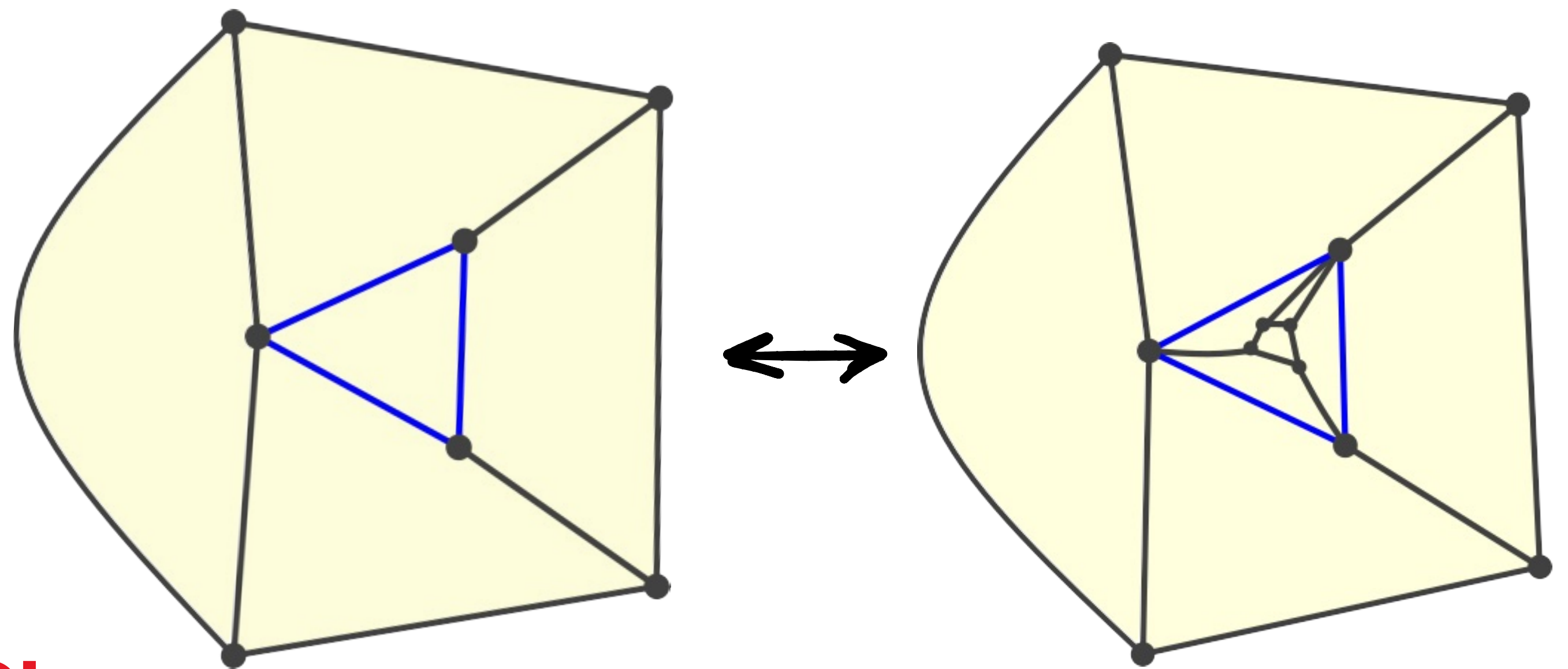


# Irreducible metric maps via substitution

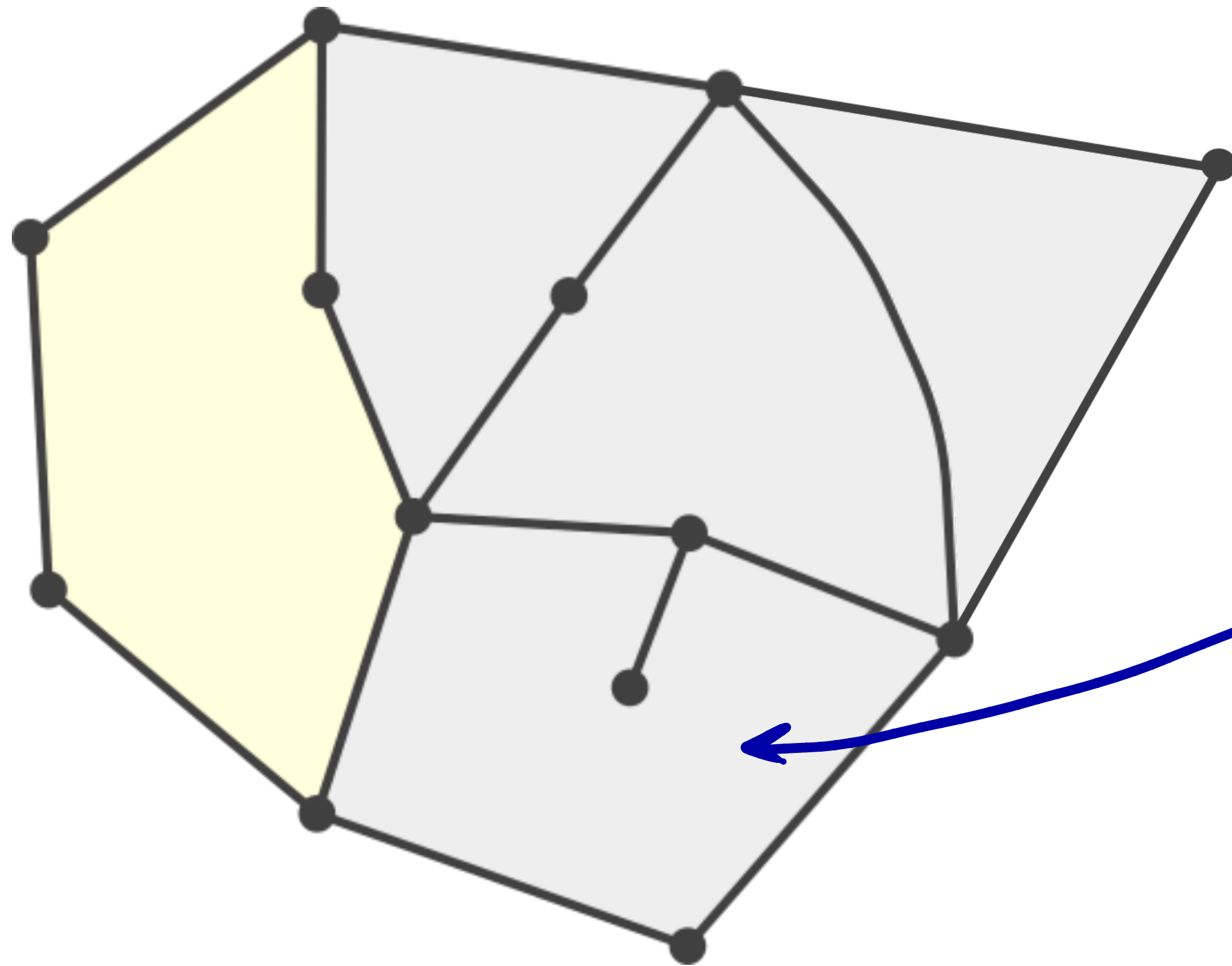


Alicia Castro



**What is an irreducible metric map?**

# What is an ~~irreducible metric~~ map?



$m$

embedded graph

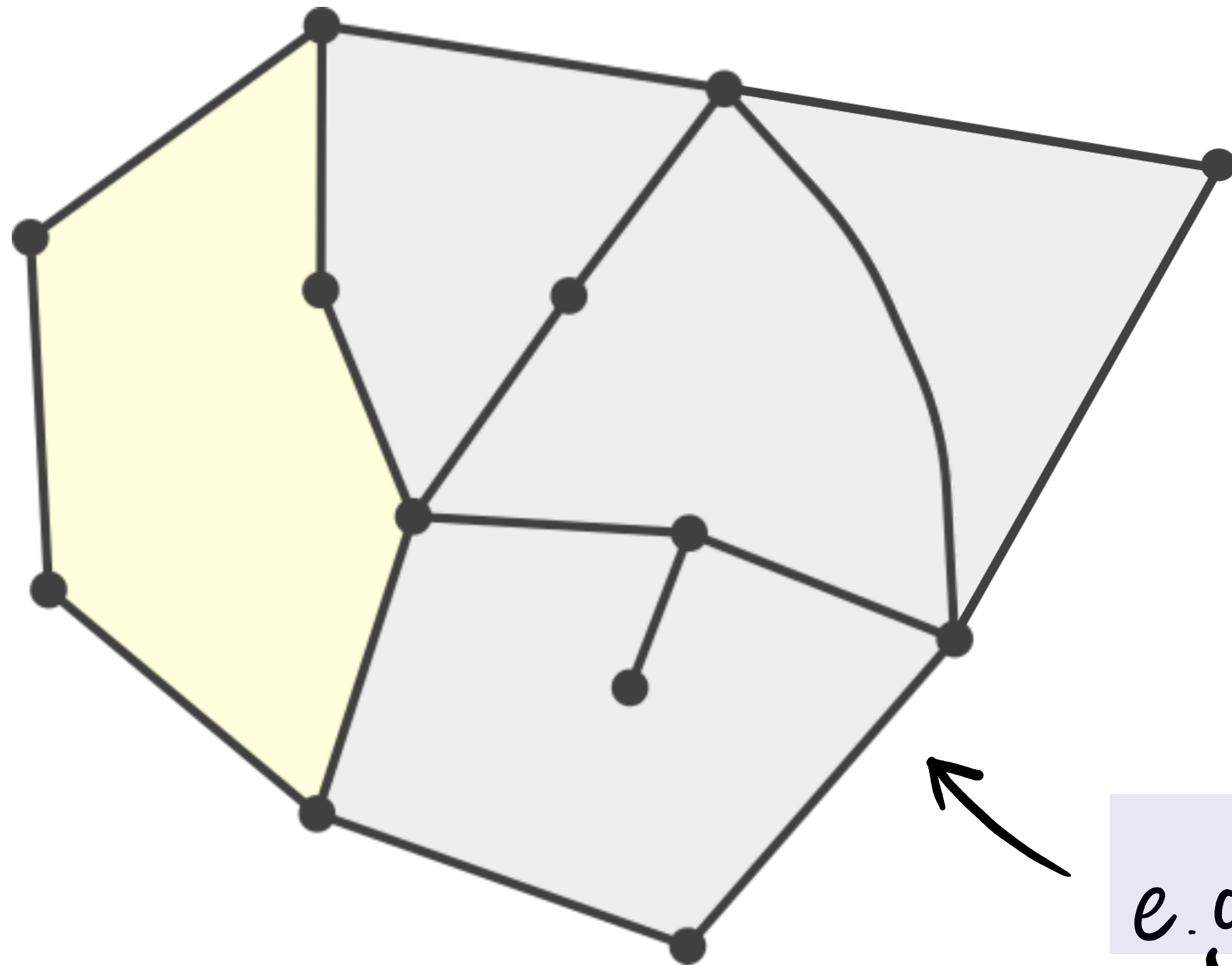
face = disk

length of face = degree of face

Euler characteristic

	# of vertices	# of faces	# of edges
	$ V(m) $	$ F(m) $	$ E(m) $
$2 - 2g - k =$	$ V(m) $	$+  F(m) $	$-  E(m) $

# What is an ~~irreducible metric~~ map?



$\mathfrak{m}$

embedded graph

face = disk

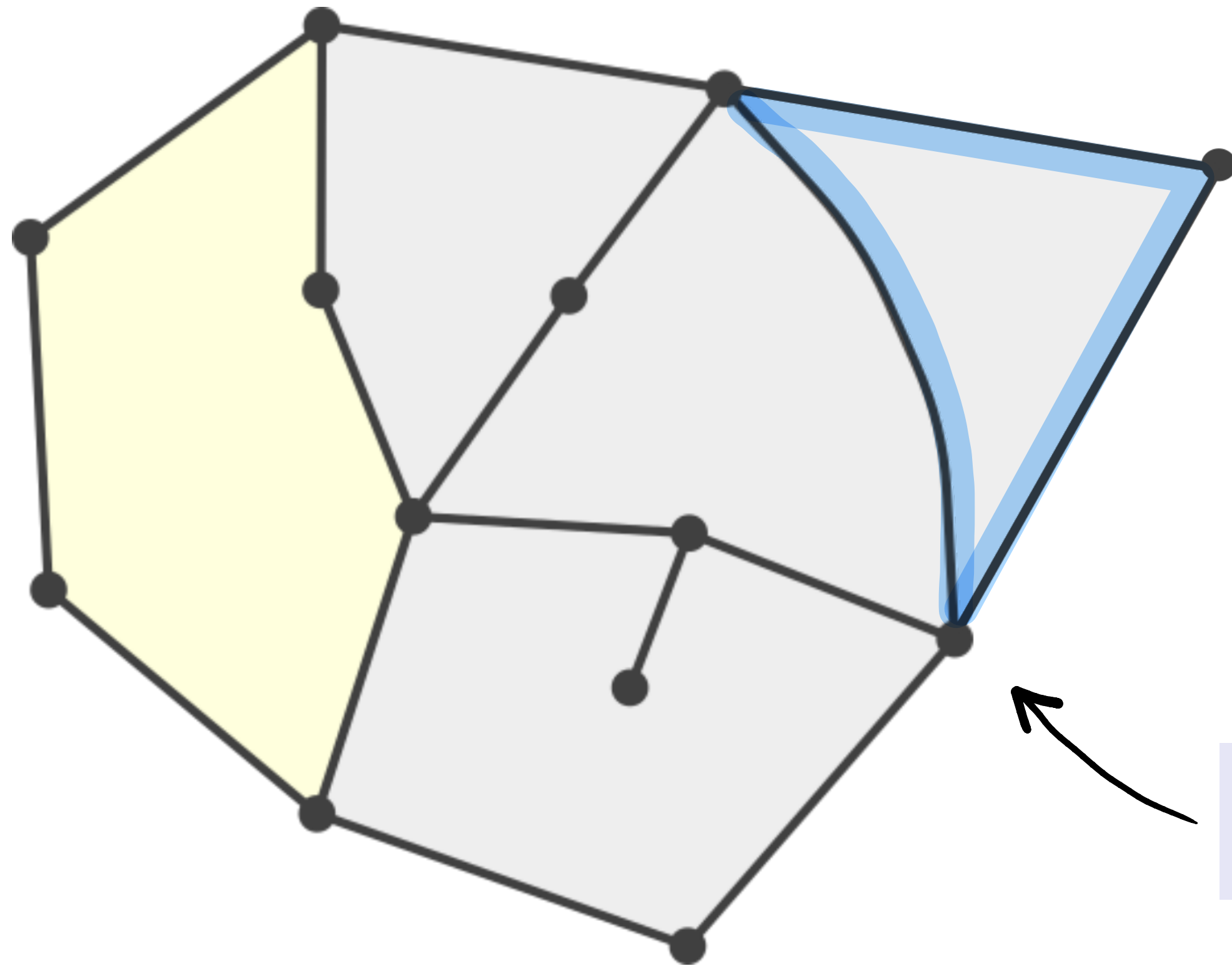
length of face = degree

e.g.  $g=0$   $k=1$   $\rightarrow$  disk w/ perimeter = 8

