

Cores and Diophantine equations

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Journées de Combinatoire de Bordeaux
3-5 Février 2025

I - Core partitions

II - A Pell-type equation

III - Generalisations using root systems

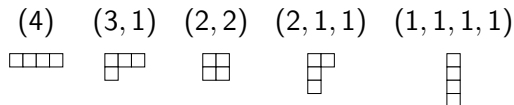
I - Core partitions

Integer partitions

Let $N \in \mathbb{N}$. A *partition* of N is a non-increasing sequence of non-negative integers of sum N :

- $\lambda = (\lambda_1, \lambda_2, \dots)$ avec $\lambda_1 \geq \lambda_2 \geq \dots$
- $|\lambda| = \lambda_1 + \lambda_2 + \dots = N$.

Denote $p(N)$ the number of partitions of N . For instance, $p(4) = 5$:



Why study partitions?

- **Representation theory** Frobenius 1900:

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- **Partition identities** Euler 1748:

$$|\{\text{part. of } N \text{ into odd parts}\}| = |\{\text{part. of } N \text{ into distinct parts}\}|$$


Generalisations: Rogers-Ramanujan 1919, Andrews-Gordon-Bressoud 1961-1980, ...

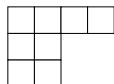
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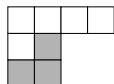
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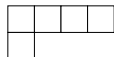
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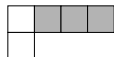
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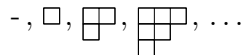
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Theorem (Brauer–Robinson 1947)

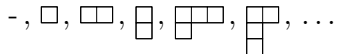
Let $e = p$ be prime. The p -cores label the blocks of $\mathbb{F}_p S_N$.

Enumerating e -cores

- 2-cores are the "staircase" partitions:



- 3-cores:



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- 2-cores are the "staircase" partitions:

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Theorem (Granville–Ono 1996)

Let $e \geq 4$. For all $N \in \mathbb{N}$, $c_e(N) > 0$.

Extensive use of arithmetical properties of the generating function

$$\sum_{N \geq 0} c_e(N) x^N = \prod_{k \geq 1} \frac{(1 - x^{ek})^e}{1 - x^k}.$$

II - A Pell-type equation

Our goal

Establish a “correspondence”

