 0]E_2(Z)$.

Proof.
$Y := Y - X$ in the patch decomposition, and similar tricks. □

Lemma. $\widetilde{\mathcal{V}^{s,t}}(x) := \widetilde{\mathcal{V}^{s,t}}(x, x, x) = \frac{(1-x^{s+2})(1-x^{t+2})}{1-x} - 1.$

Corollary. $\widetilde{\mathcal{G}^{s,t}_{\neq 2}}(x, x, x)$ is symmetric in s and t.

Proof.

$$\widetilde{\mathcal{M} \circ \mathcal{V}^{s,t}} = Z_{\mathcal{M}}\big(\widetilde{\mathcal{V}^{s,t}}(x), \widetilde{\mathcal{V}^{s,t}}(x^2), \dots\big).$$

□

new idea: a refined patch decomposition

Definition. The *reduction* of a graph other than K_2 is obtained by repeatedly removing leaves and contracting each group of siblings to a single vertex.

- $\mathcal{G}_{\neq 2}^{R,s}(X, Y, Z)$ . . . graphs with leaves Y, siblings Z, sibling number $\leq s$, that reduce to R
- $\mathcal{M}^R$ . . . co-mating graphs that reduce to R

new idea: a refined patch decomposition

Definition. The *reduction* of a graph other than K_2 is obtained by repeatedly removing leaves and contracting each group of siblings to a single vertex.

- $\mathcal{G}_{\neq 2}^{R,s}(X,Y,Z)$ . . . graphs with leaves Y, siblings Z, sibling number $\leq s$, that reduce to R
- $\mathcal{M}^R$ . . . co-mating graphs that reduce to R

Theorem (Fürnsinn & Gangl & R. 2024).

$$\mathcal{G}_{\neq 2}^{R,s}(X,X+Y,Z) = \begin{cases} \mathcal{M}^\bullet \circ \mathcal{P}^s - XY - [s>0]E_2(Z) & R = \bullet \\ \mathcal{M}^R \circ \mathcal{P}^s & \text{otherwise.} \end{cases}$$

Theorem (Fürnsinn & Gangl & R. 2024). On the set of graphs that reduce to R,

$$(s,t) \sim (t,s)$$

questions

- find a bijection!

questions

- find a bijection!
- find any bijection!

questions

- find a bijection!
- find any bijection!
- do tropical species exist?