

Parking sur des arbres de Bienaymé—Galton—Watson géométriques surcritiques

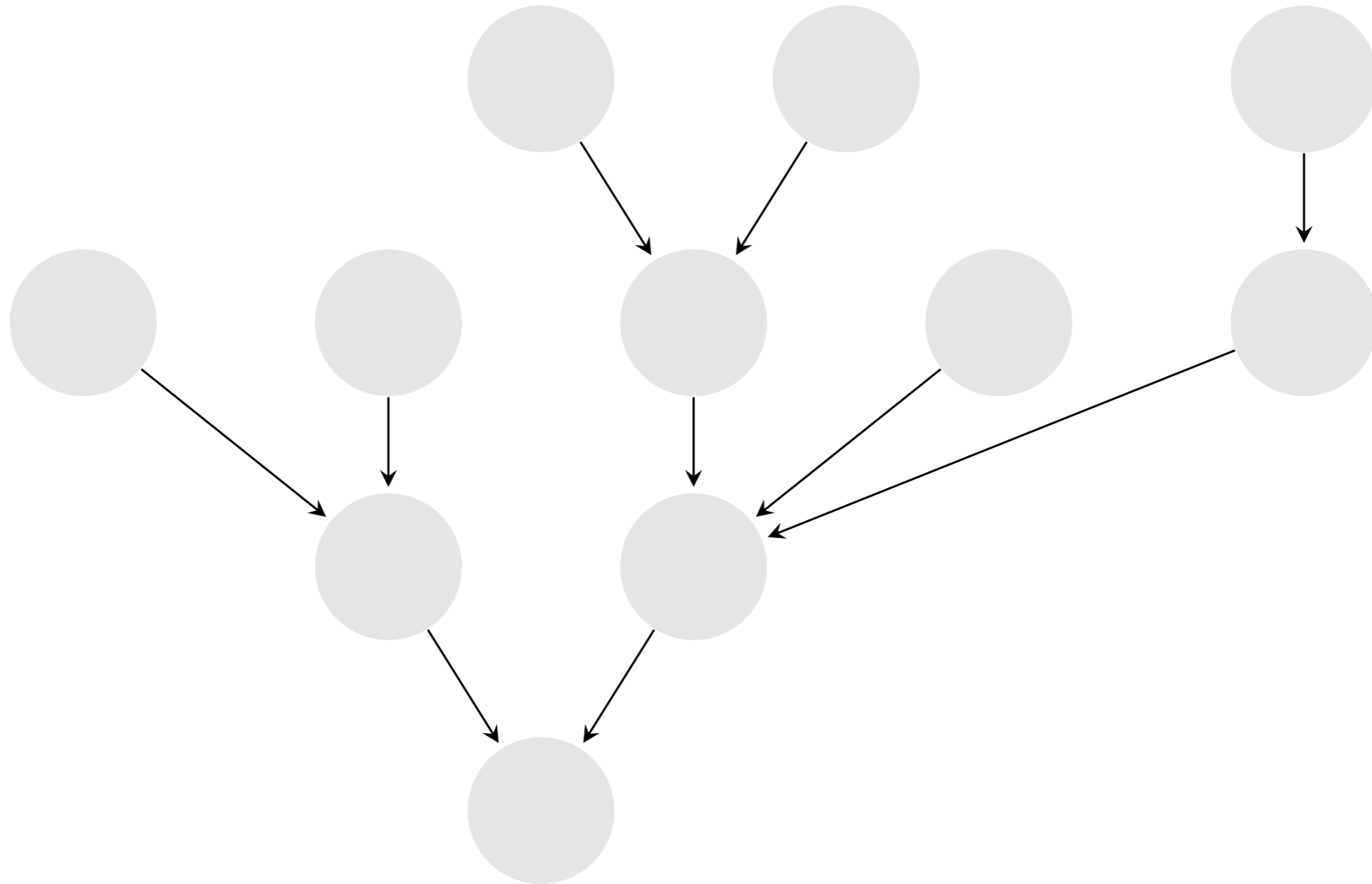
Alice CONTAT

LAGA – Université Sorbonne Paris Nord

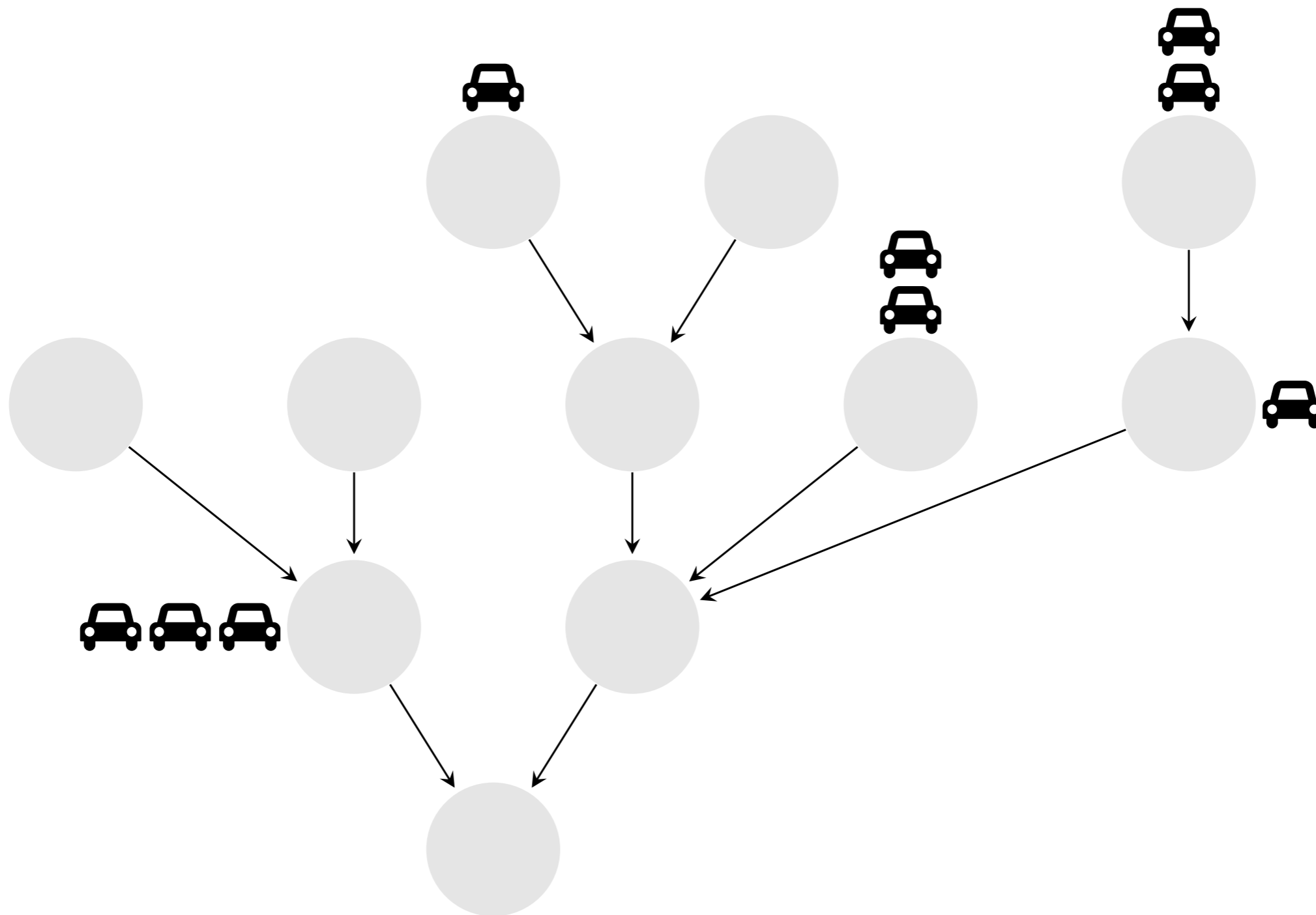
avec Linxiao CHEN

Journées de combinatoire de Bordeaux
3 Février 2025

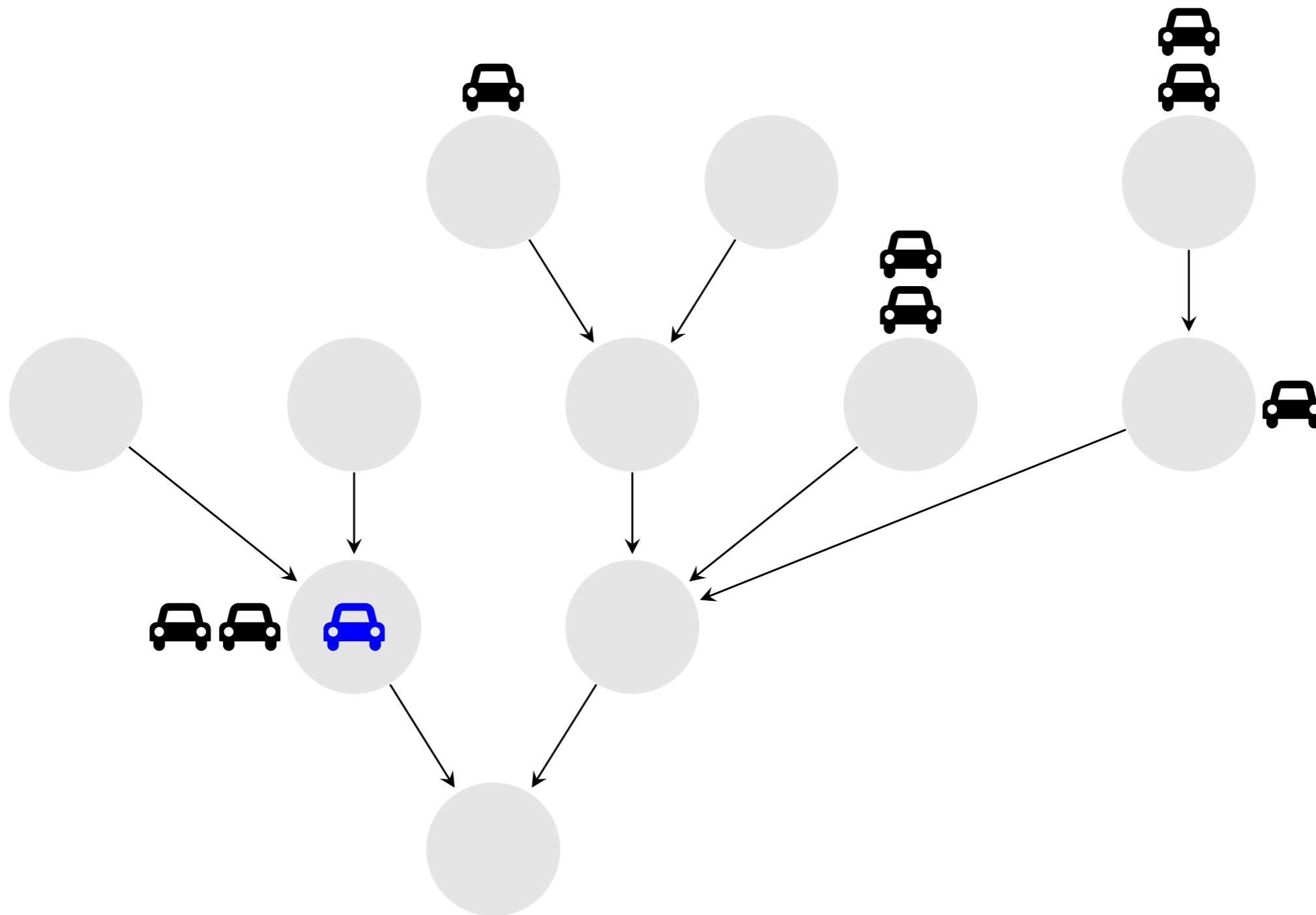
Parking rules on a tree