$$2 - 2g - k = |V(\mathfrak{m})| + |F(\mathfrak{m})| - |E(\mathfrak{m})|$$



# What is an ~~irreducible metric~~ map?



$\mathfrak{m}$

girth = length of the shortest closed path

e.g.  $\text{girth}(\mathfrak{m}) = 3$

$$2 - 2g - k = |V(\mathfrak{m})| + |F(\mathfrak{m})| - |E(\mathfrak{m})|$$

# What is an irreducible ~~metric~~ map?

Bouttier  
Quitter '13

A map is  $q$ -irreducible

if its girth  $\geq q$

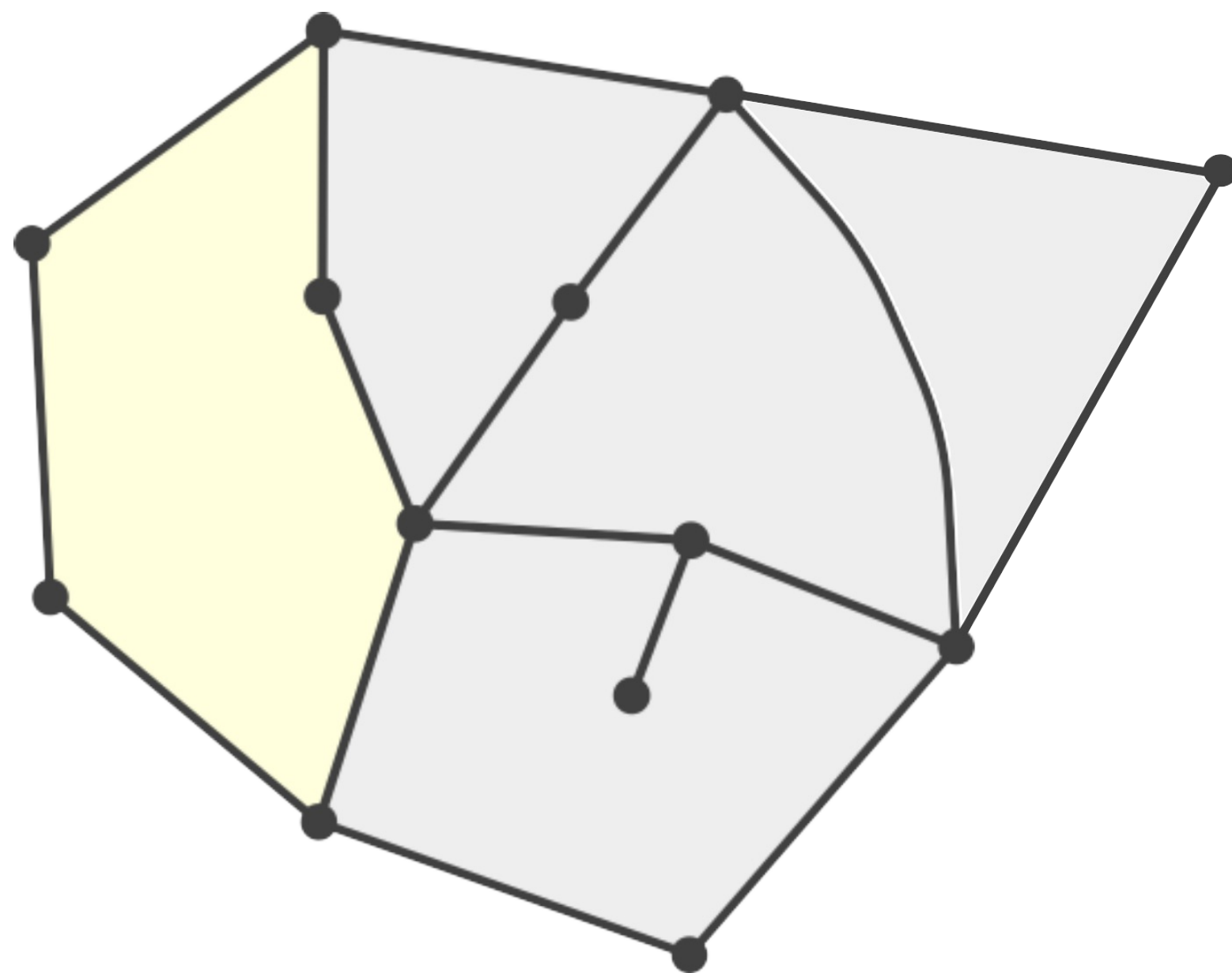
and the closed paths of length  $= q$  are faces

# What is an irreducible ~~metric~~ map?

A map is  $q$ -irreducible

if its girth  $\geq q$

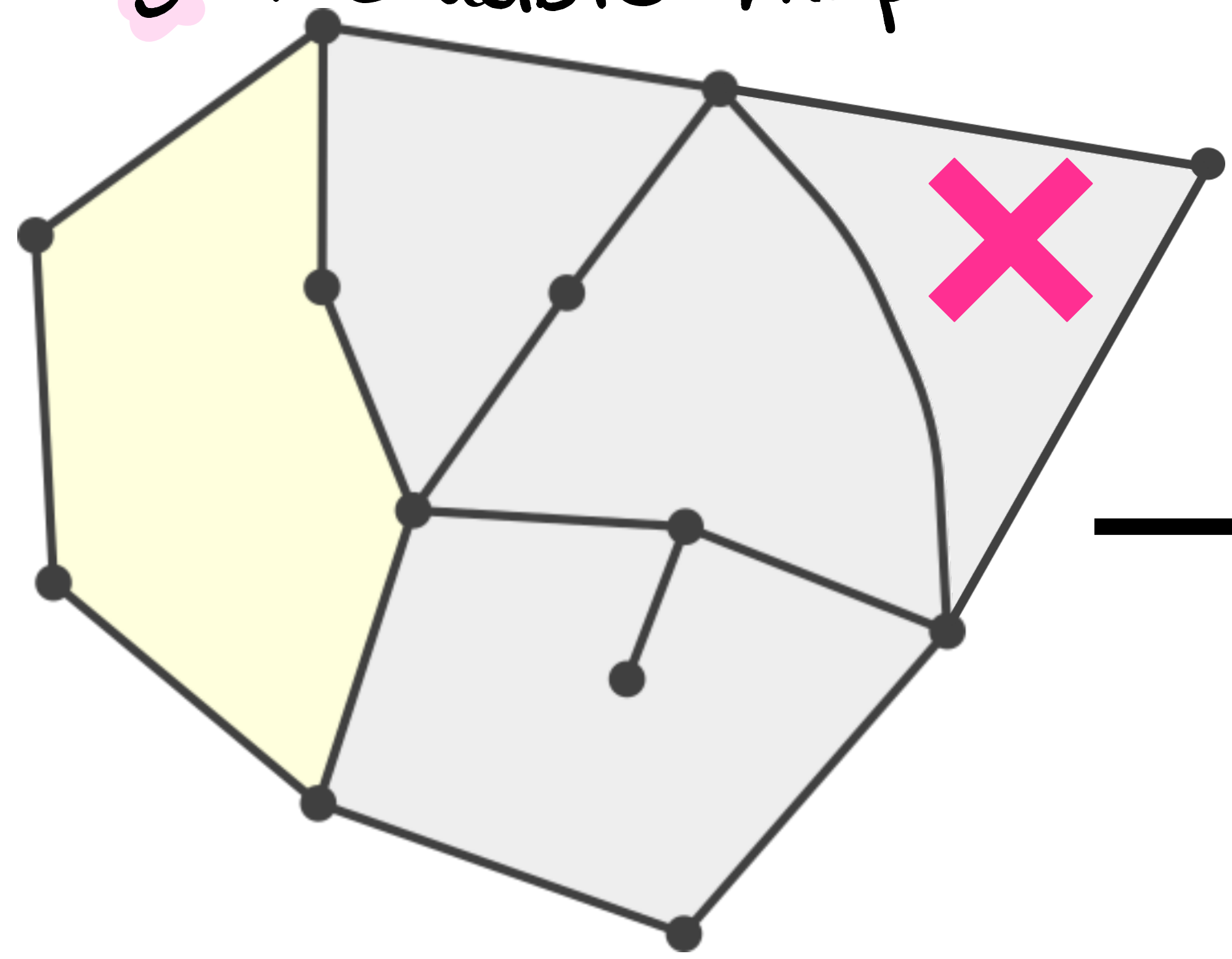
and the closed paths of length  $= q$  are faces