e -cores of N



solutions of a certain Diophantine equation

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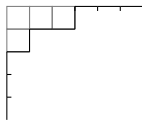
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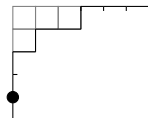
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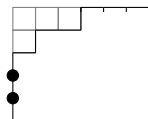
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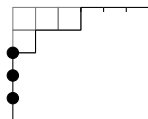
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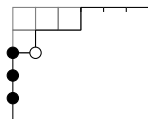
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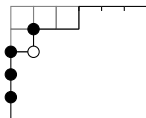
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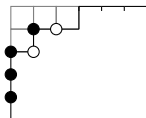
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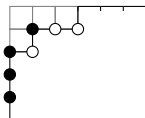
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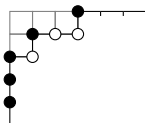
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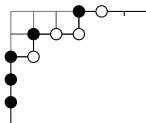
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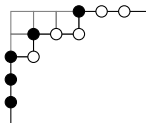
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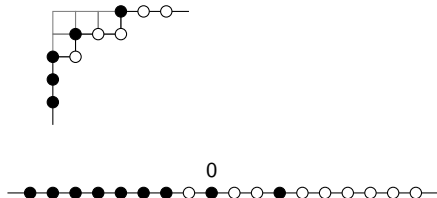
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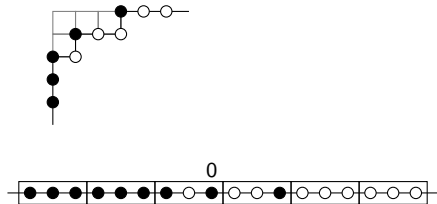
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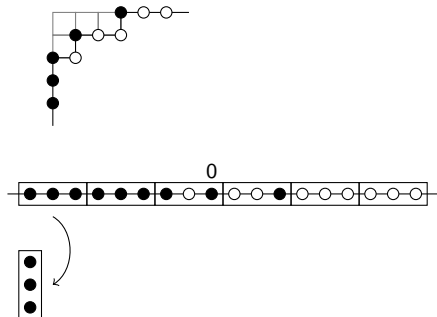
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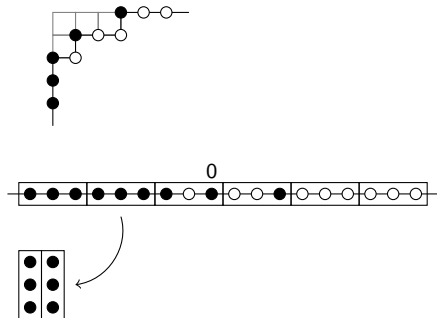
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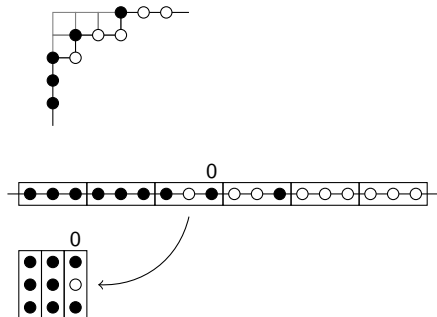
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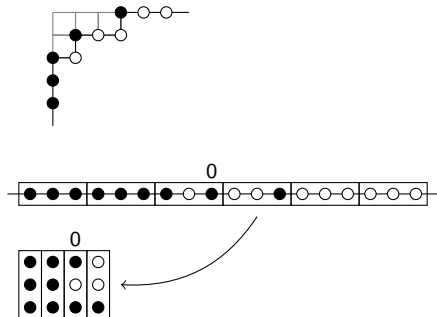
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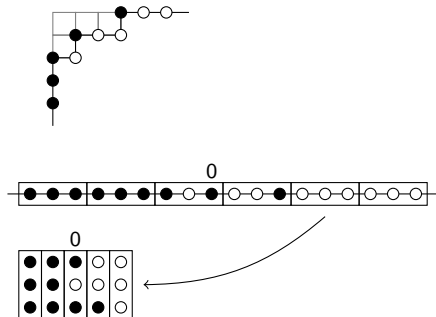
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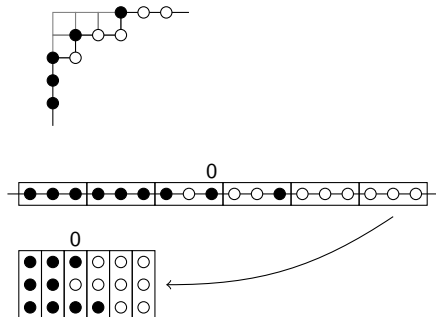
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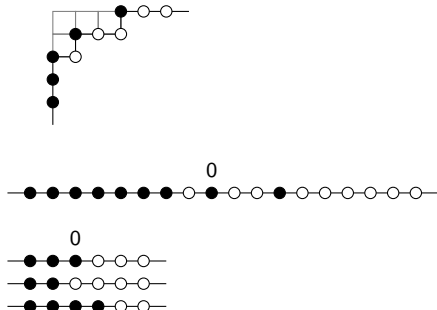
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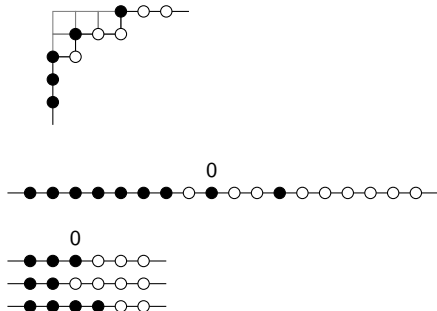
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We see that λ is encoded by the vector $q = (0, -1, 1)$.