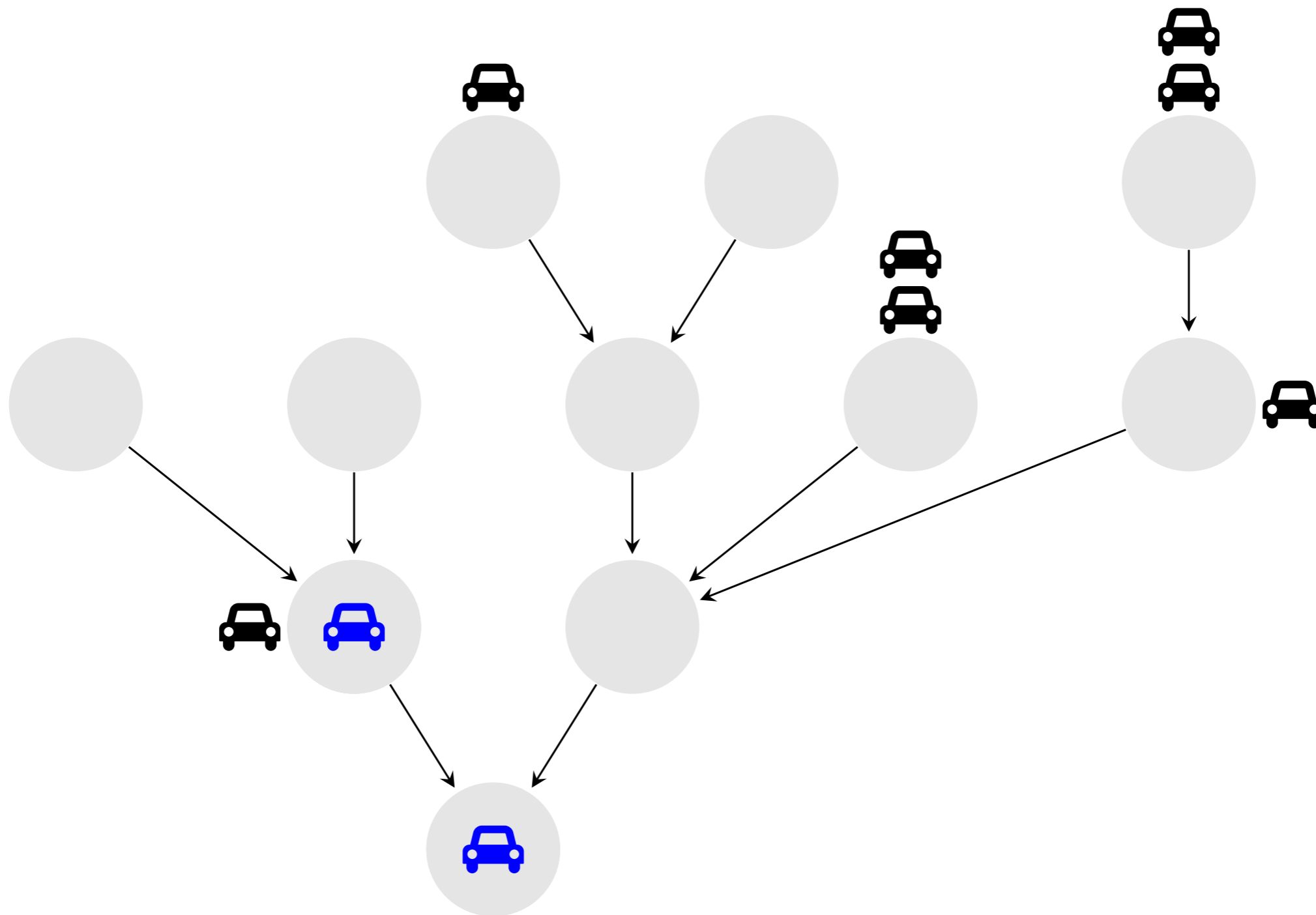
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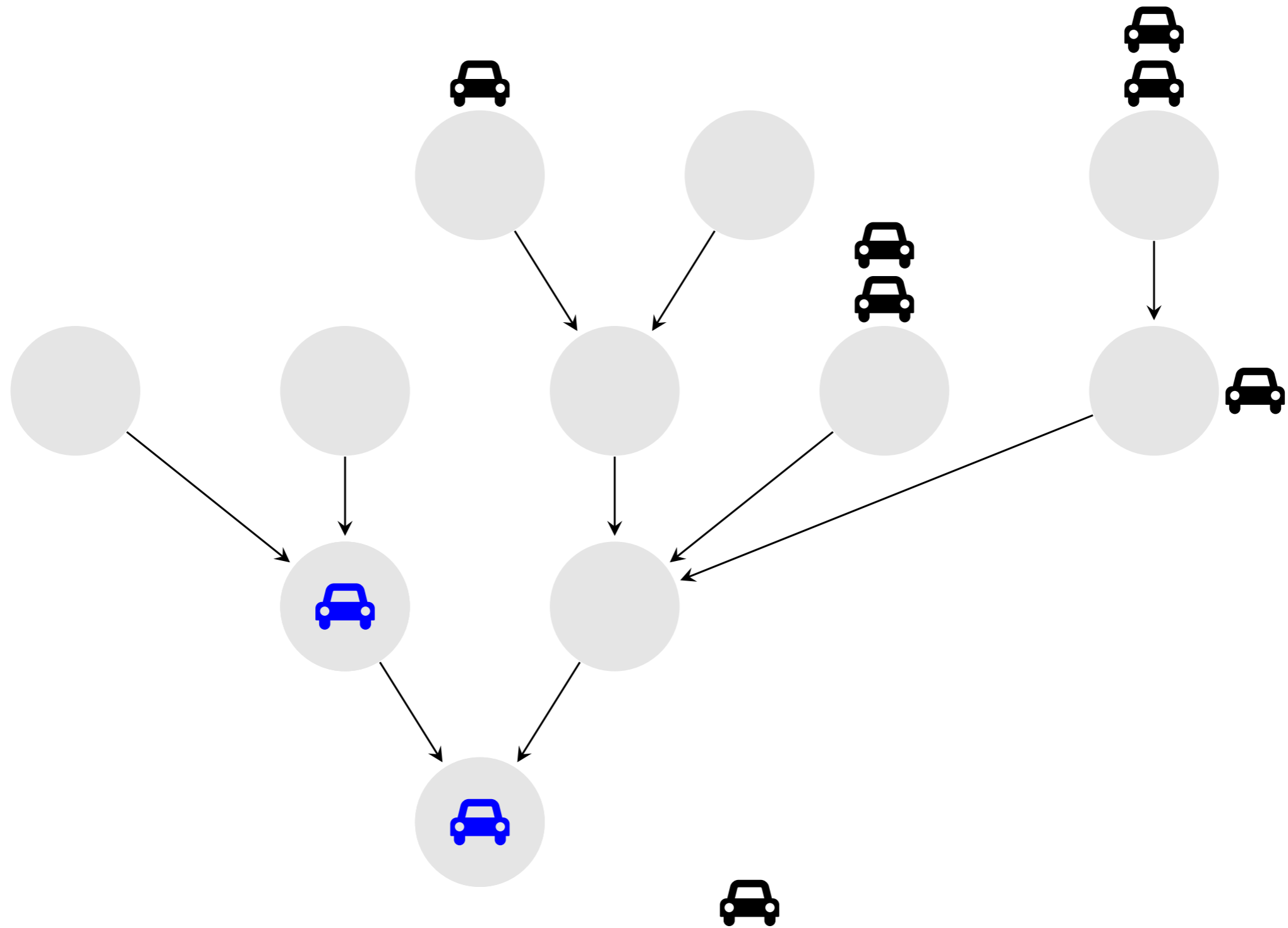
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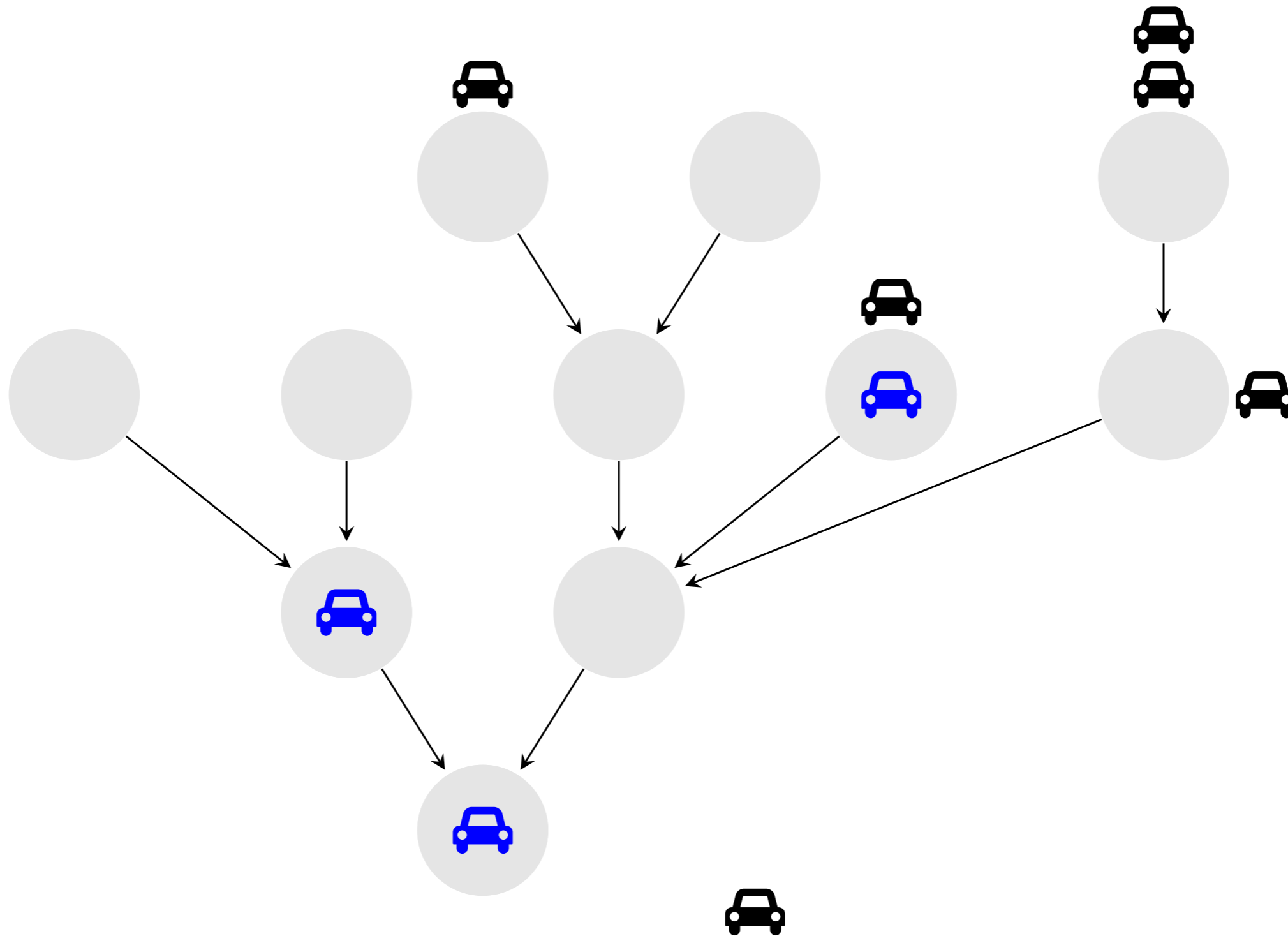
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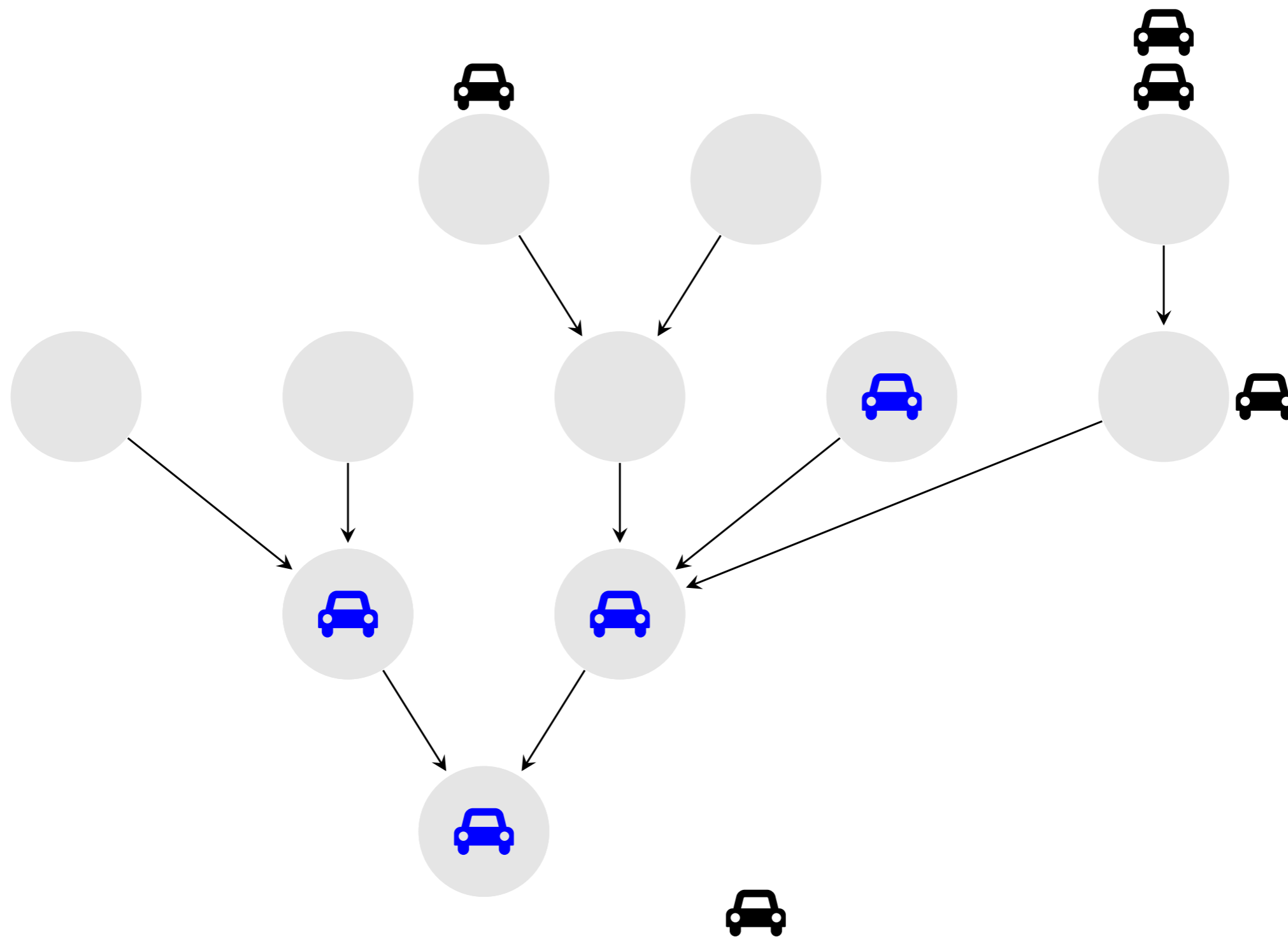
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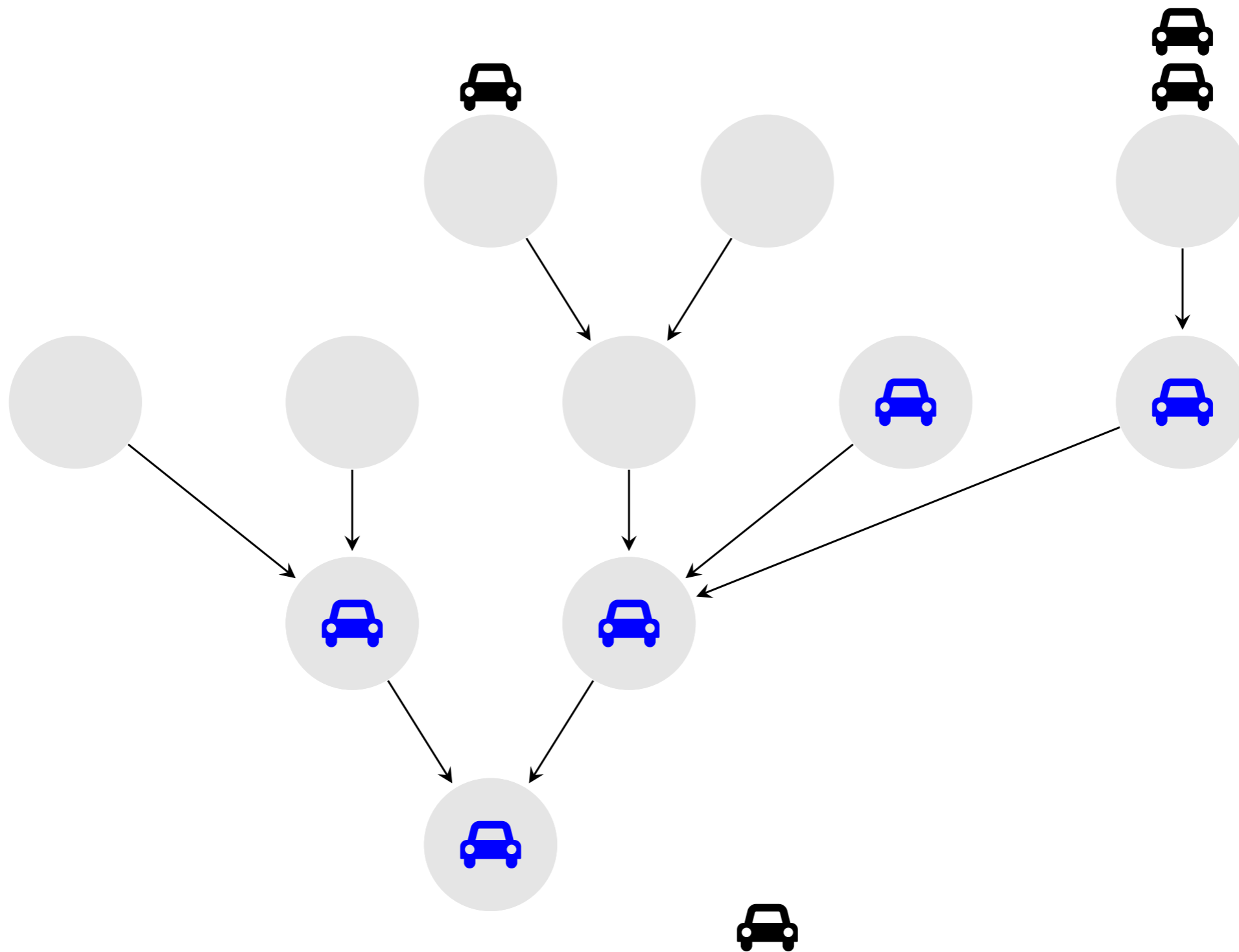