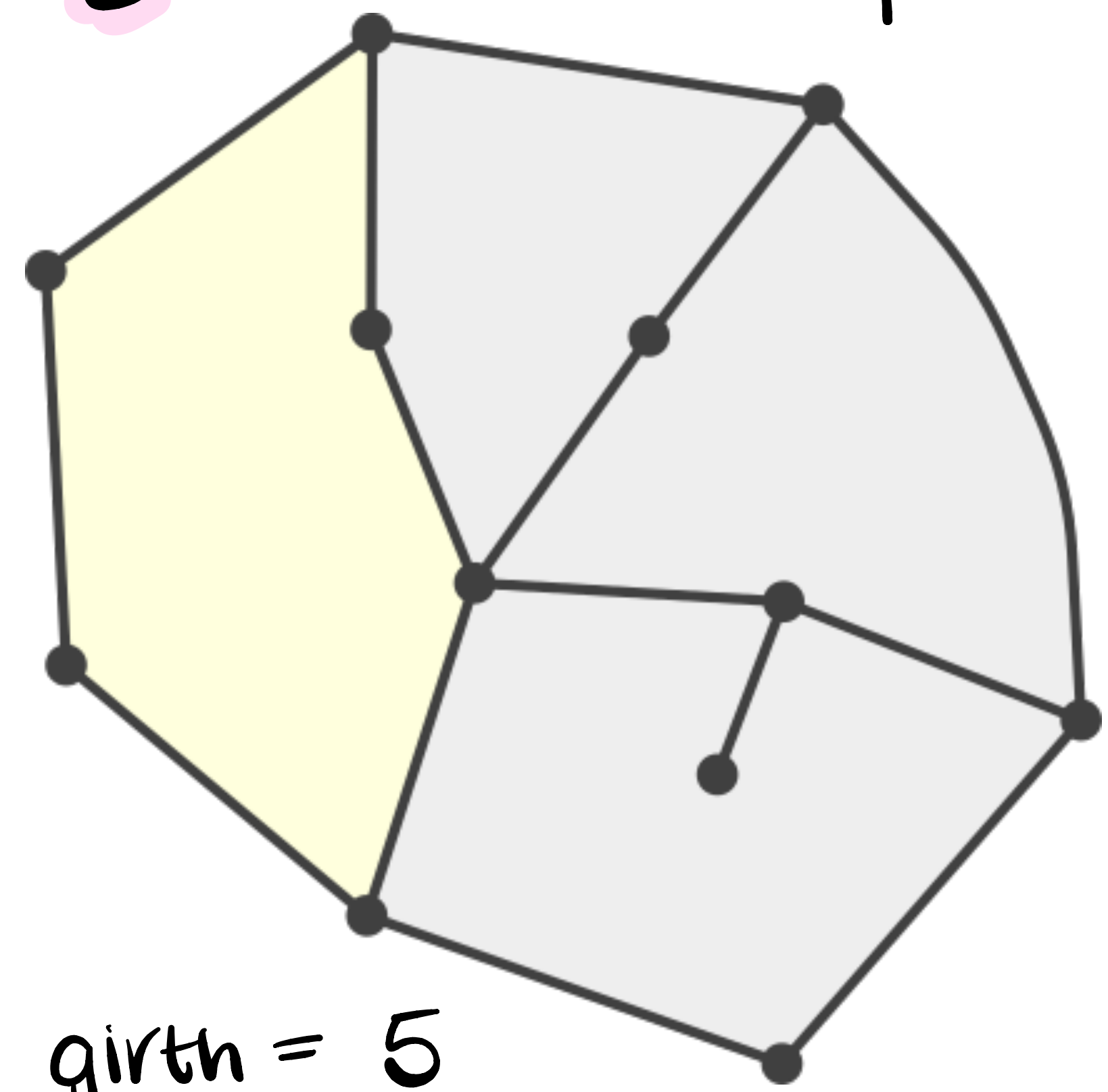
← e.g. 3-irreducible map

# What is an irreducible ~~metric~~ map?

3-irreducible map



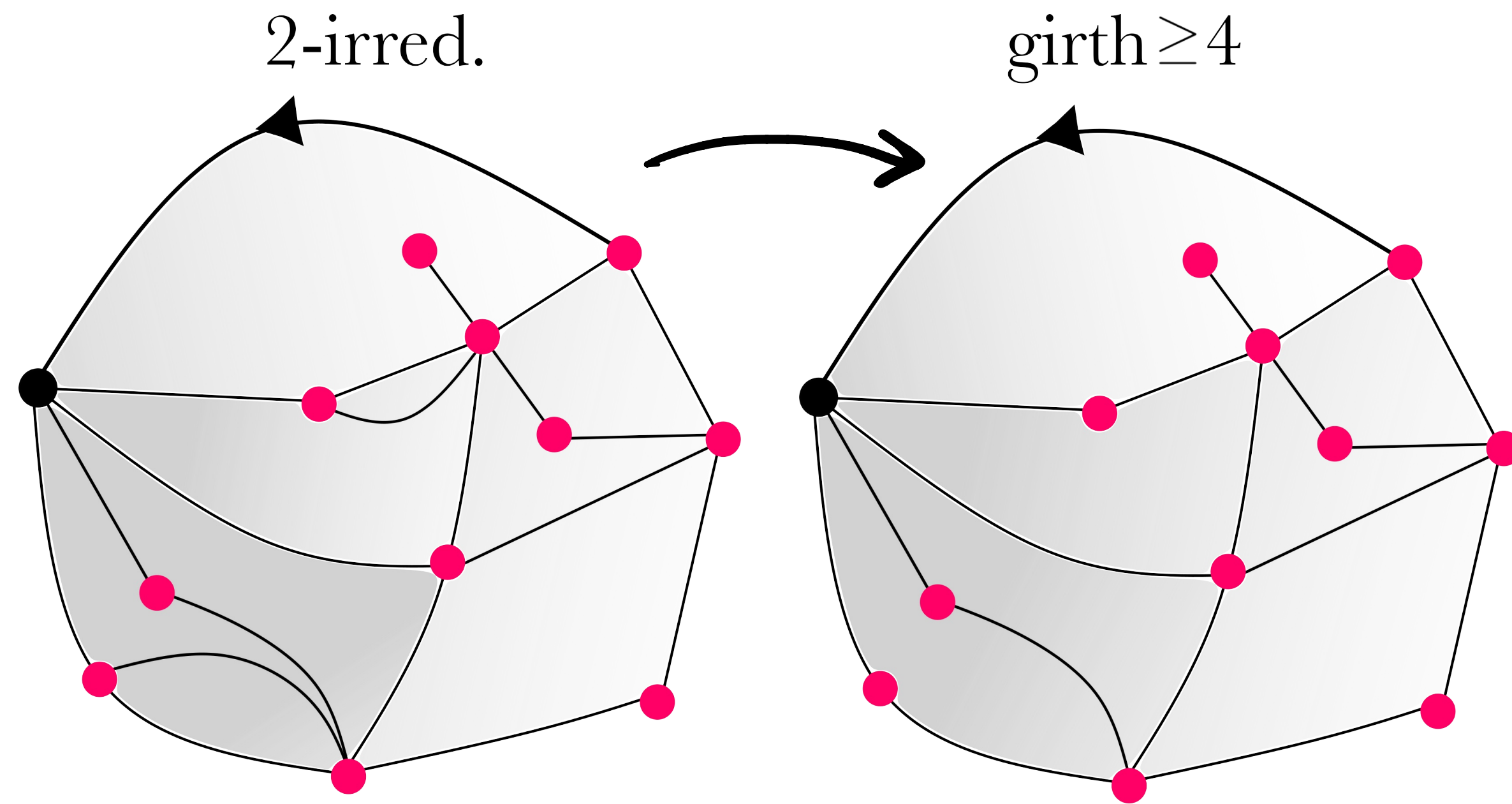
5-irreducible map



girth = 5  
girth  $\geq$  5  
length = 5 closed paths = faces

# Enumeration

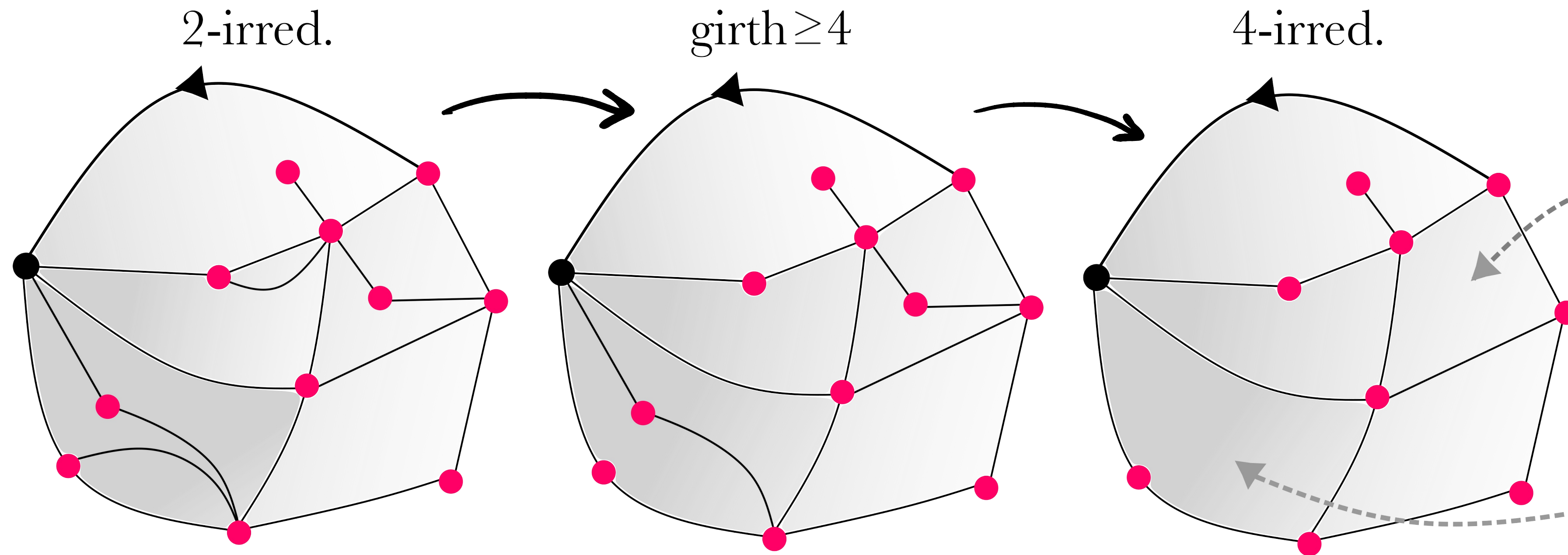
## $q$ -IRREDUCIBLE MAPS





# Enumeration

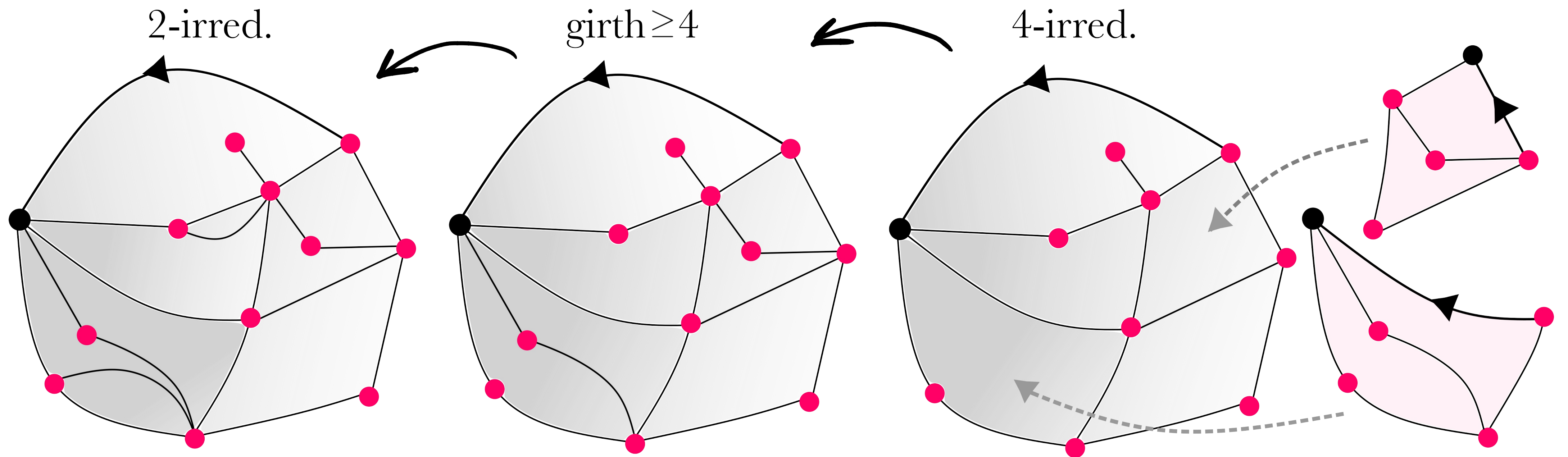
## $q$ -IRREDUCIBLE MAPS





# Enumeration

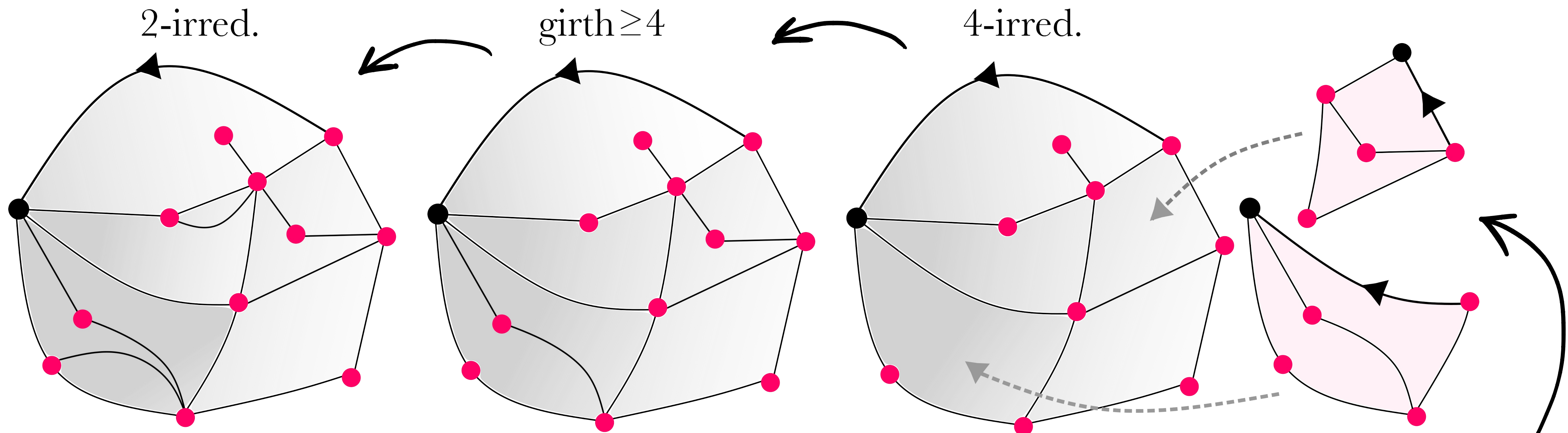
## $q$ -IRREDUCIBLE MAPS



$\rightarrow$  0-irreducible maps = **maps**

# Enumeration

## $q$ -IRREDUCIBLE MAPS



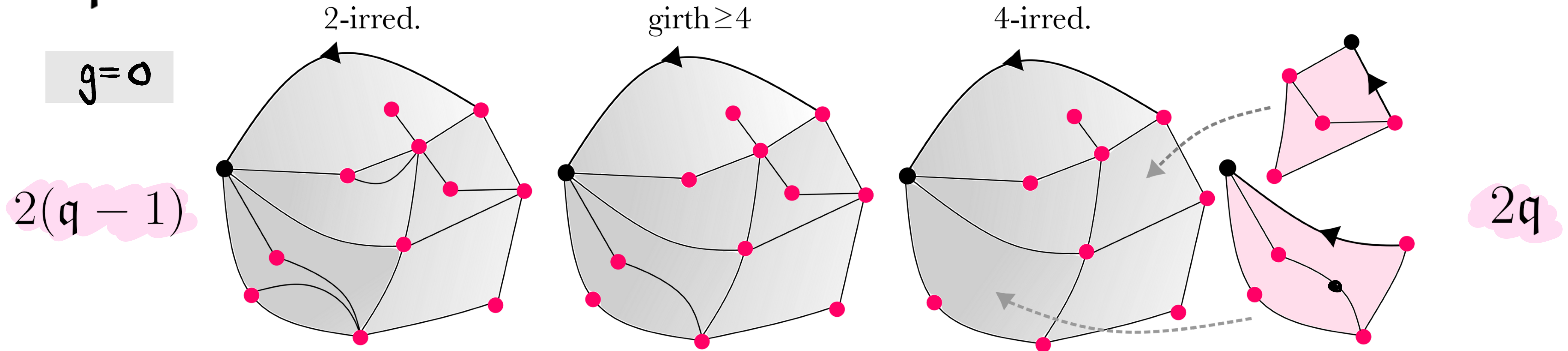
$\rightarrow$  0-irreducible maps = maps

Use results on enumeration of maps + these substitutions

# Enumeration

J. Bouttier, E. Quittner (2013)