A formula for the size

Theorem (Garvan–Kim–Stanton 1990, Chapelier–G. 2022)

Let λ be an e -core of size N , and $q = (q_1, \dots, q_e)$ its coding vector. Then

$$N = \frac{e}{2} \sum_{i=1}^e q_i^2 - \sum_{i=1}^e (e - i)q_i$$

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Example For $e = 3$ we obtain

$$N = 3(q_1^2 + q_2^2 + q_1 q_2) - 2q_1 - q_2.$$

A Diophantine equation for $e = 3$

We can perform a Gaussian reduction and obtain the equivalent identity

$$(3q_1 + 6q_2 - 1)^2 + 3(3q_1 - 1)^2 = 12N + 4.$$

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... Actually, we have a much stronger result.

Theorem (Brunat-Nath 2022)

$$\{3\text{-cores of } N\} \times \mathbb{Z}/6\mathbb{Z} \xleftrightarrow{1:1} \{\text{integer solutions of (E)}\}.$$

G -set structure of the solution set

More precisely, the group

$$G = \left\langle \frac{1}{2} \begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix} \right\rangle \simeq \mathbb{Z}/6\mathbb{Z}$$

acts freely on the solutions, and each orbit has a unique representative of the form $\varphi(\mathbf{q}) = (3q_1 + 6q_2 - 1, 3q_1 - 1)$.



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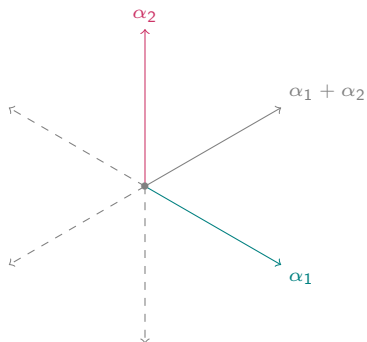
Example Let $N = 4$. There are two 3-cores λ of size 4:

λ		
q	$(0, -1, 1)$	$(-1, 1, 0)$
$\varphi(q)$	$(-7, -1)$	$(2, -4)$
$G \cdot \varphi(q)$	$\left\{ \begin{array}{l} (-7, -1), (-2, -4), (5, -3), \\ (7, 1), (2, 4), (-5, 3) \end{array} \right\}$	$\left\{ \begin{array}{l} (2, -4), (7, -1), (5, 3), \\ (-2, 4), (-7, 1), (-5, -3) \end{array} \right\}$

III - Generalisations using root systems

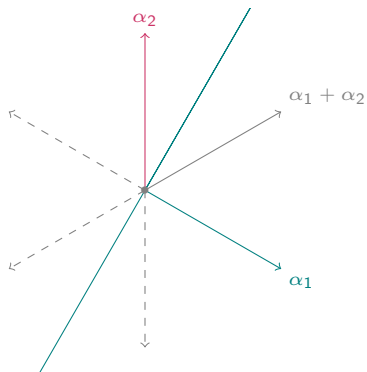
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Root system of type $A_{e-1}^{(1)}$. Example for $e = 3$:



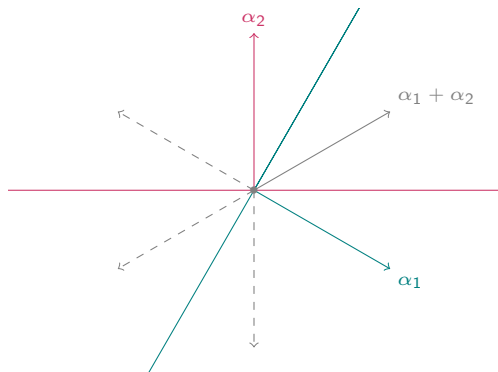
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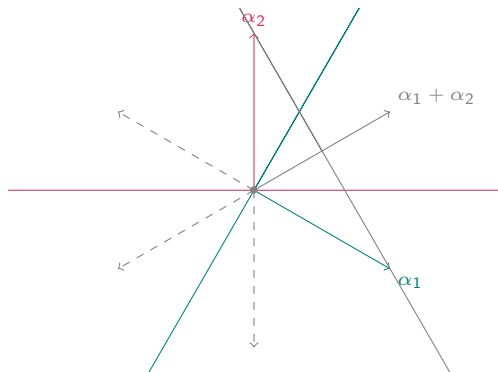
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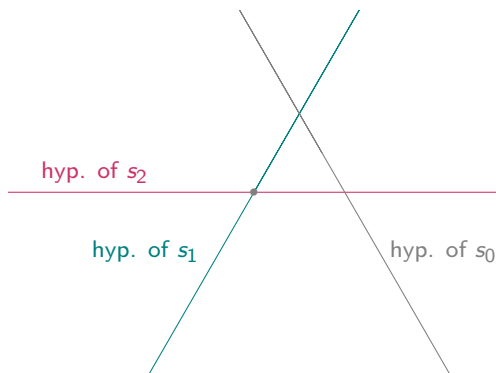
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Cores and root systems

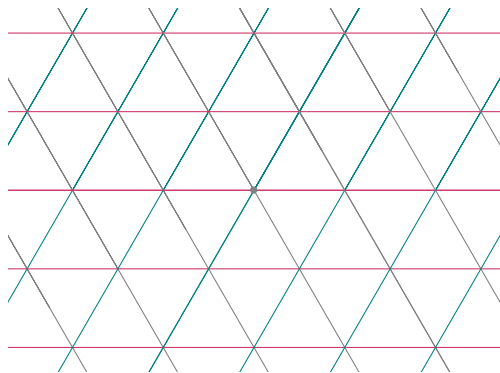
Root system of type $A_{e-1}^{(1)}$. Example for $e = 3$:



s_i = reflection through these hyperplanes

Cores and root systems

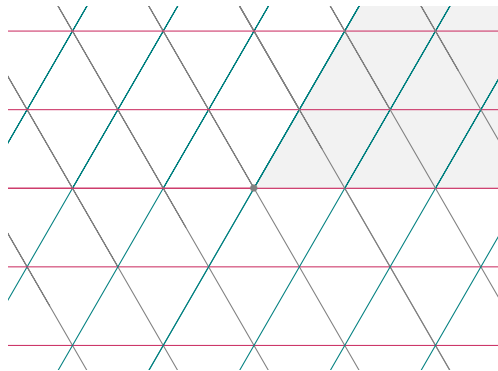
Root system of type $A_{e-1}^{(1)}$. Example for $e = 3$:



$$W = \langle s_0, s_1, \dots, s_{e-1} \rangle = \langle s_1, \dots, s_{e-1} \rangle \rtimes T(\mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_{e-1}).$$

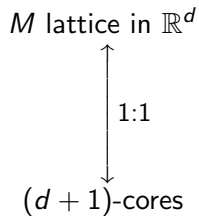
Cores and root systems

Root system of type $A_{e-1}^{(1)}$. Example for $e = 3$:

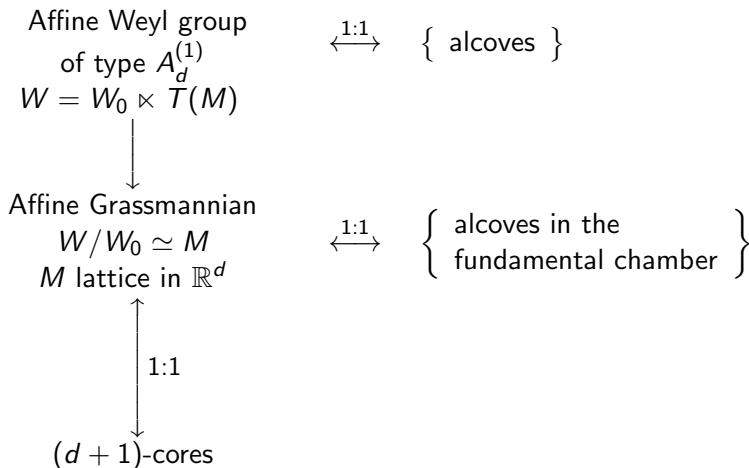


The fundamental chamber

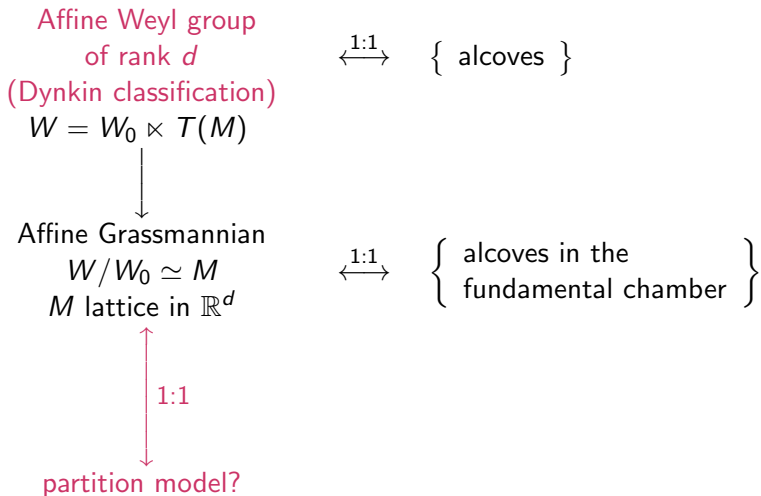
Cores and root systems



Cores and root systems



Cores and root systems



Some new results

Focus on rank 2 Dynkin types.

Theorem (Brunat–Chapelier–G. 2024)

$$\begin{array}{l} \text{Type } C_2^{(1)} \quad \left\{ \begin{array}{l} \text{self-conjugate} \\ \text{4-cores of } N \end{array} \right\} \times D_8 \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{l} \text{integer solutions of} \\ x^2 + y^2 = 8N + 5 \end{array} \right\} \\ \text{Type } D_3^{(2)} \quad \mathcal{D}_6(N) \quad \times D_8 \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{l} \text{integer solutions of} \\ x^2 + y^2 = 12N + 5 \end{array} \right\} \\ \text{Type } D_4^{(3)} \quad \mathcal{D}_4^b(N) \quad \times (\mathbb{Z}/2\mathbb{Z})^2 \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{l} \text{integer solutions of} \\ x^2 + 3y^2 = 12N + 7 \end{array} \right\} \end{array}$$

Here, $\mathcal{D}_{2d+2}(N)$ and $\mathcal{D}_{2d}^b(N)$ are partition models realised in terms of $2d + 2$ cores (Lecouvey–Wahiche 2024).

Example

Type $C_2^{(1)}$ Let $N = 40$.

s.c. 4-cores



q

$(-2, 4)$

$(2, -4)$

$(-4, 0)$

solution

$(-10, 15)$

$(6, -17)$

$(-18, -1)$

orbit

$\left\{ \begin{array}{l} (\pm 10, \pm 15) \\ (\pm 15, \pm 10) \end{array} \right\}$

$\left\{ \begin{array}{l} (\pm 6, \pm 17) \\ (\pm 17, \pm 6) \end{array} \right\}$

$\left\{ \begin{array}{l} (\pm 18, \pm 1) \\ (\pm 1, \pm 18) \end{array} \right\}$

Type $D_4^{(3)}$ Let $N = 7$.

$\mathcal{D}_4^b(N)$



q

$(-1, 1)$

$(0, 1)$

solution

$(-4, -5)$

$(8, -3)$

orbit

$\left\{ \begin{array}{l} (-4, -5), (-4, 5) \\ (4, -5), (4, 5) \end{array} \right\}$

$\left\{ \begin{array}{l} (-8, -3), (-8, 3) \\ (8, -3), (8, 3) \end{array} \right\}$