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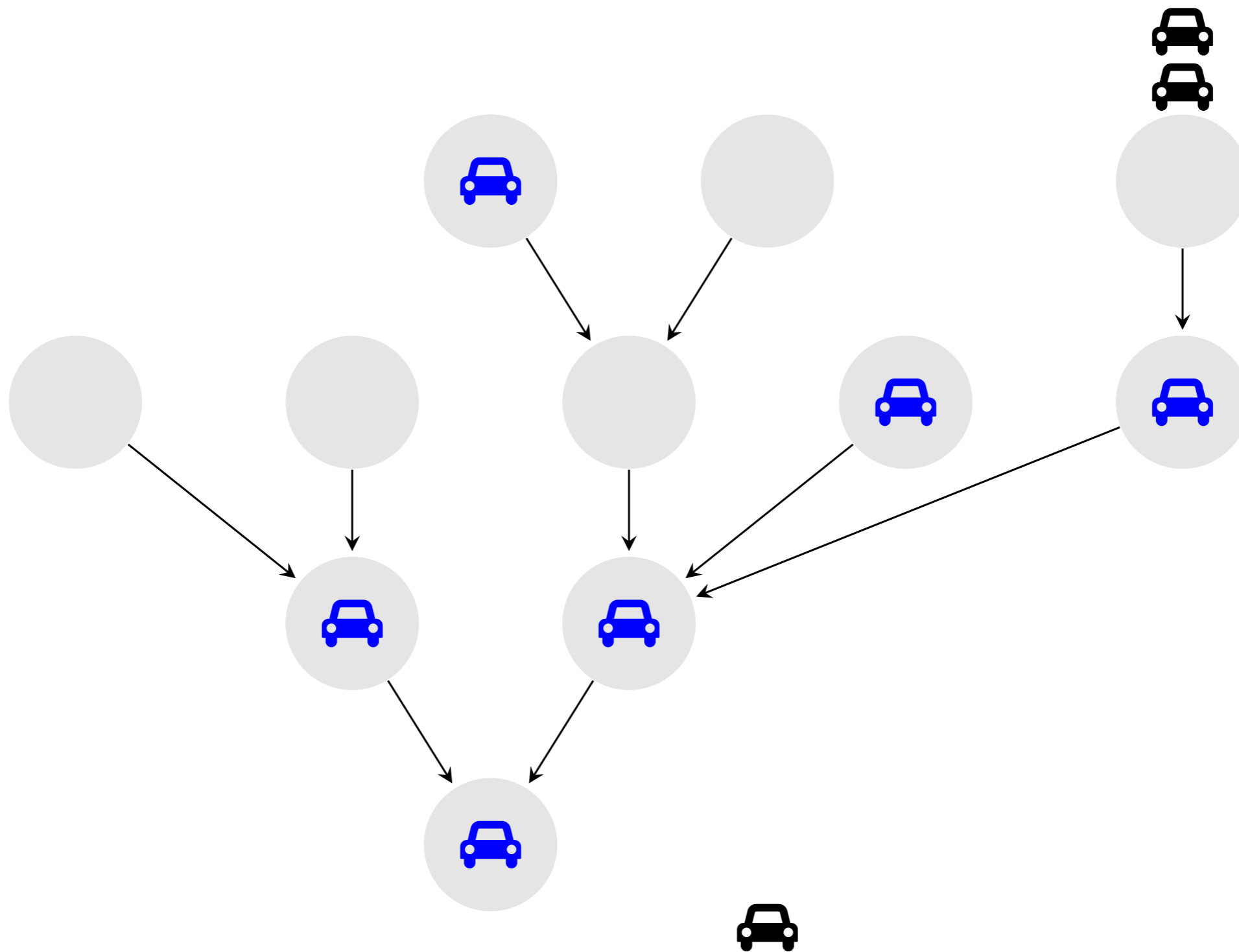
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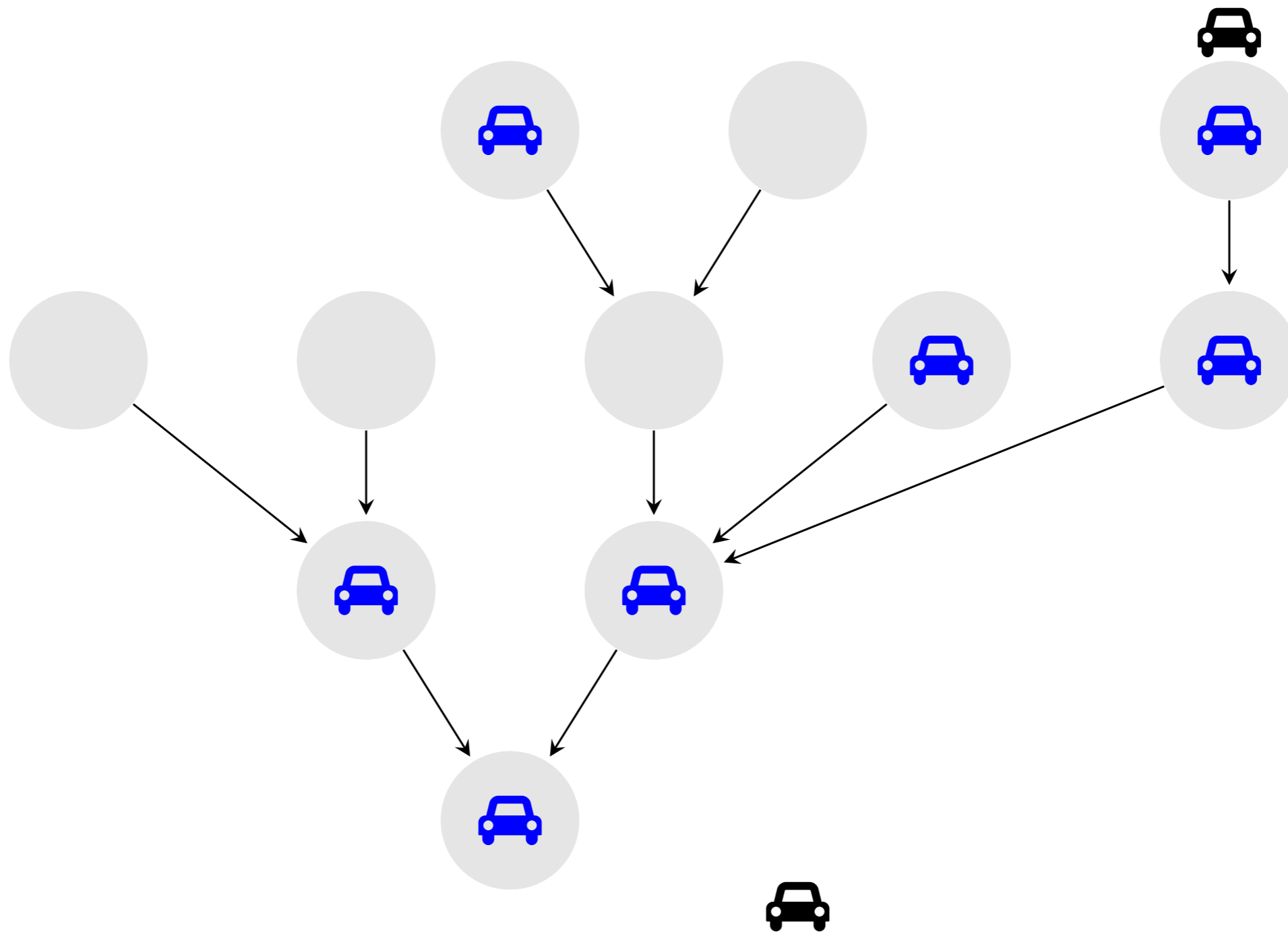
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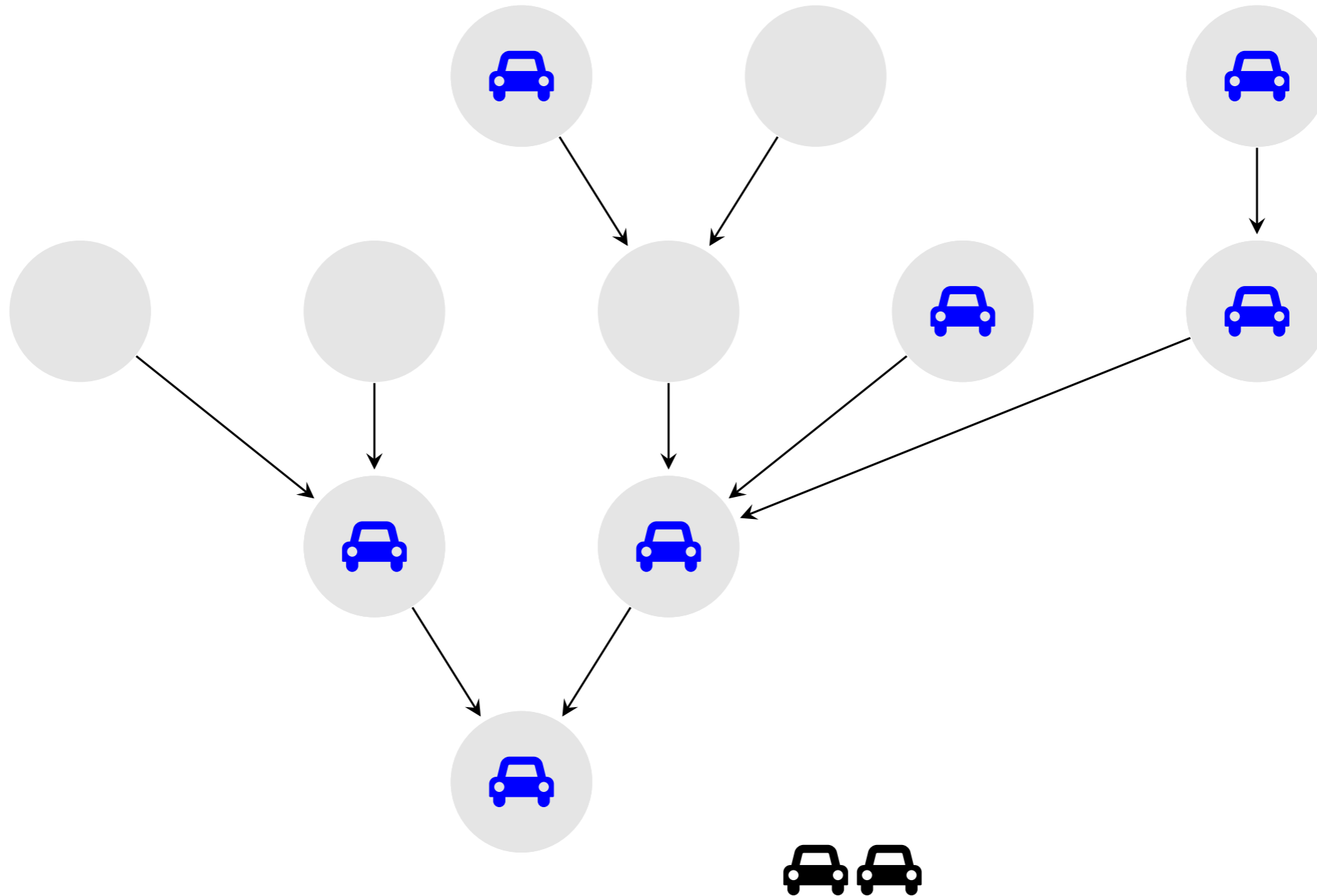
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General model