## $q$ -IRREDUCIBLE MAPS



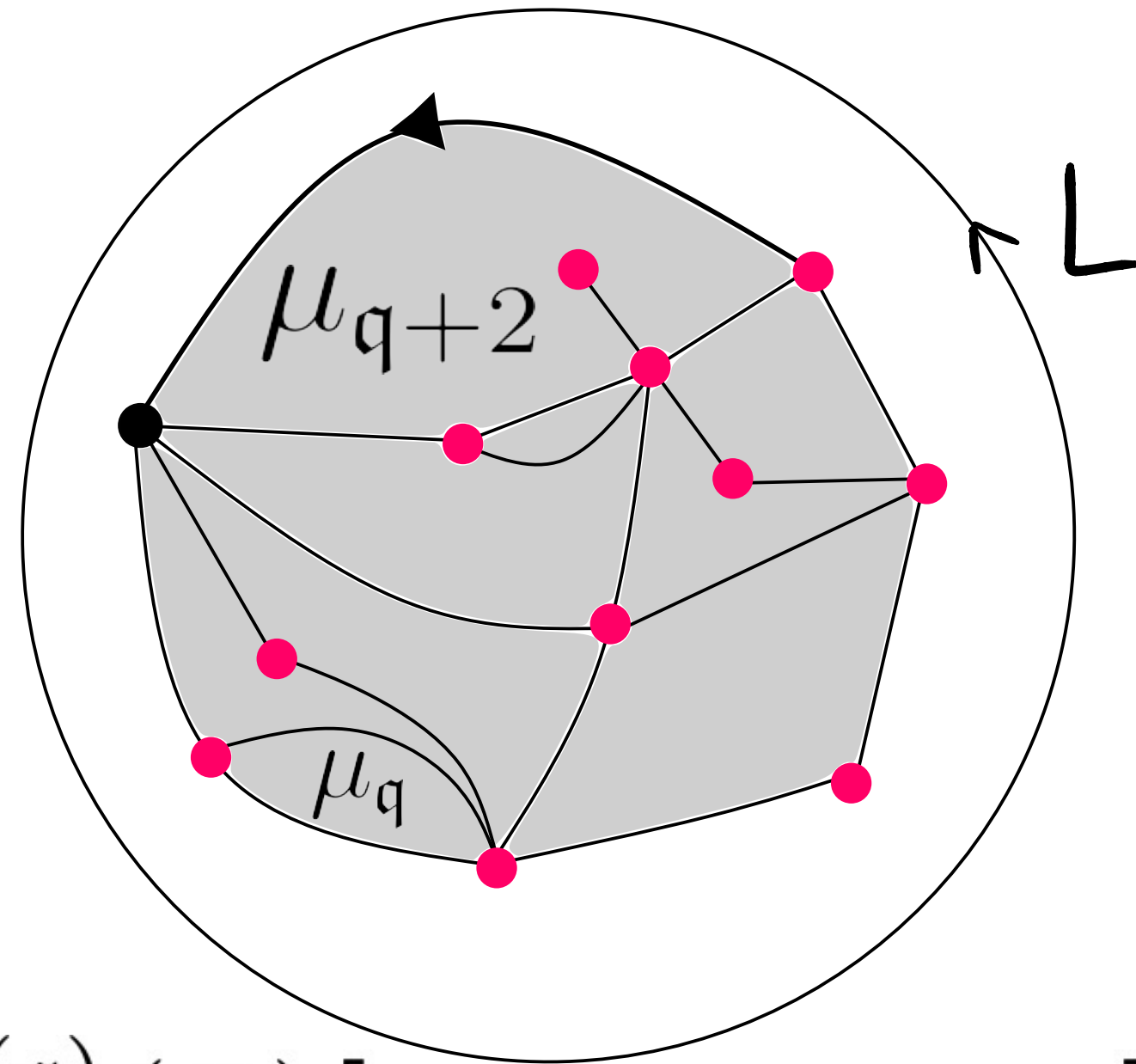
Use generating function of  $q$ -irred maps  
with a marked face of length  $L$

$$W^{(q)}(L) [\mu_q, \mu_{q+1}, \dots]$$



# Enumeration

## $q$ -IRREDUCIBLE MAPS



$$W^{(q)}(L) [\mu_q, \mu_{q+1}, \dots]$$

$$W^{(q)}(L) [\mu_q, \mu_{q+1}, \dots] = \underbrace{\text{Cat}(L)}_{\text{rooted plane tree}} + \underbrace{\mu_q \mathbb{I}_{L=q}}_{\text{rooted map w/one face of length } 2q} \quad \text{for } 1 \leq L \leq q$$

# Enumeration

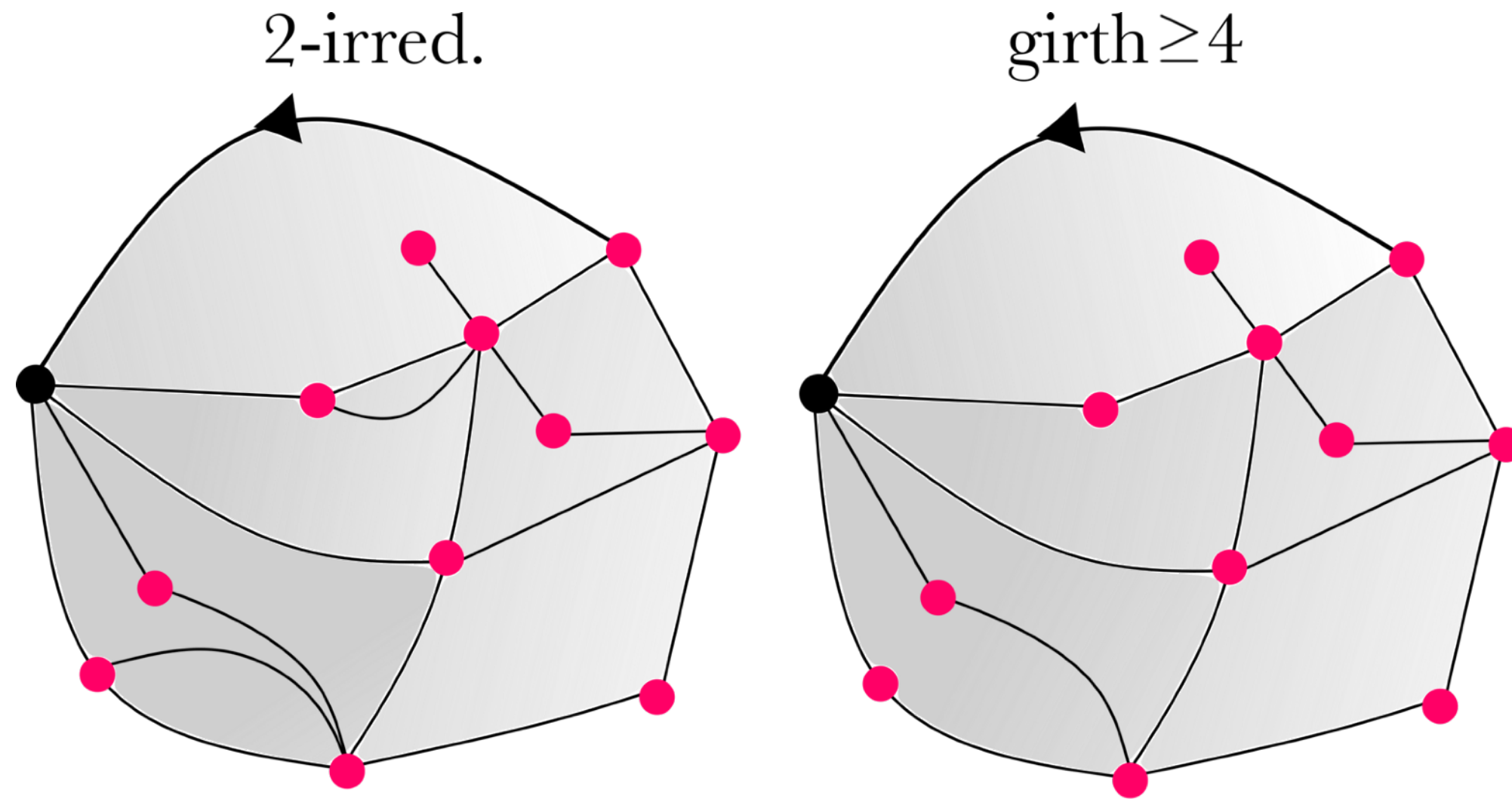
## $q$ -IRREDUCIBLE MAPS

J. Bouttier, E. Quittner (2013)

$$L > q$$

$$g=0$$

$$2(q-1)$$



$$G^{(\geq q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(q-1)}(L)[0, \mu_q, \mu_{q+1}, \dots]$$