- ▶ First, fix a **rooted** tree t (deterministic or random, finite or infinite).

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Example and motivation: Family of laws $(\mu^\alpha : \alpha > 0)$ stochastically increasing with $\mathbb{E}[\mu^\alpha] = \alpha$.

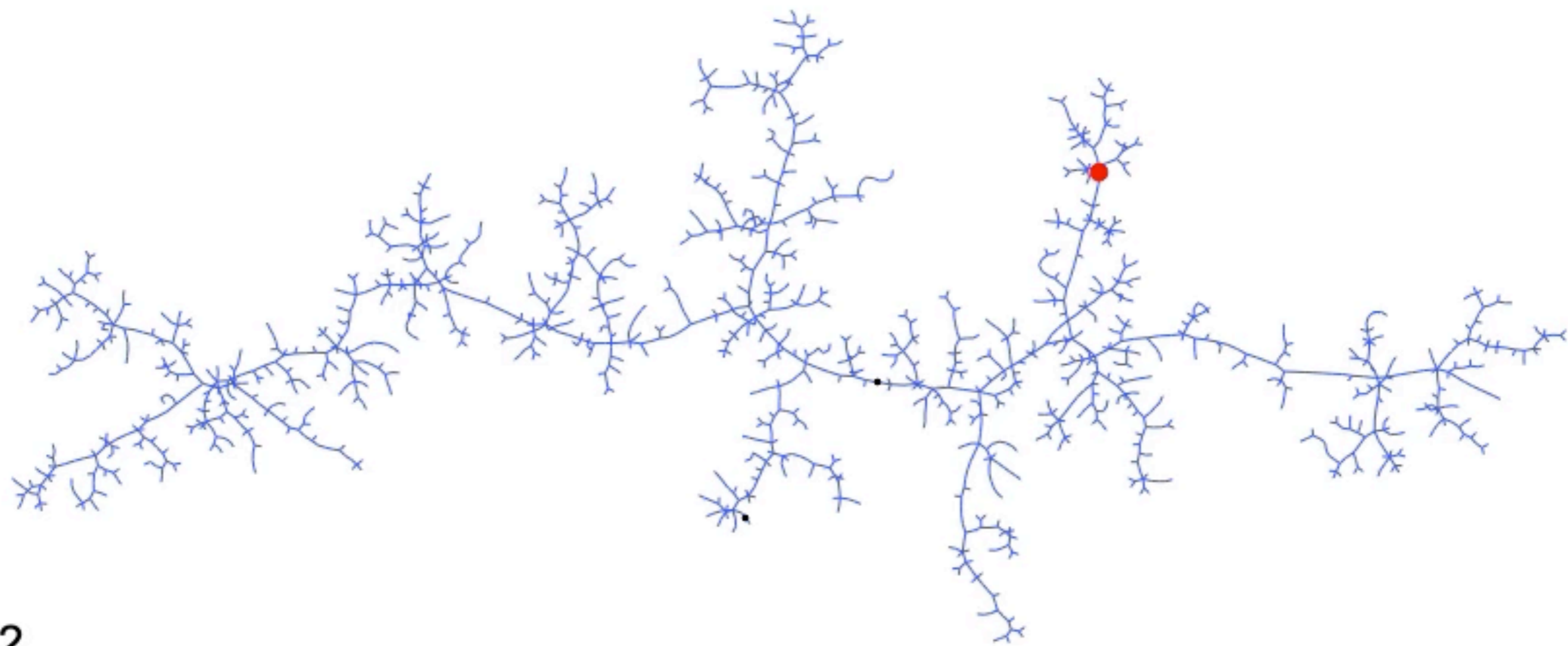
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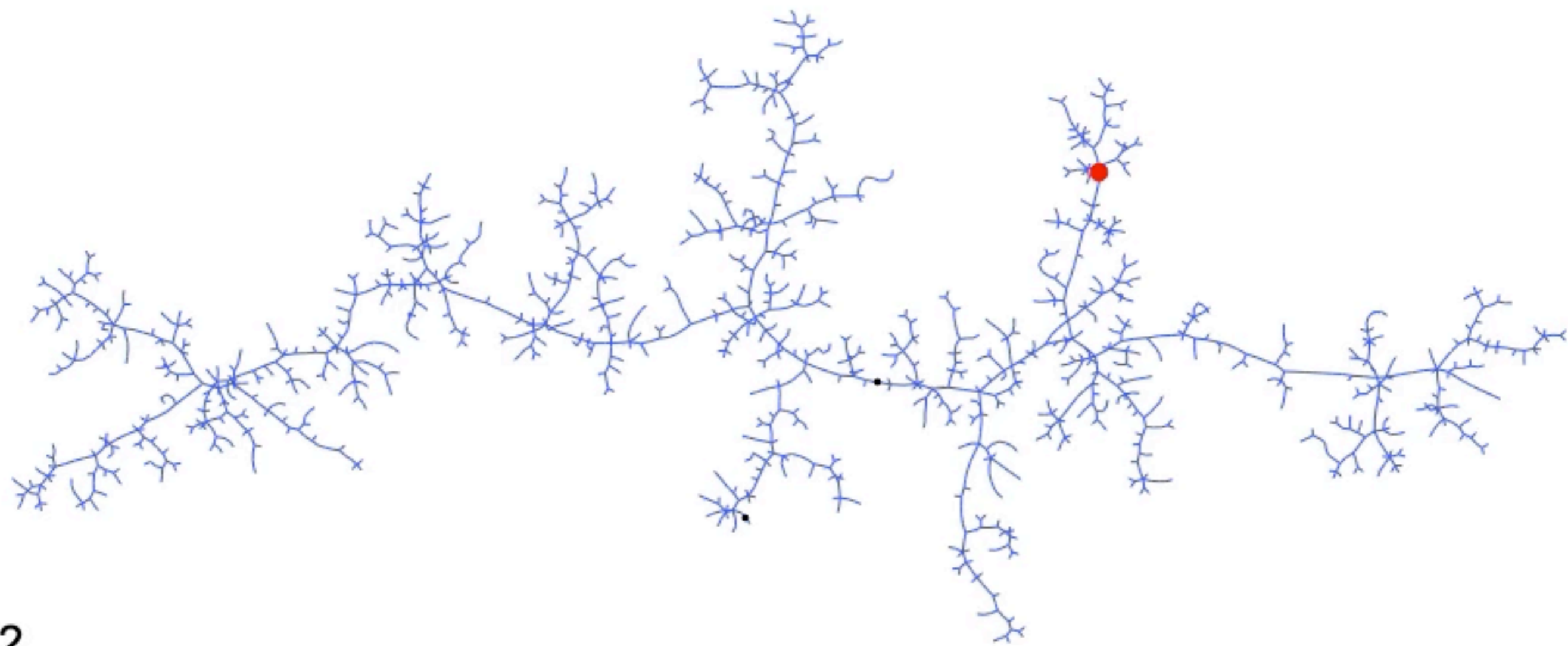
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→ Phase transition