forbid faces of length  $2(q-1)$

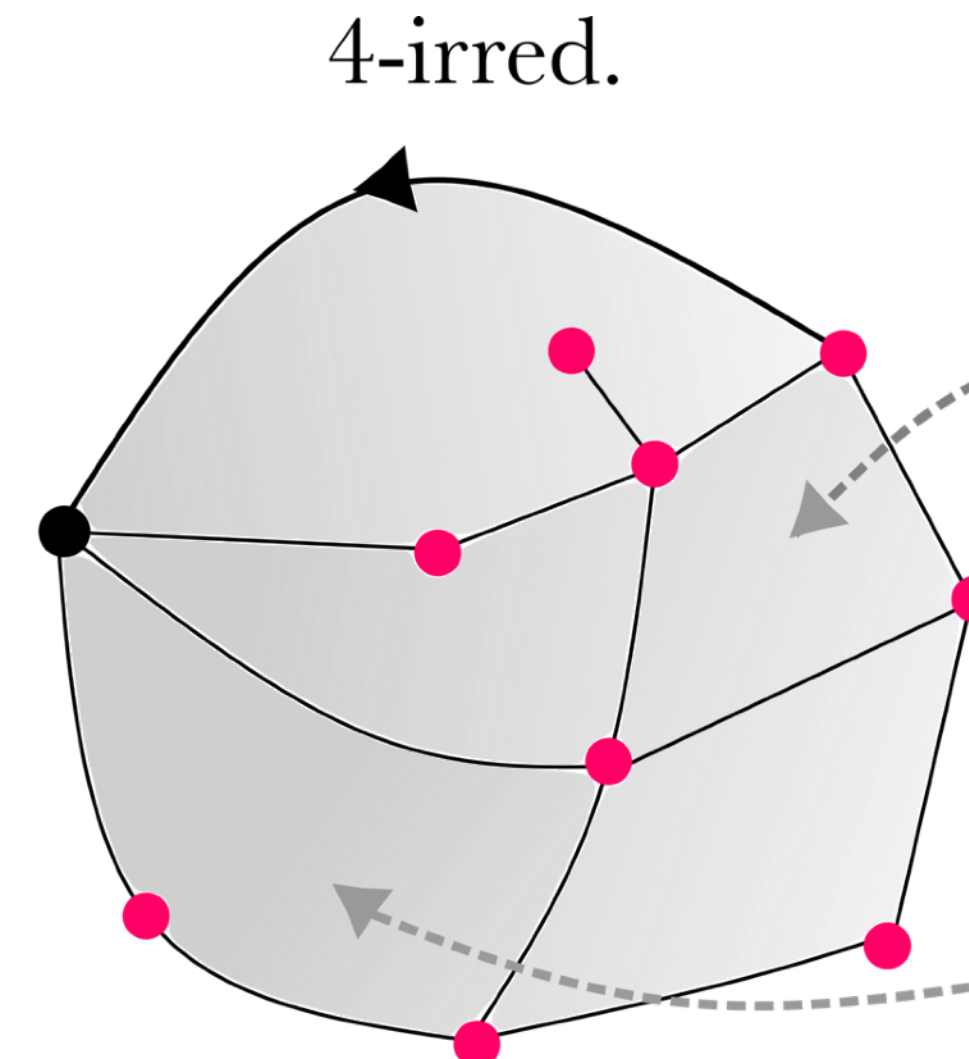
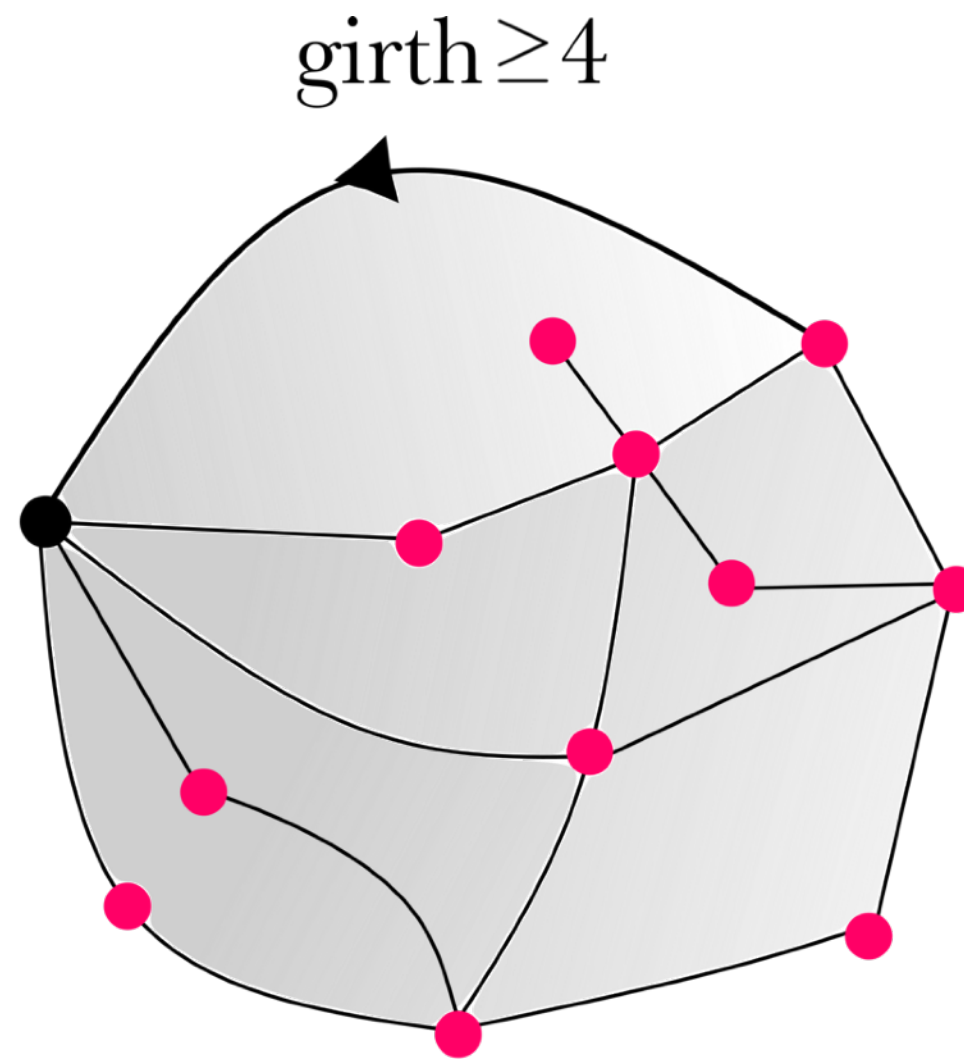
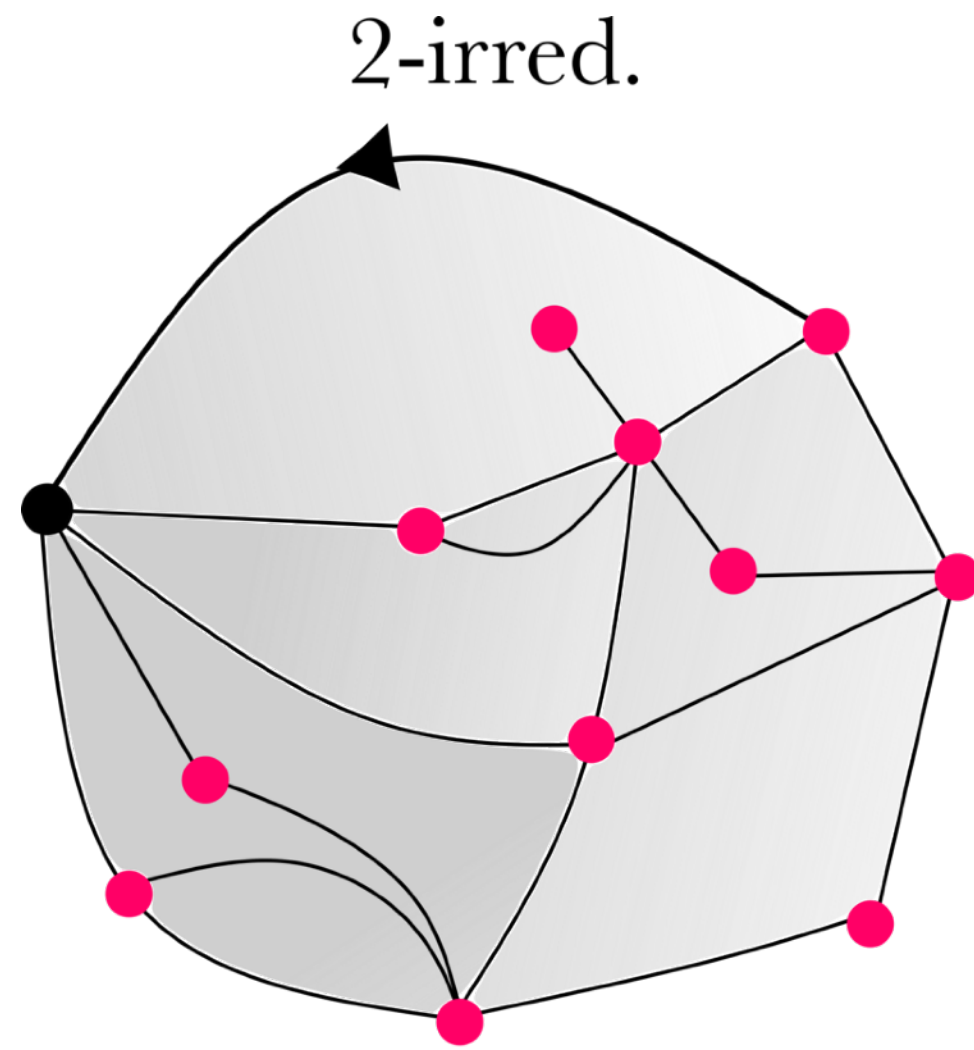
# Enumeration

J. Bouttier, E. Quittner (2013)

## $q$ -IRREDUCIBLE MAPS

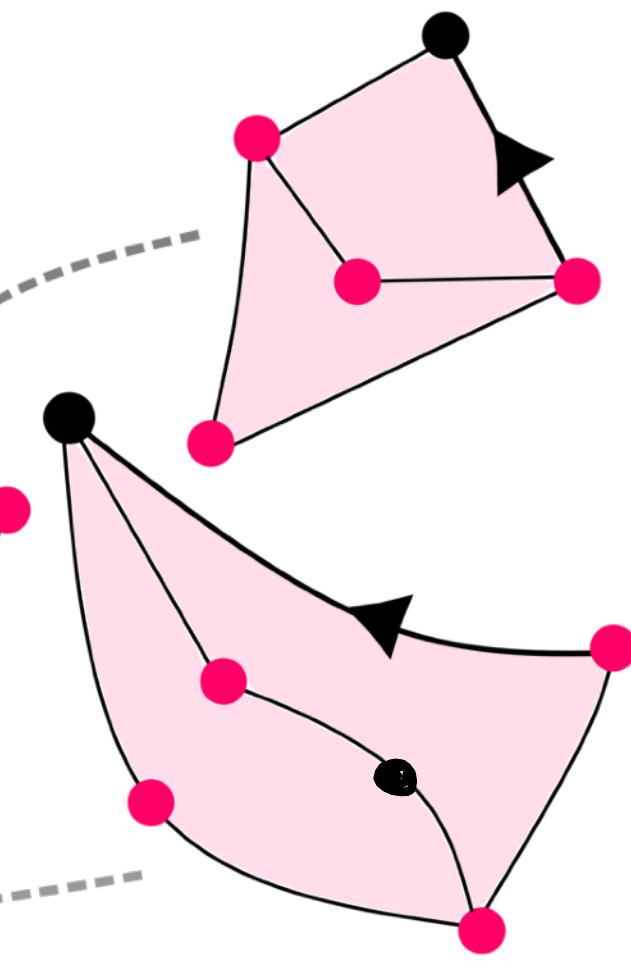
$g=0$

$2(q-1)$



$L > q$

$2q$



$$G^{(\geq q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(q-1)}(L)[0, \mu_q, \mu_{q+1}, \dots]$$

forbid faces of length  $2(q-1)$

$$G^{(\geq q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(q)}(L)[G^{(q)}(q)[\mu_q, \mu_{q+1}, \dots], \mu_q, \mu_{q+1}, \dots]$$

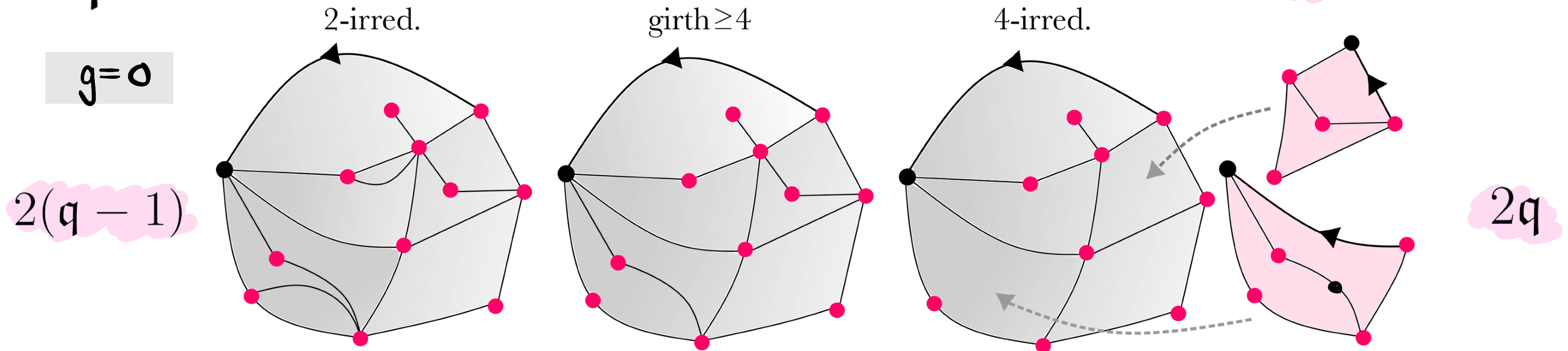
add disk of girth  $2q$  and perimeter  $2q$



# Enumeration

J. Bouttier, E. Quittier (2013)

## $q$ -IRREDUCIBLE MAPS



$$G^{(\geq q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(q-1)}(L)[0, \mu_q, \mu_{q+1}, \dots]$$

forbid faces of length  $2(q-1)$

$$G^{(\geq q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(q)}(L)[G^{(q)}(q)[\mu_q, \mu_{q+1}, \dots], \mu_q, \mu_{q+1}, \dots]$$

add disk of girth  $2q$  and perimeter  $2q$

$$G^{(q)}(q)[\mu_q, \mu_{q+1}, \dots] = G^{(\geq q)}(q)[\mu_q, \mu_{q+1}, \dots] - \text{Cat}(q),$$

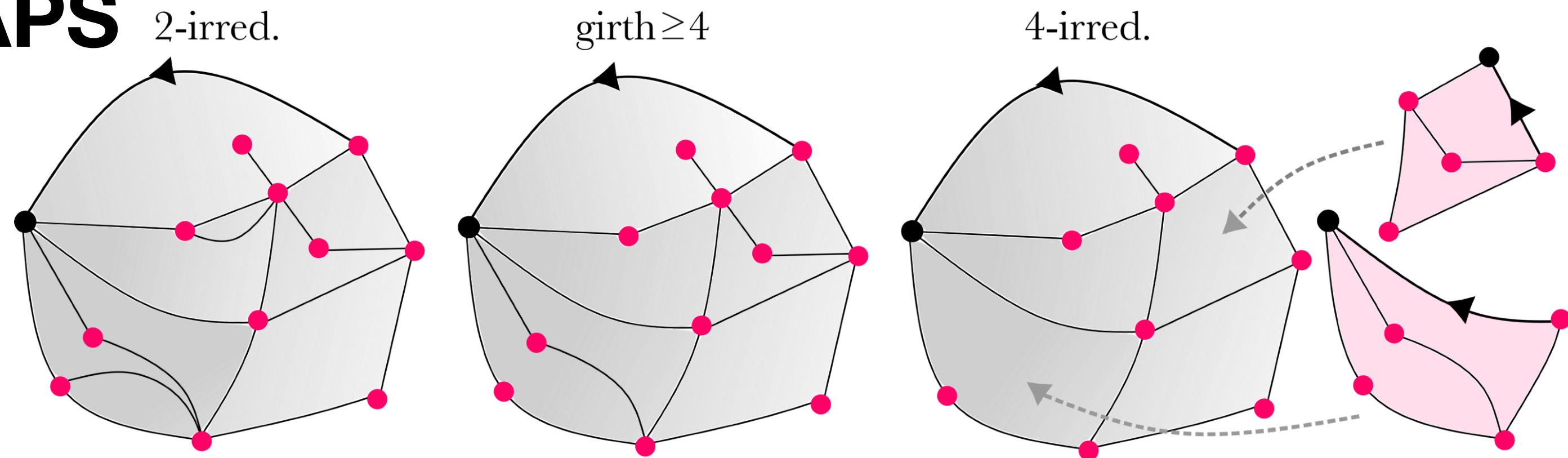
girth =  $2q$  disks are either  
girth  $\geq 2q$  disks or trees

# Enumeration

J. Bouttier, E. Quittner (2013)

## $q$ -IRREDUCIBLE MAPS

$q=0$



$$G^{(\geq q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(q-1)}(L)[0, \mu_q, \mu_{q+1}, \dots]$$

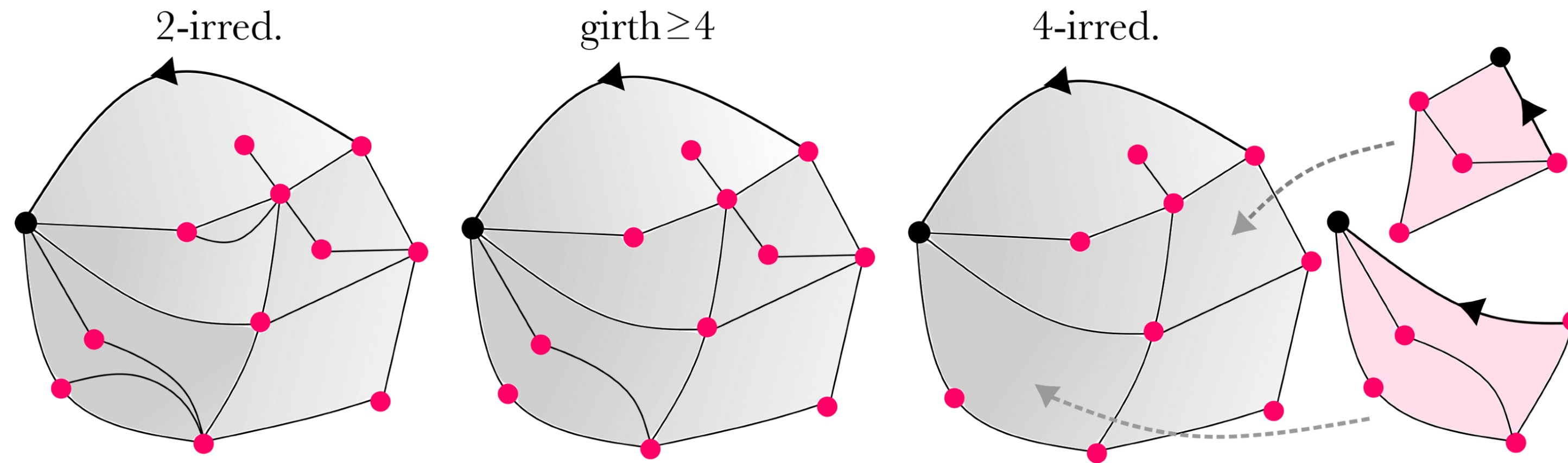
$$G^{(\geq q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(q)}(L)[G^{(q)}(\mathfrak{q})[\mu_q, \mu_{q+1}, \dots], \mu_q, \mu_{q+1}, \dots]$$

$$G^{(q)}(\mathfrak{q})[\mu_q, \mu_{q+1}, \dots] = G^{(\geq q)}(\mathfrak{q})[\mu_q, \mu_{q+1}, \dots] - \text{Cat}(\mathfrak{q}),$$

$$W^{(q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(0)}(L)[\underbrace{\mu_1^{(q)}, \mu_2^{(q)}, \dots, \mu_q^{(q)}}_{\forall L}, \mu_{q+1}, \dots]$$

# Enumeration

## q - IRREDUCIBLE MAPS



$g=0$  Bouttier  
Quitier '13

$$W^{(q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(0)}(L)[\mu_1^{(q)}, \mu_2^{(q)}, \dots, \mu_q^{(q)}, \mu_{q+1}, \dots]$$

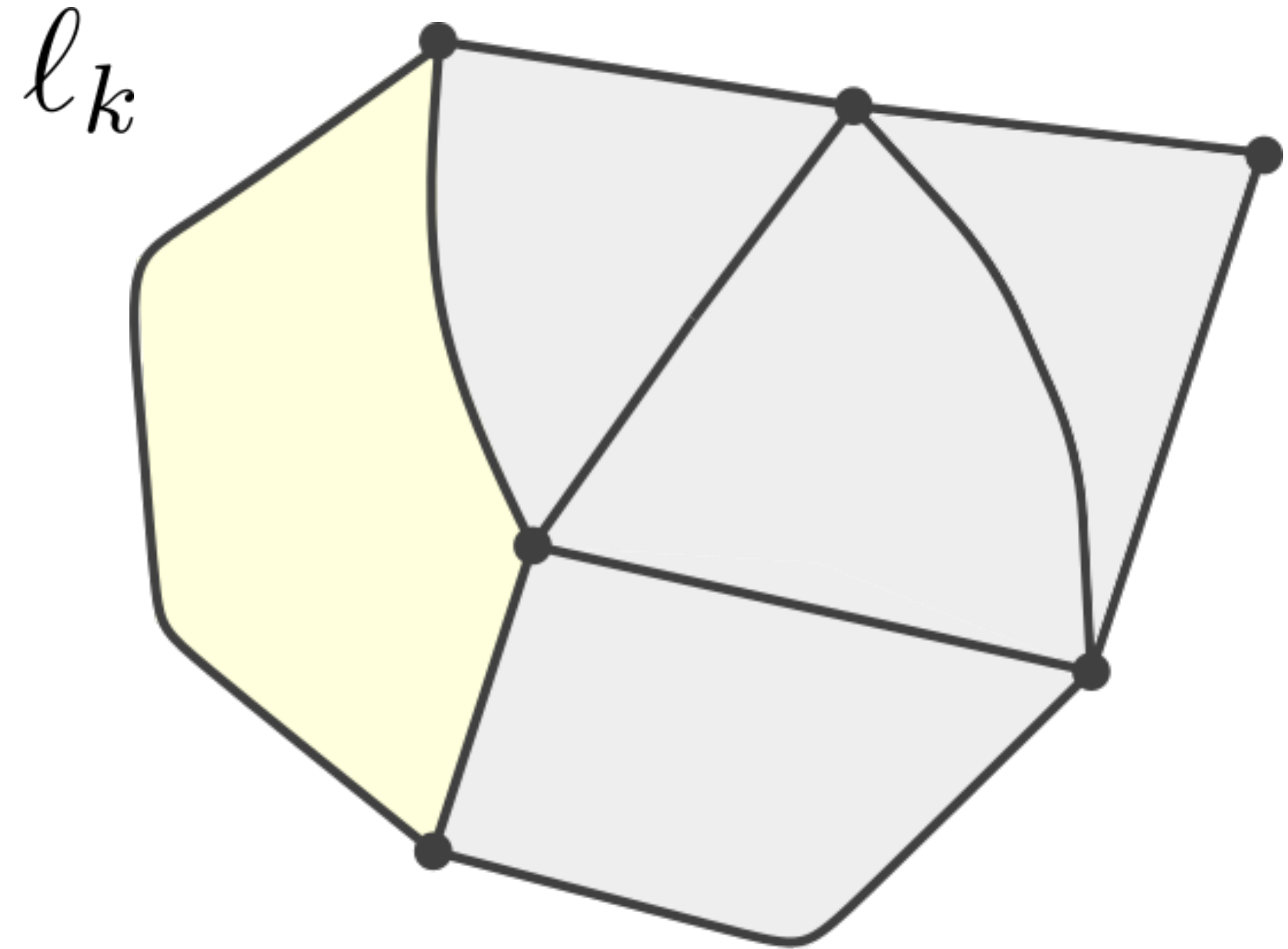
$g>0$  Budd '20

$$\|\mathcal{M}_{g,k}^{(q)}(L_1, \dots, L_k)\| = \underbrace{\hat{N}_{g,k}^{(q)}(L_1, \dots, L_k)}_{\text{Polynomial on } \{L_i^2\}_{i=1}^k \text{ and } q} + \mathbf{1}_{\{g=0, k \geq 4, L_1 = \dots = L_k = q\}} \frac{(k-1)!}{2} (-1)^k$$