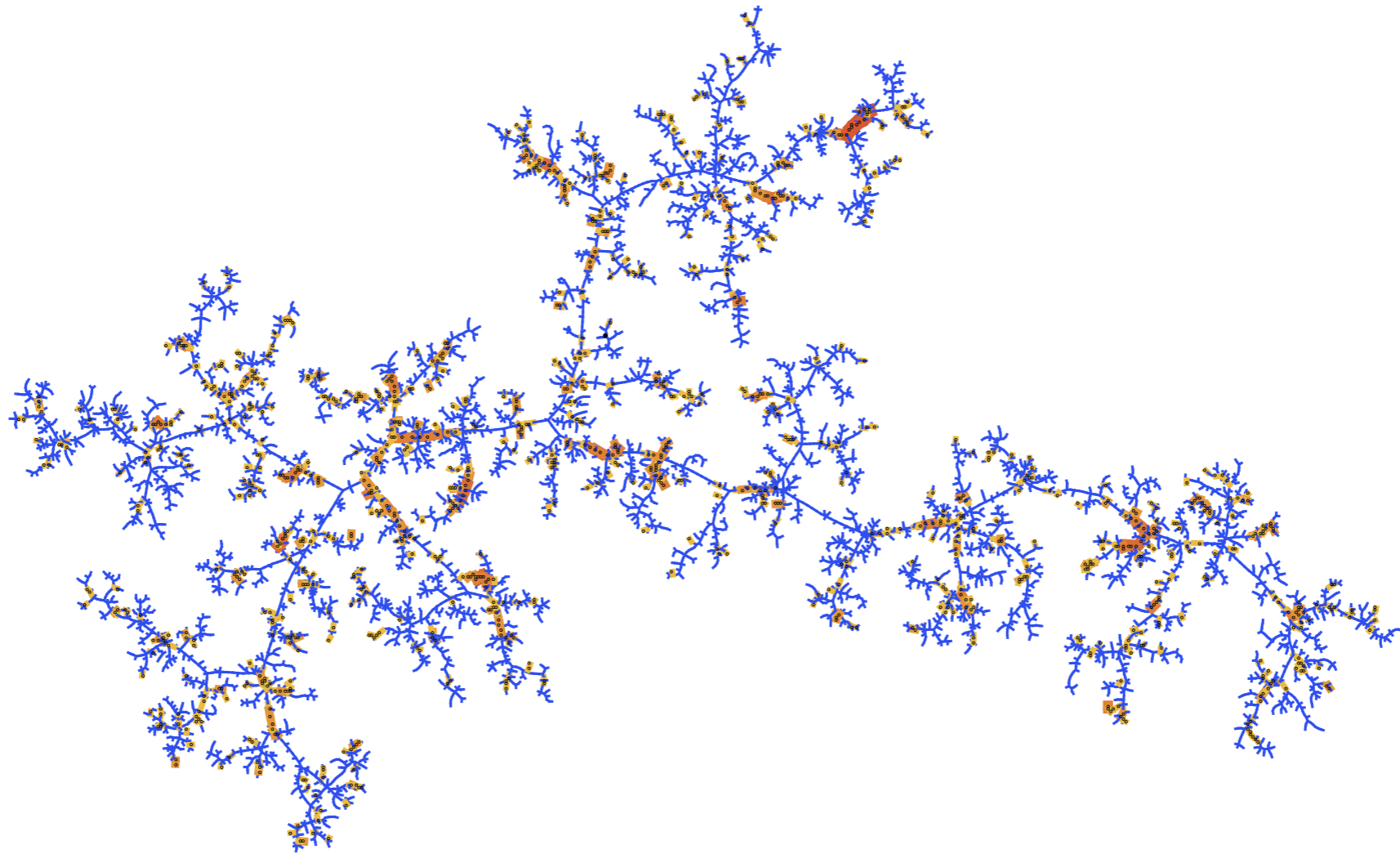


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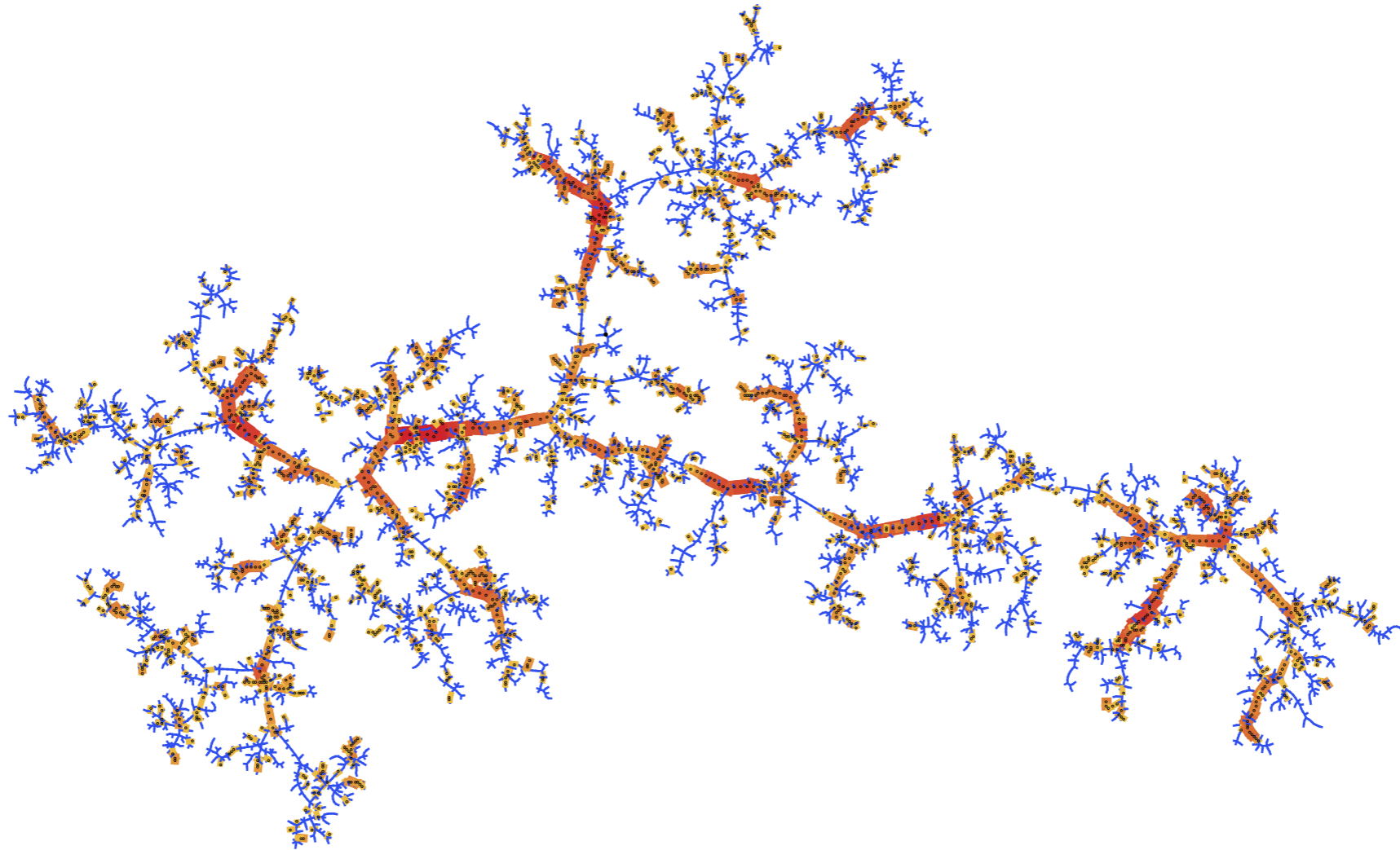
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Subcritical regime



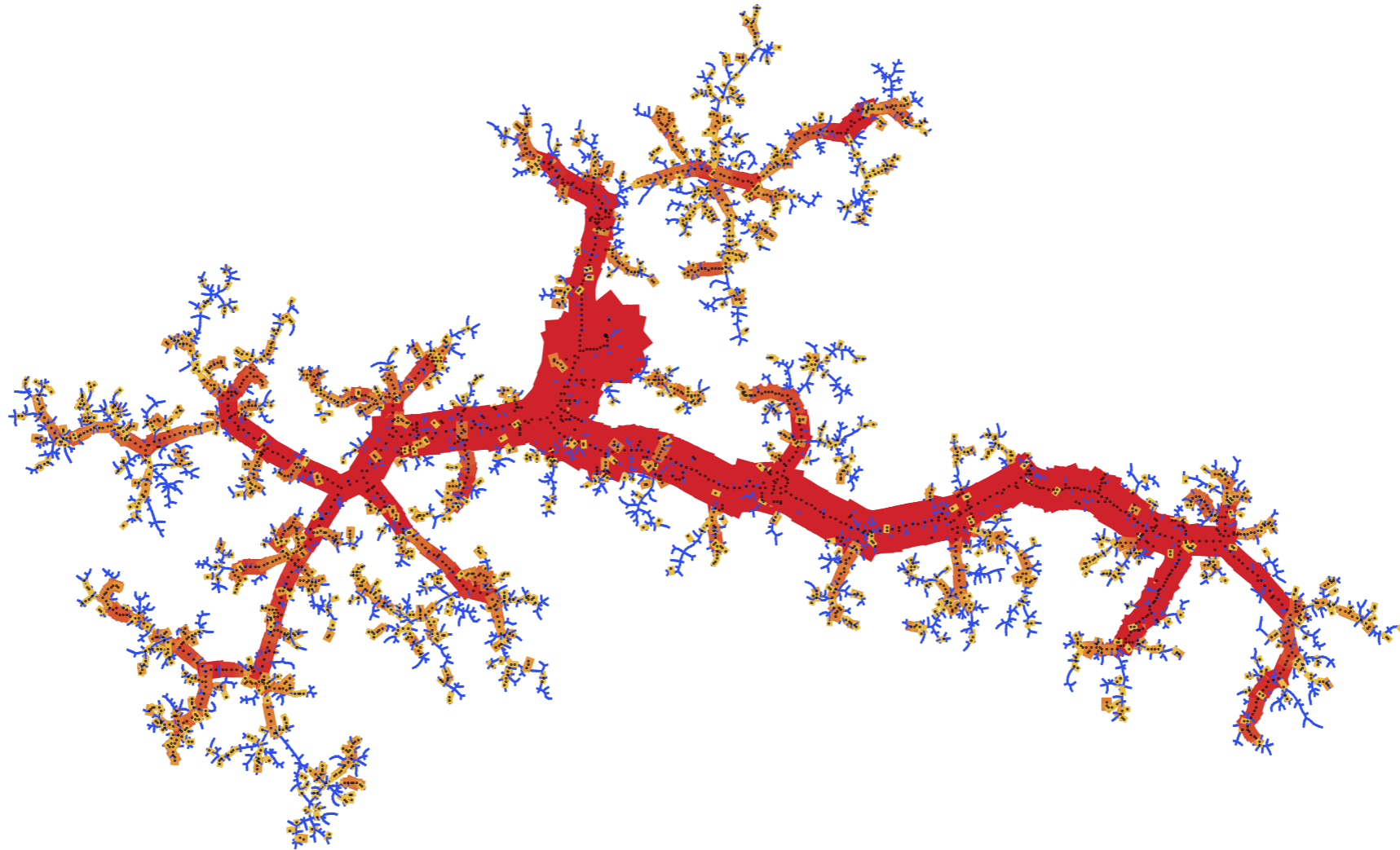
Flux of outgoing cars = $o_{\mathbb{P}}(n)$

Critical regime



Flux of outgoing cars $= o_{\mathbb{P}}(n)$

Supercritical regime



$$\text{Flux of outgoing cars} = (c + o_{\mathbb{P}}(1))n$$

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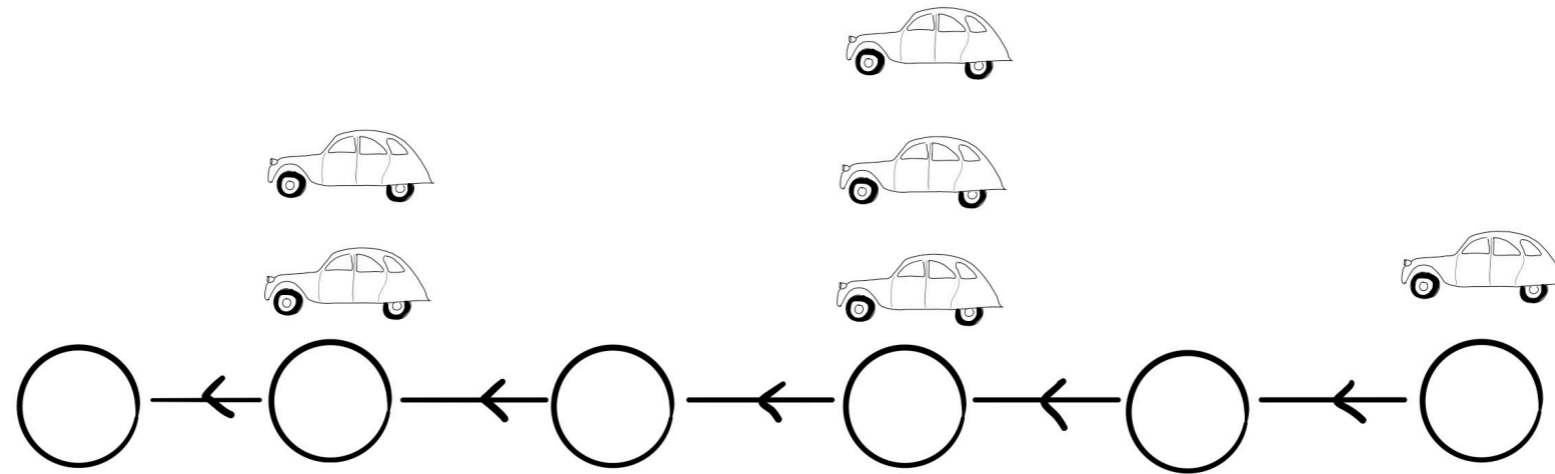
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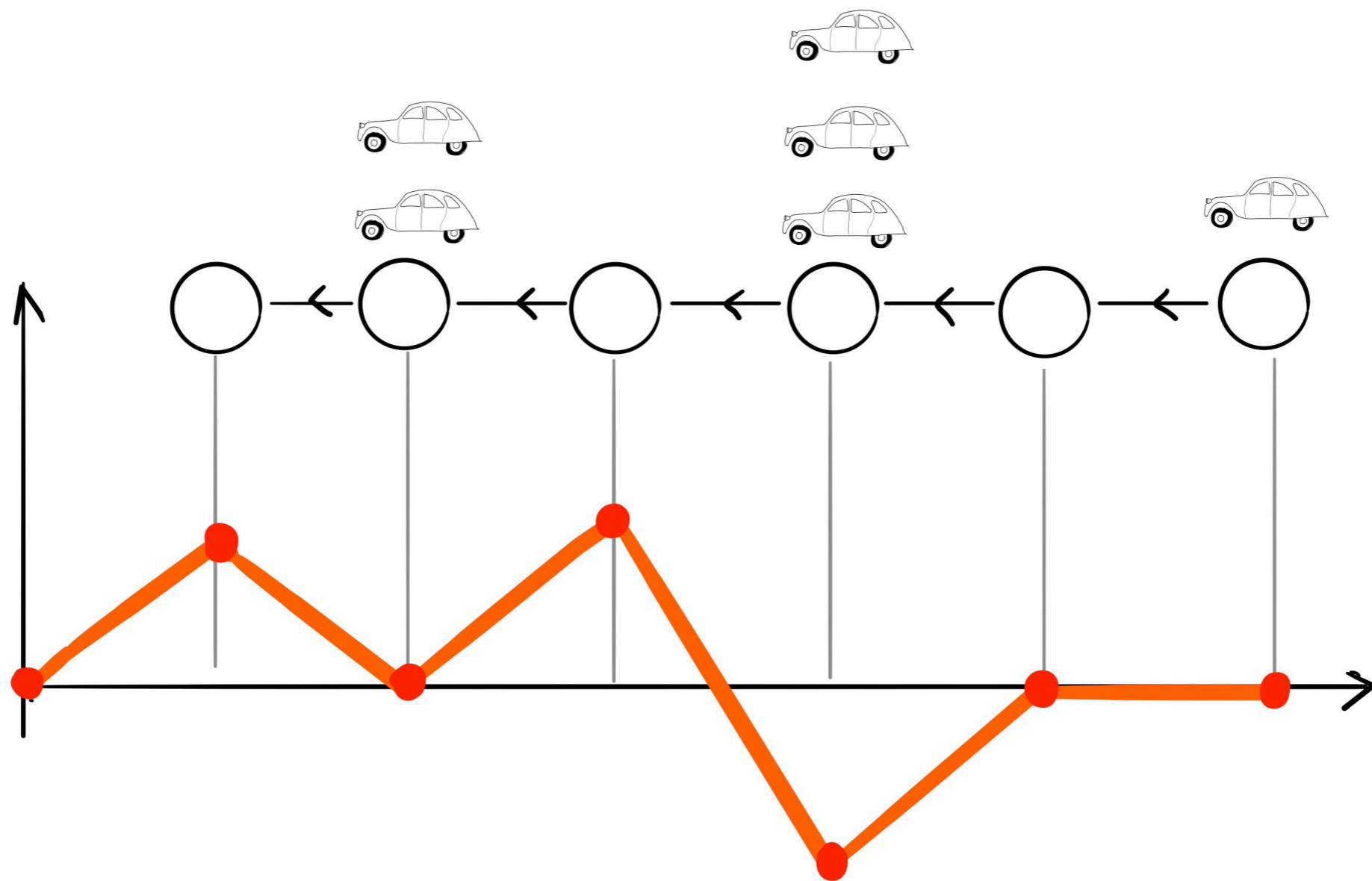
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We can define a “phase transition” for finite but large trees (see later).

Case of the line



Case of the line



“Trivial” phase transition: always at $\alpha = 1$ whatever the distribution.

And on random trees ?

- ▶ Fix $t = \mathcal{T}_n$ a Bienaymé–Galton–Watson tree conditioned to have n vertices with offspring distribution

$$\nu = \sum_{k=0}^{\infty} \nu_k \delta_k \text{ aperiodic with mean 1 and finite variance } \Sigma^2.$$

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- ▶ The car arrivals on each vertex are independent.
- ▶ The law of the car arrivals only depends on the degree of the vertex.

Phase transition on critical random trees

Building on [Curien, Hénard 2019]

Theorem (C. 2020)

We observe a phase transition which depends only on

$$\Theta = (1 - \alpha)^2 - \Sigma^2(\alpha + \alpha^2 - \sigma^2) \quad \text{or} \quad \Theta = \Theta(\Sigma^2, \dots)$$

	<i>subcritical</i> $\Theta > 0$	<i>critical</i> $\Theta = 0$	<i>supercritical</i> $\Theta < 0$
$\varphi(\mathcal{T}_n)$ when $n \rightarrow \infty$	<i>finite</i>	$o(n)$	$\sim cn$ with $c > 0$
$\mathbb{E}[\varphi(\mathcal{T})]$			
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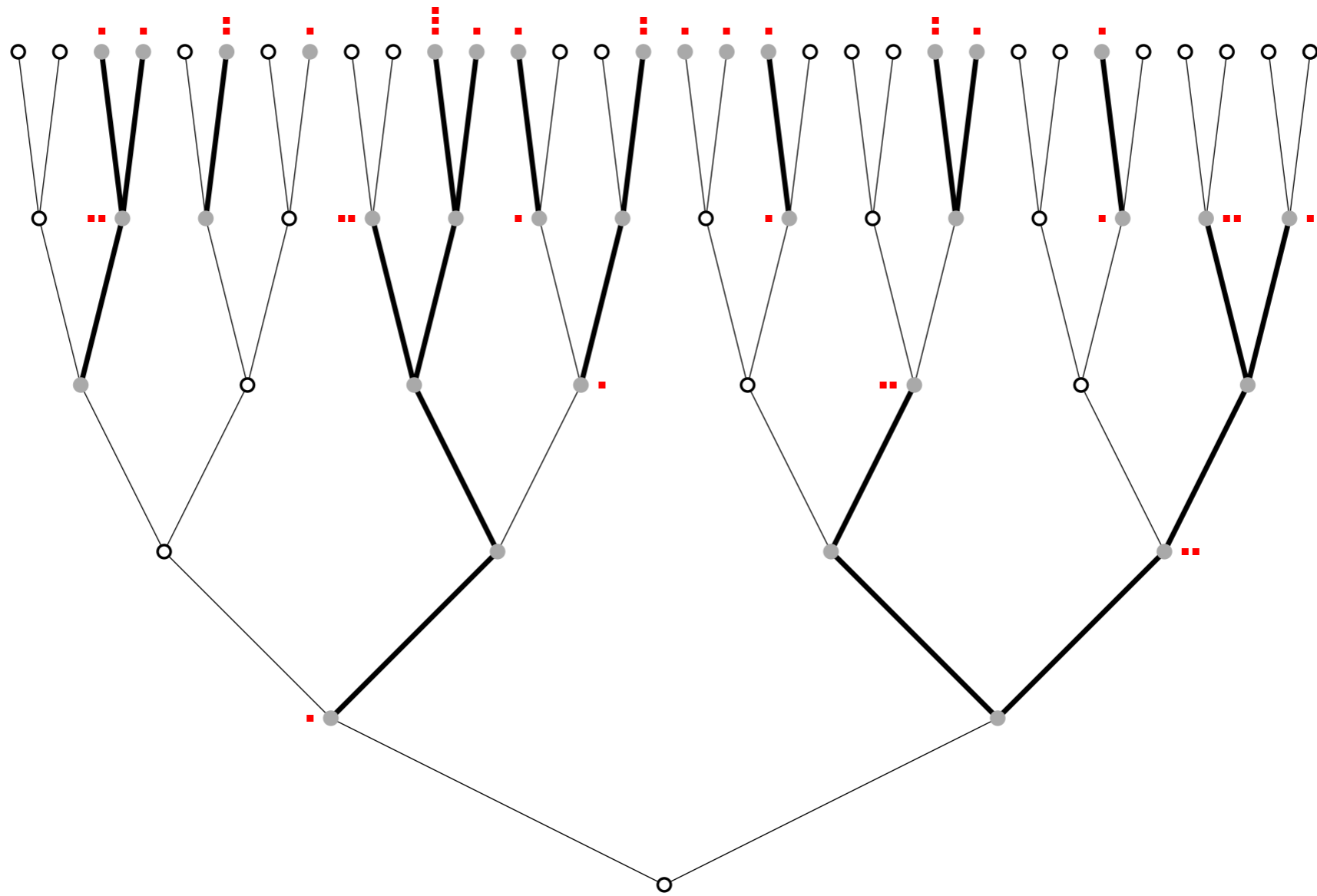
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$ C_{\max}(n) $ when $n \rightarrow \infty$	$\leq A \log(n)$?	$\sim Cn$ avec $C > 0$

And for SUPERCRITICAL
trees ?

Location of the transition in the binary case



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Take t the infinite binary tree. Let G be the generating function of the law μ of the car arrivals.

Theorem (Aldous, C., Curien, Hénard, 2022)

Suppose there exists

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In the generic situation, the time t_c exists.

Examples

Car arrivals	Critical value α_c
Binary 0/2 $\mu^\alpha = (1 - \frac{\alpha}{2})\delta_0 + \frac{\alpha}{2}\delta_2$	
Binary 0/k $\mu^\alpha = (1 - \frac{\alpha}{k})\delta_0 + \frac{\alpha}{k}\delta_k$	
Poisson $G_\alpha(t) = \exp(t(\alpha - 1))$	
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Geometric $G_\alpha(t) = \frac{1}{1 + \alpha - \alpha t}$	$\frac{1}{8}$

And for supercritical
RANDOM trees ?

- Consider a Bienaymé—Galton—Watson tree \mathcal{T} with geometric offspring distribution

$$\nu_q = \sum_{k=0}^{+\infty} q^k (1 - q) \delta_k$$

with $q > 1/2$.

- Again, we denote by X the number of outgoing cars.
 - ▶ **Subcritical** : X is almost surely finite.
 - ▶ **Supercritical** : X is infinite as soon as \mathcal{T} is infinite.