# What is an ~~irreducible~~ metric map?

Kontsevich  
'92

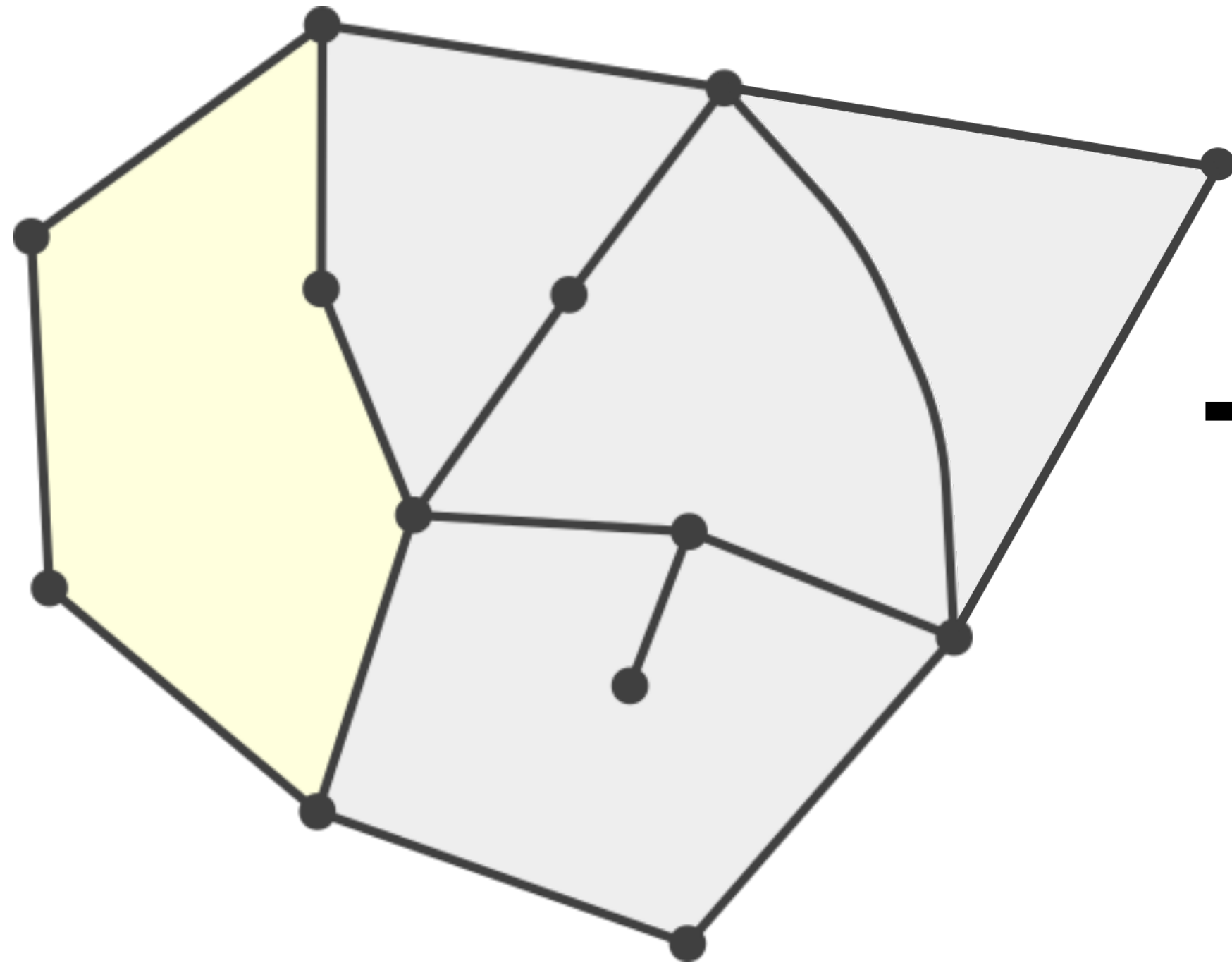


map w/ vertices of  
degree  $\geq 3$  and positive real  
lengths on its edges

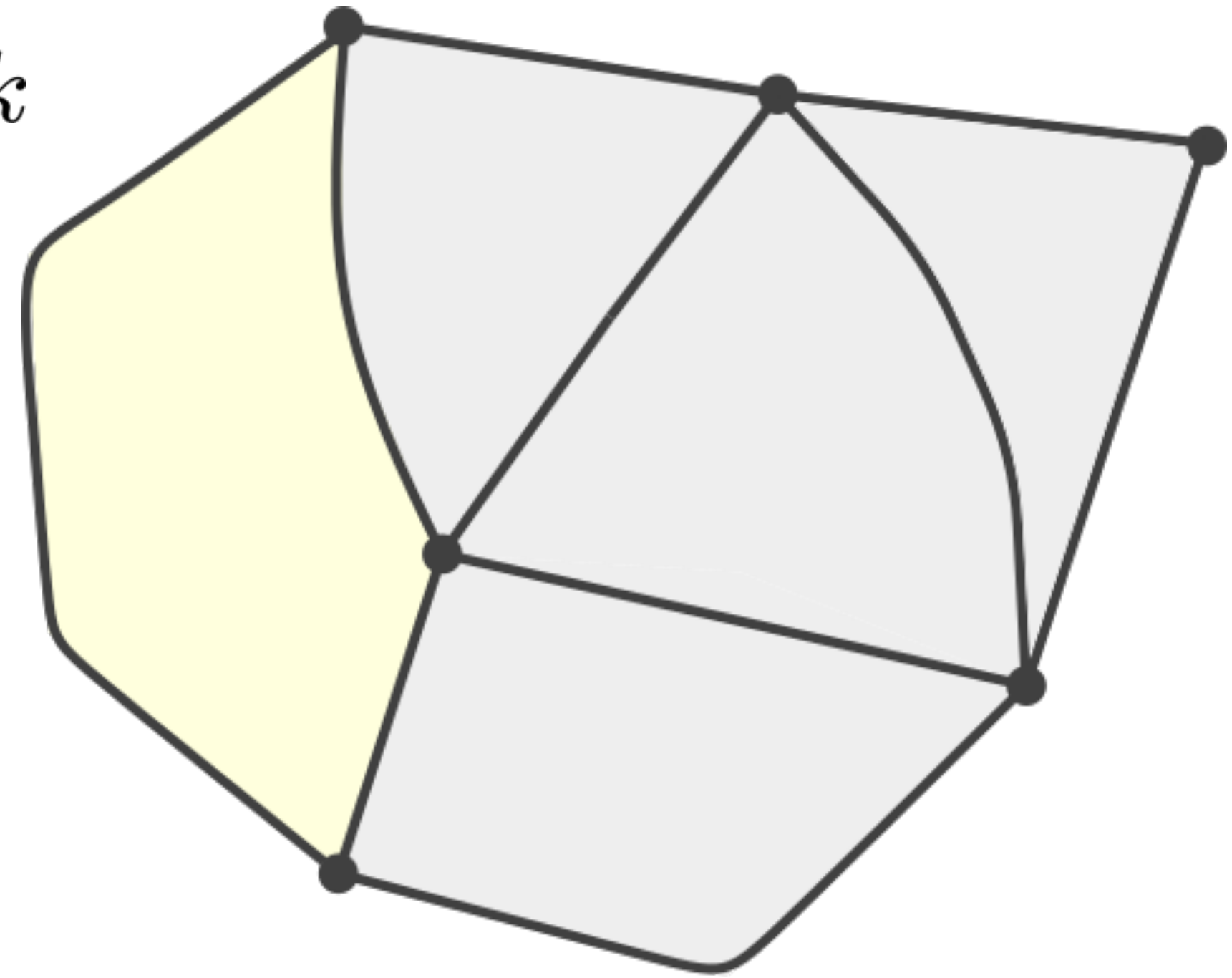
# What is an ~~irreducible~~ metric map?