Sketch of proof: combinatorial decomposition

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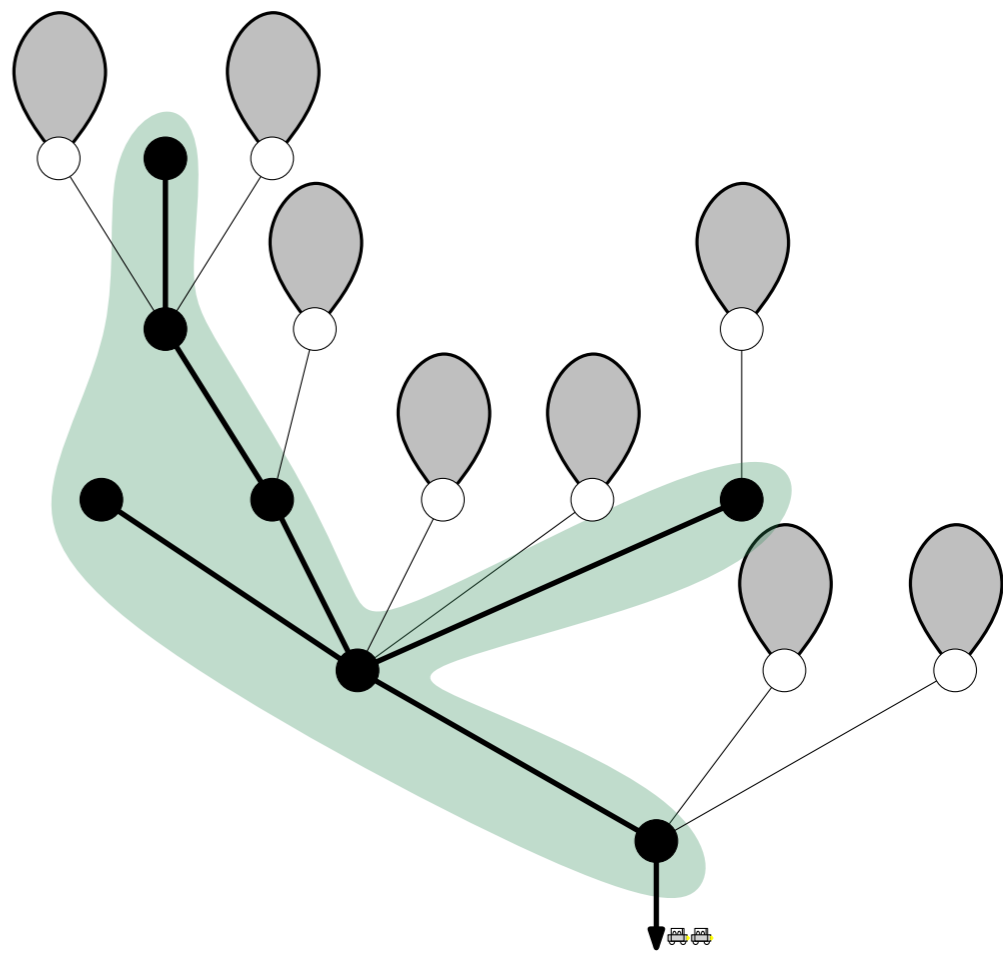
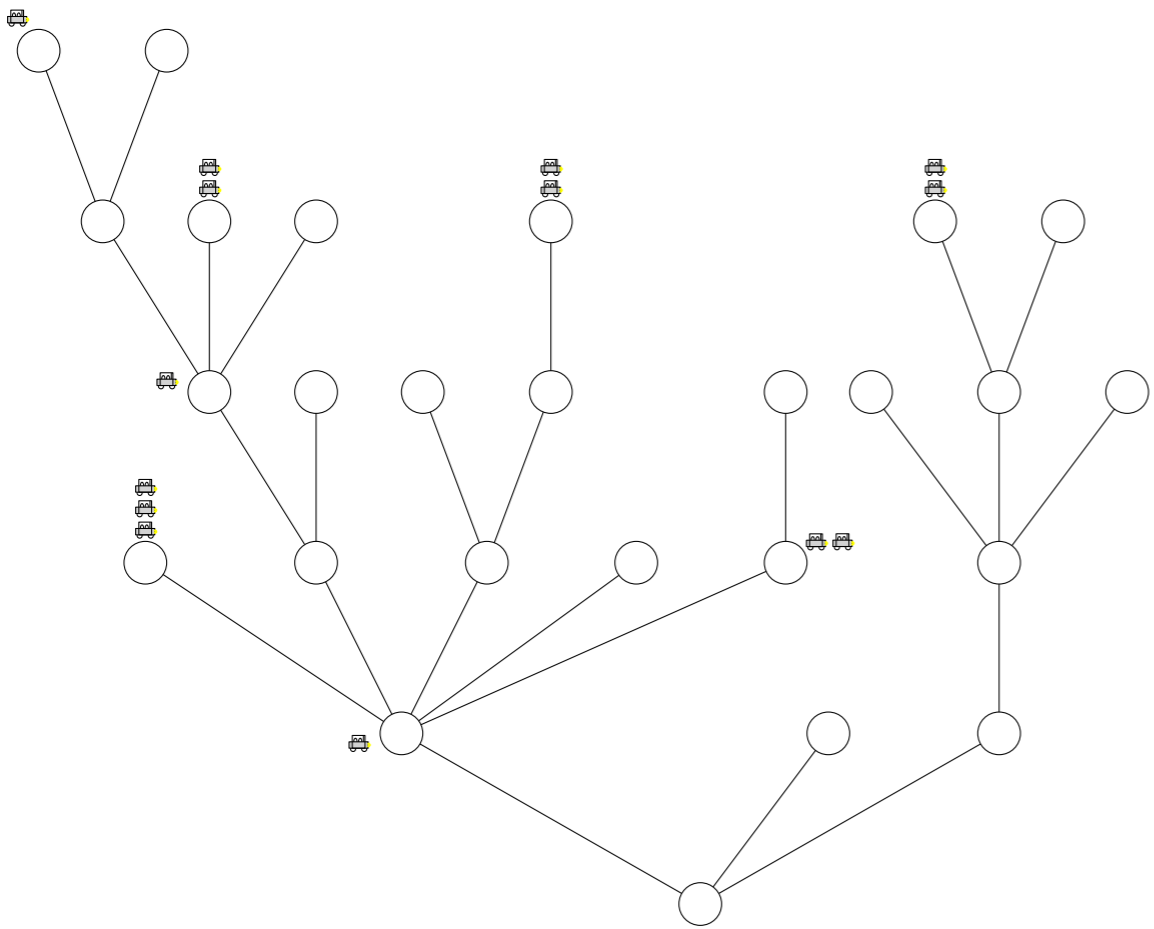
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We denote by

$$p_{\circ} = \mathbb{P}(\text{the root is empty}), \text{ and}$$

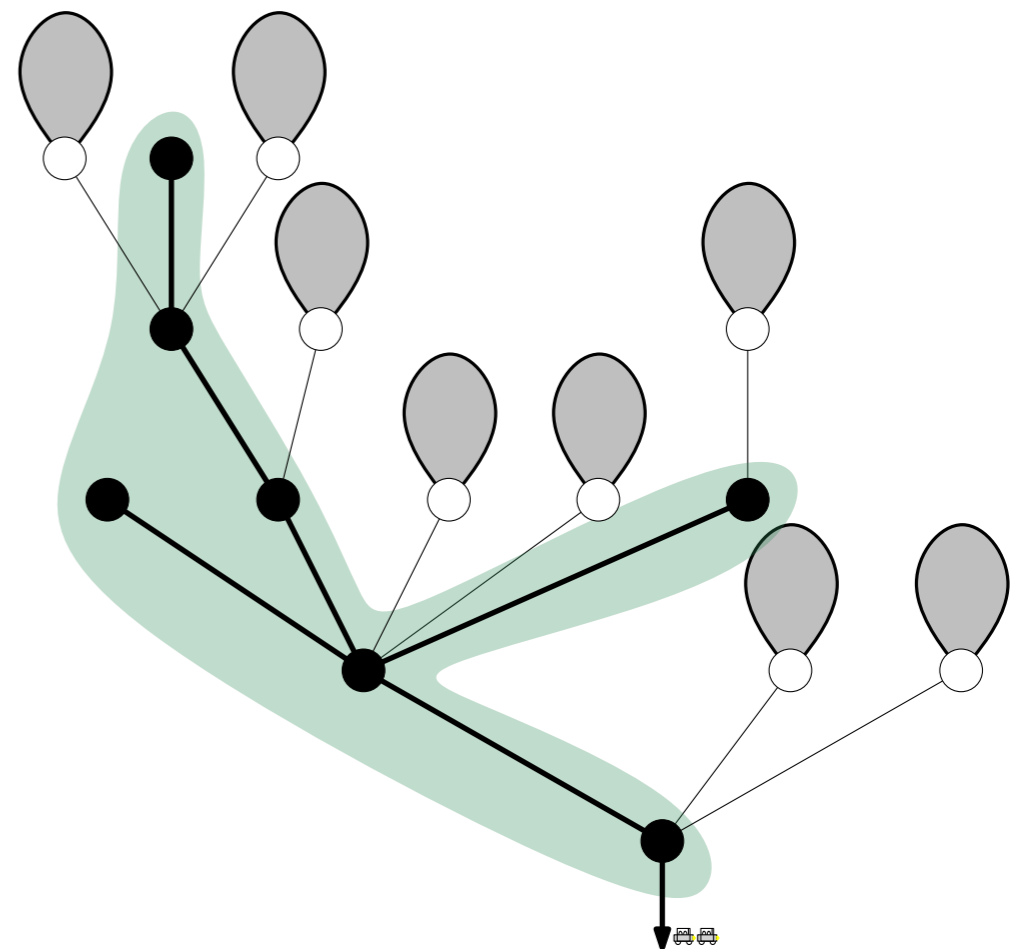
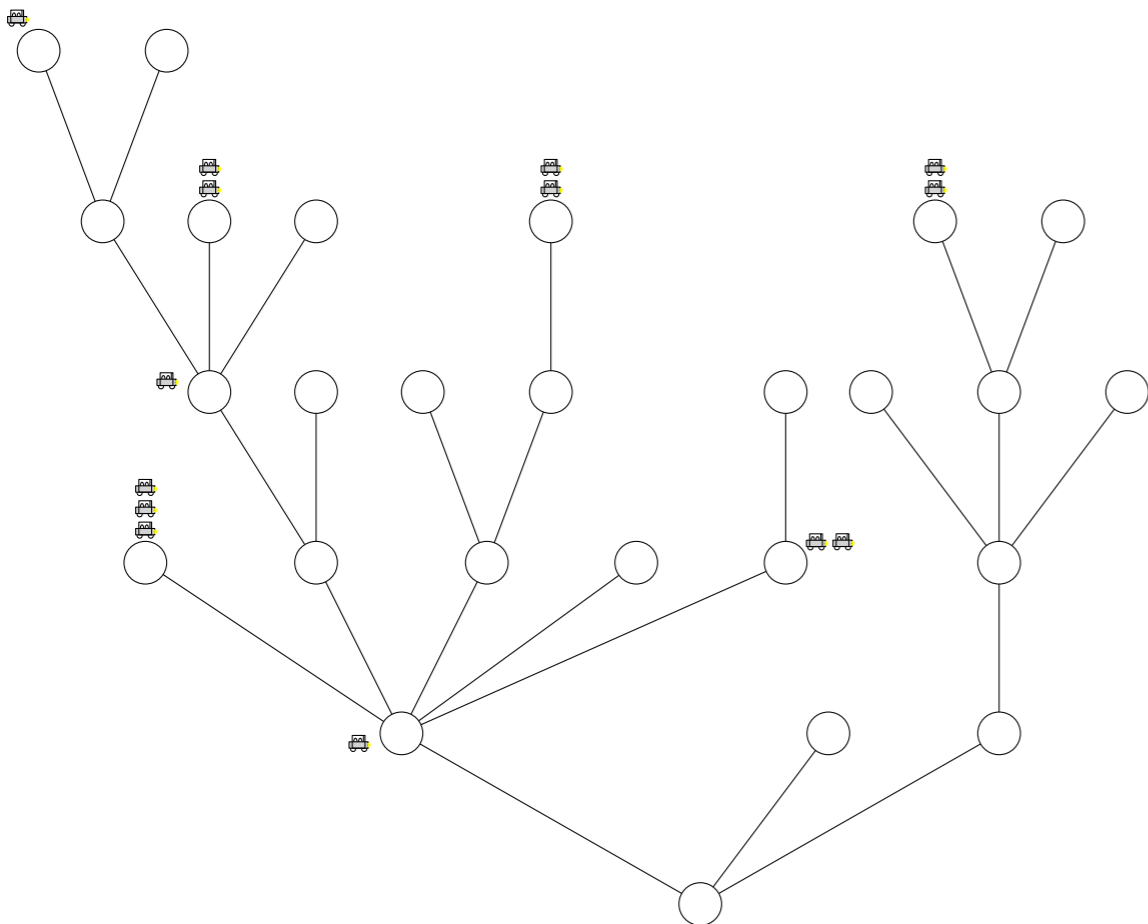
$$p_{\bullet} = \mathbb{P}(X = 0 \text{ and the root is parked}).$$

$$F(x, y) = \sum_{n \geq 0} \sum_{p \geq 0} \sum_{\mathbf{t} \in \text{FPT}_n^p} w(\mathbf{t}) x^n y^p$$



- We obtain a collection of equations:

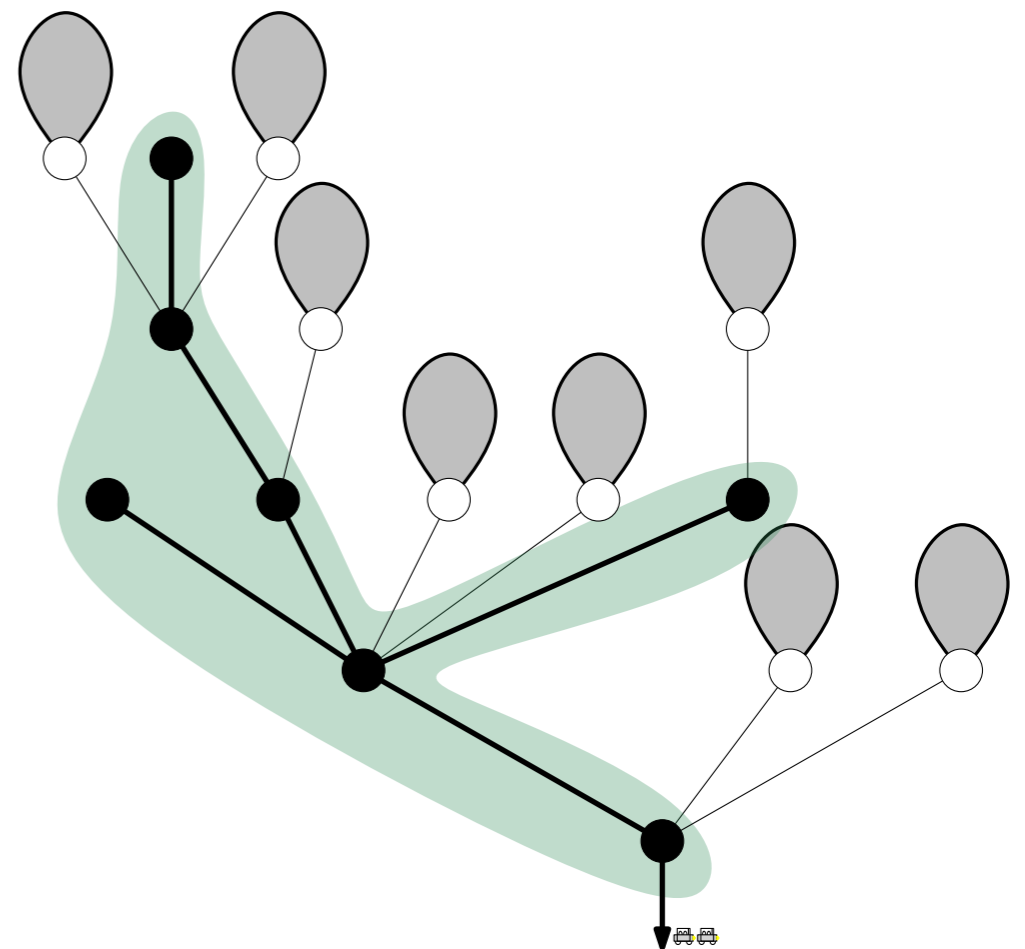
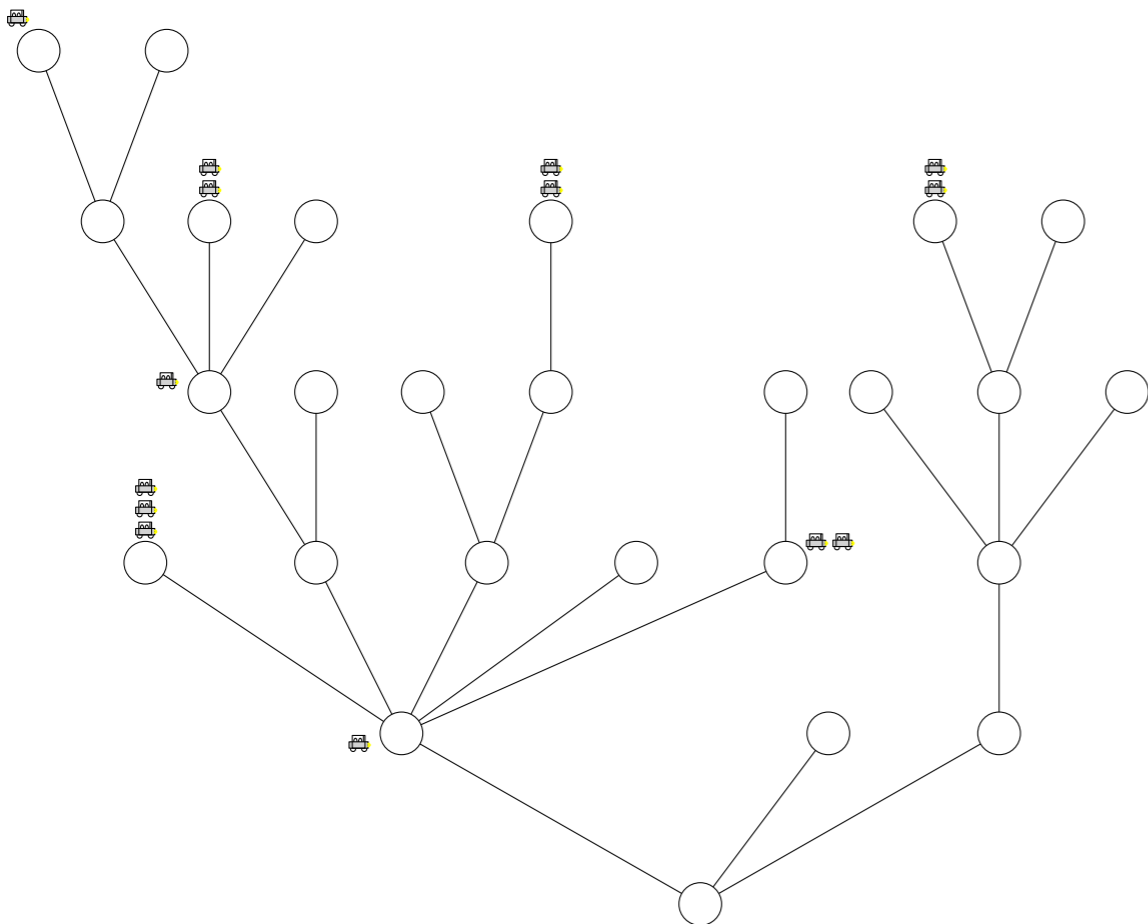
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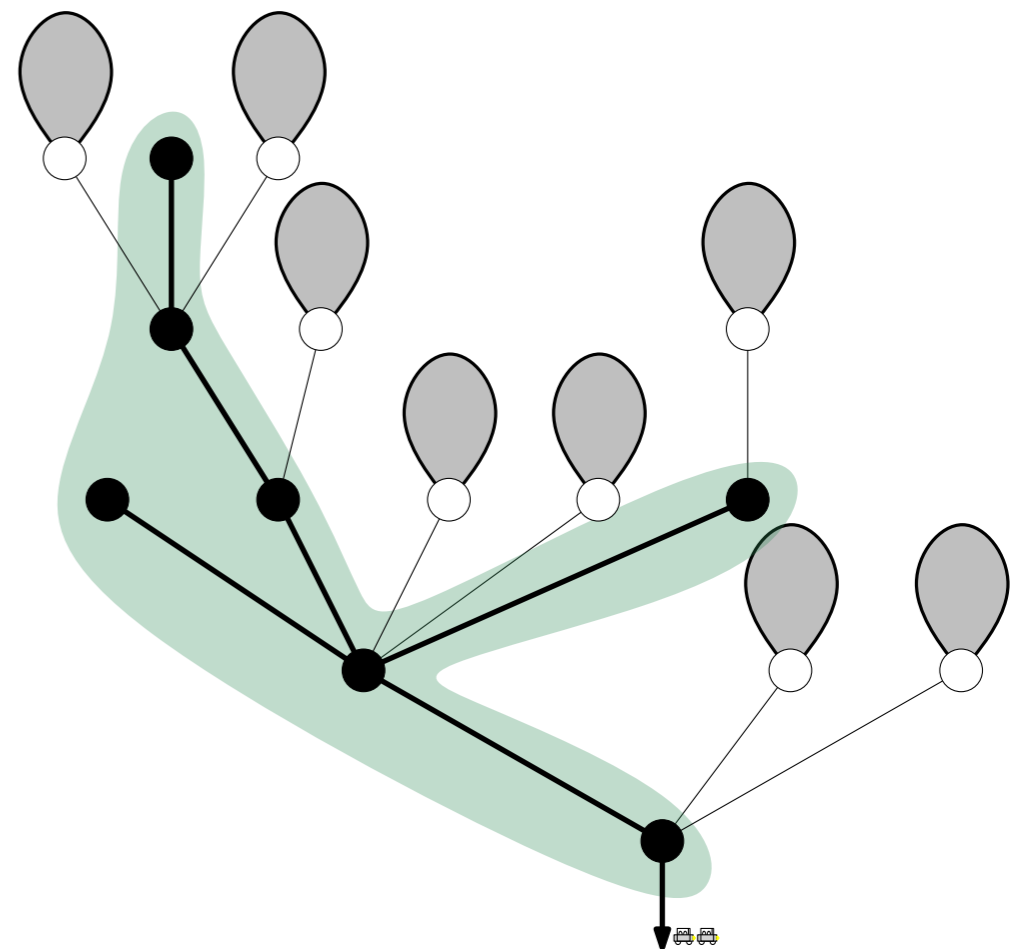
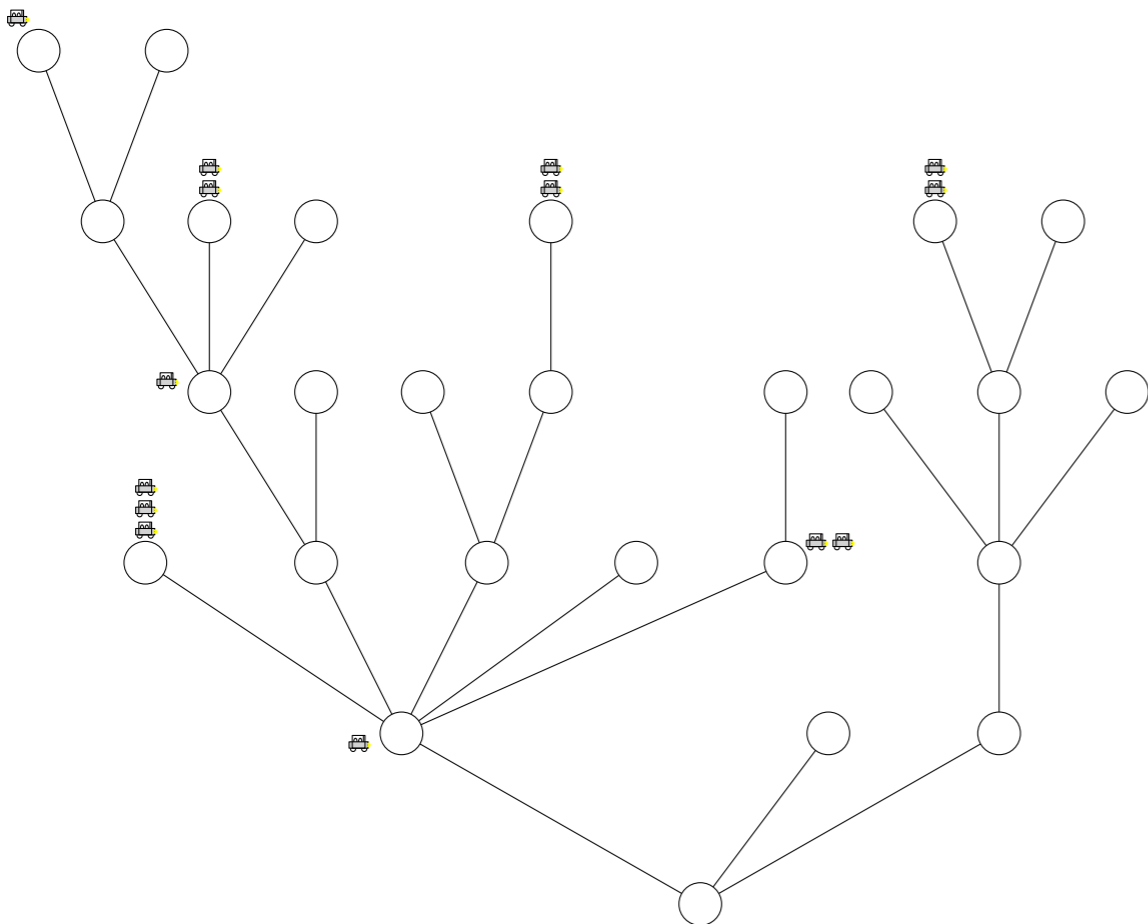
$$\forall k \geq 0, \quad \mathbb{P}(X = k + 1) = \frac{1 - qp_{\bullet}}{q} [y^k] F \left(\frac{q(1 - q)}{(1 - qp_{\bullet})^2}, y \right)$$



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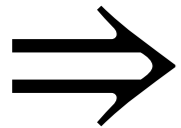
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- Solution: in the subcritical regime, we have $\mathbb{P}(X < \infty) = 1$

The parking process is subcritical if and only if there exists a positive solution p_0 to the equation

$$\frac{1 - qp}{q} \cdot F\left(\frac{q(1 - q)}{(1 - qp)^2}, 1\right) + p = 1$$

Theorem (Chen, C., 2024)

Suppose that there exists t_c such that

$$t_c := \inf\{t > 0, (G(t) - tG'(t))^2 = 2t^2 G(t)G''(t)\}.$$

Then the parking process is subcritical if and only if

$$t_c > 1 \quad \text{and} \quad \frac{t_c G(t_c)}{\varphi(t_c)^2} \leq q(1 - q),$$

where $\varphi(y) = (y + 1)G(y) - y(y - 1)G'(y)$.

Thank you for
your attention !