Kontsevich  
'92

$L_k$

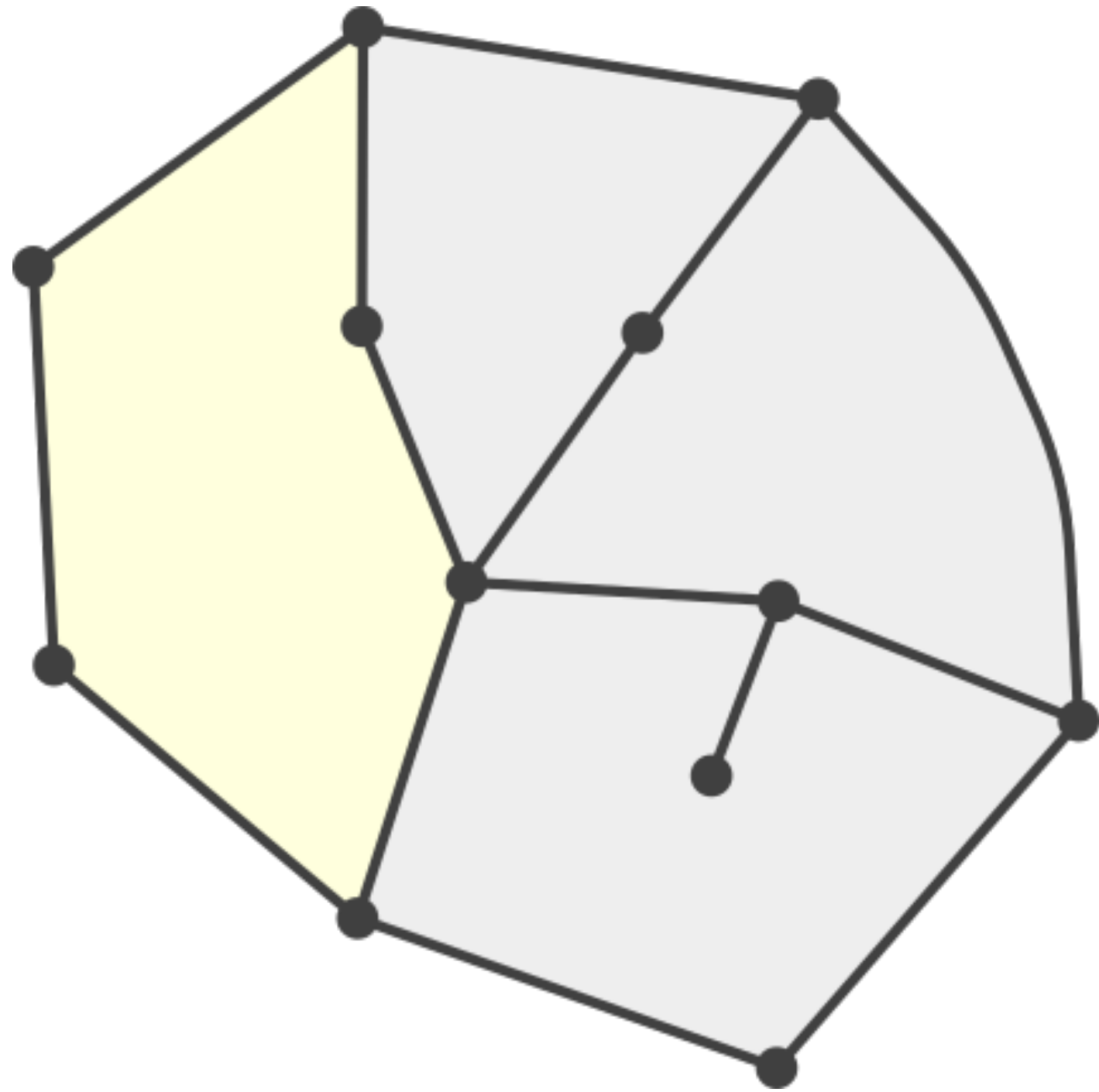


$l_k$



map w/ vertices of  
degree  $\geq 3$  and positive real  
lengths on its edges

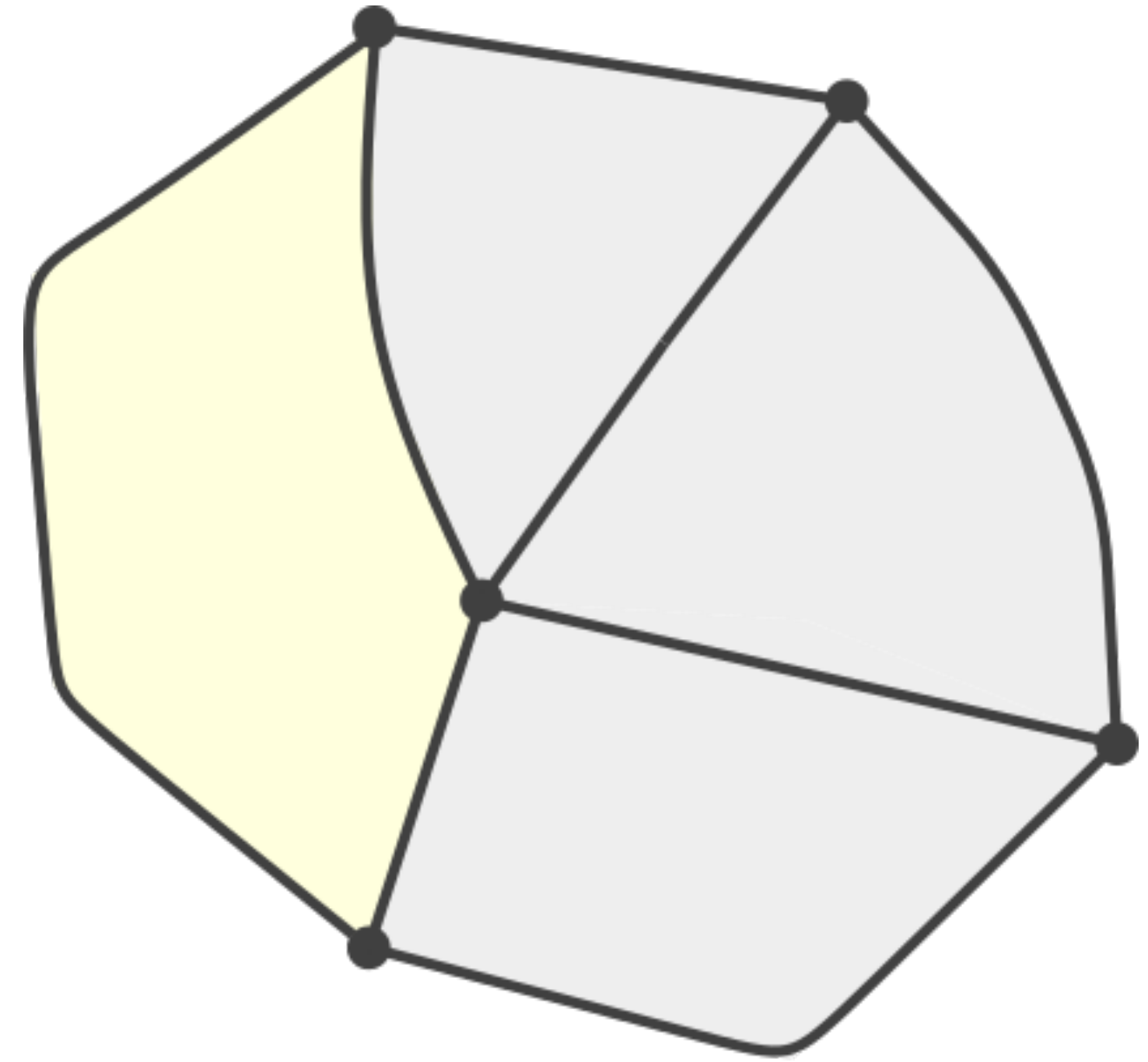
# What is an irreducible metric map?



5-irred. map



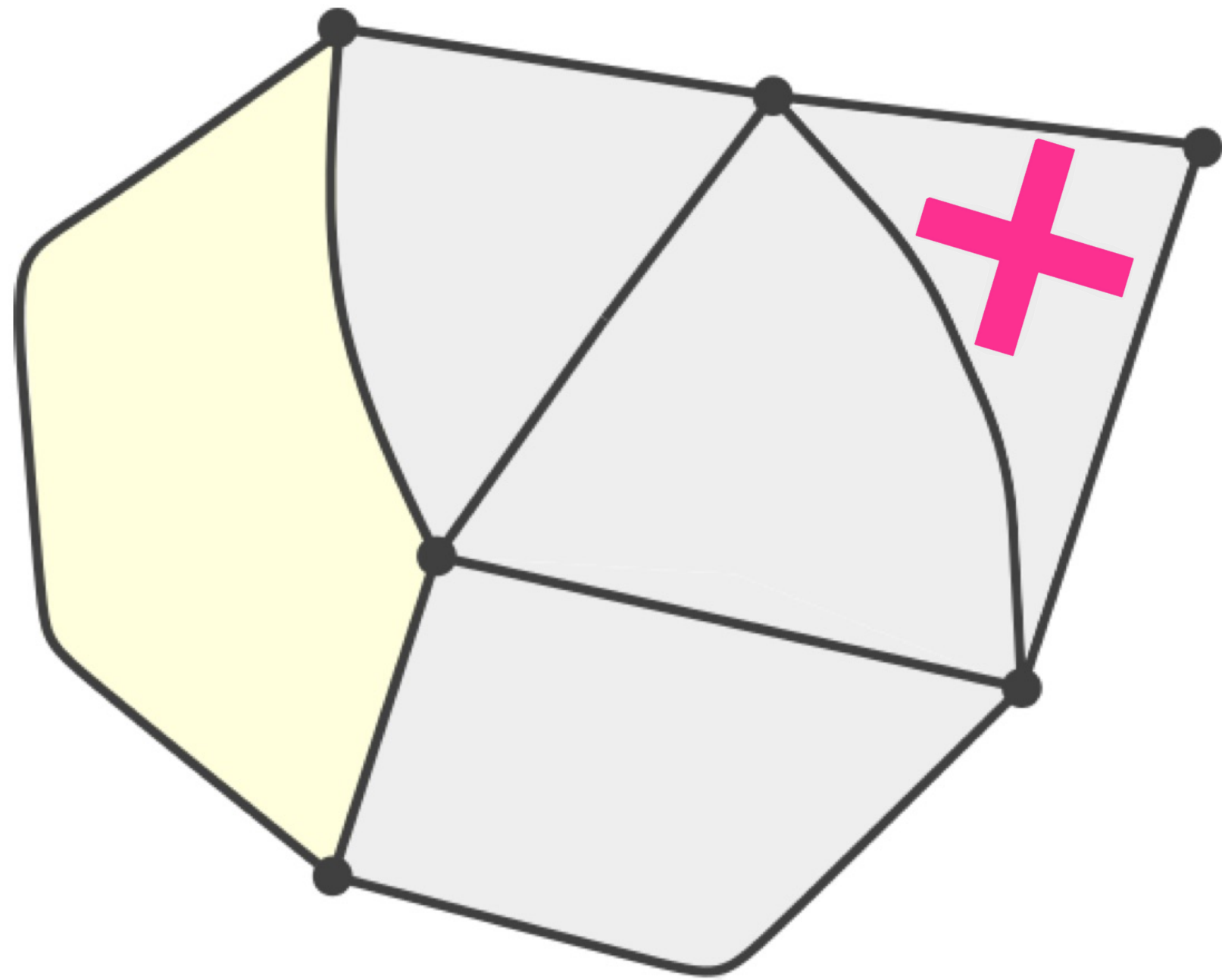
T. Budd.  
'22



9-irred. metric  
map



# What is an irreducible metric map?

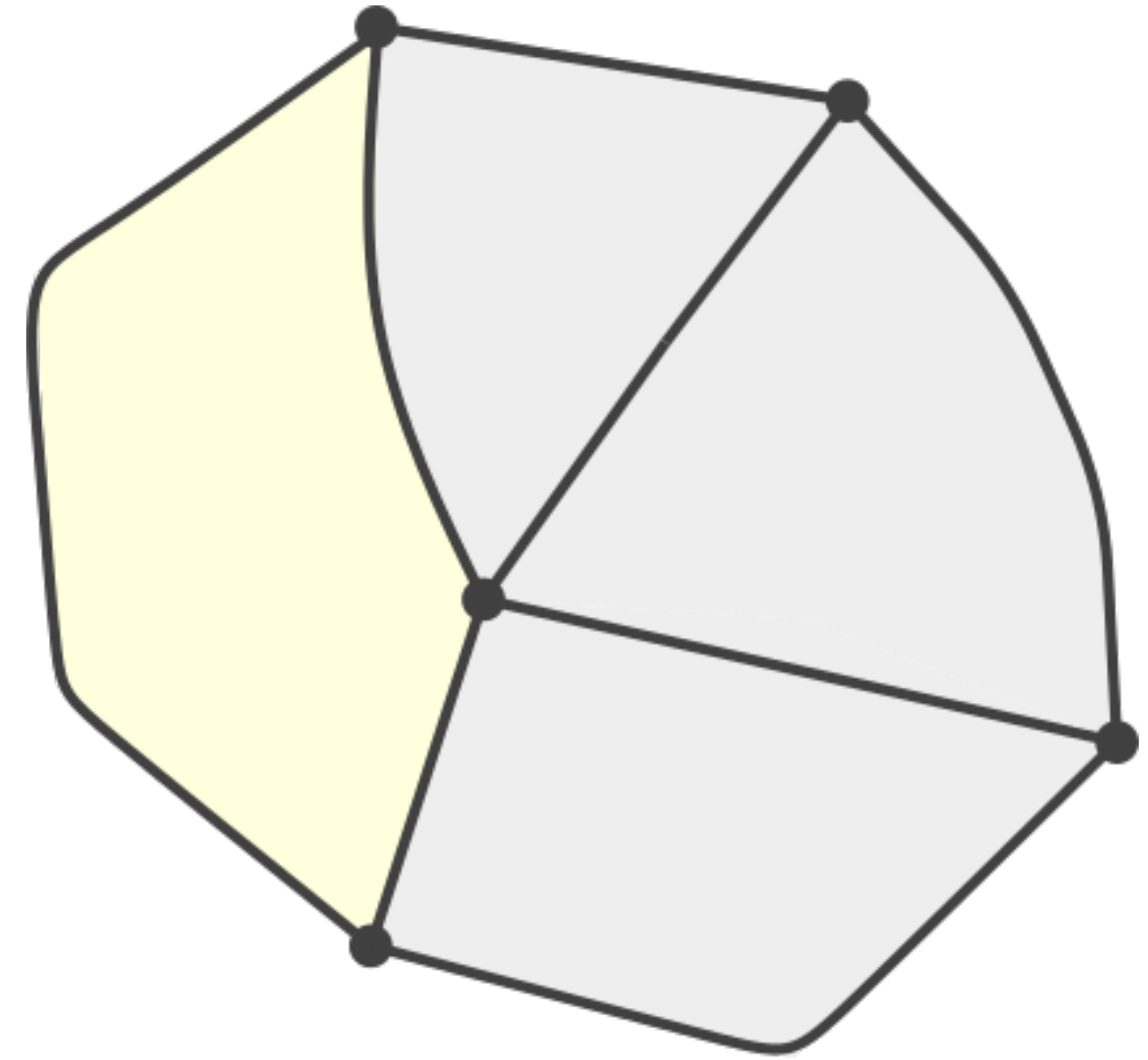


metric map



This  
work

T. Budd,  
A. Castro  
to appear

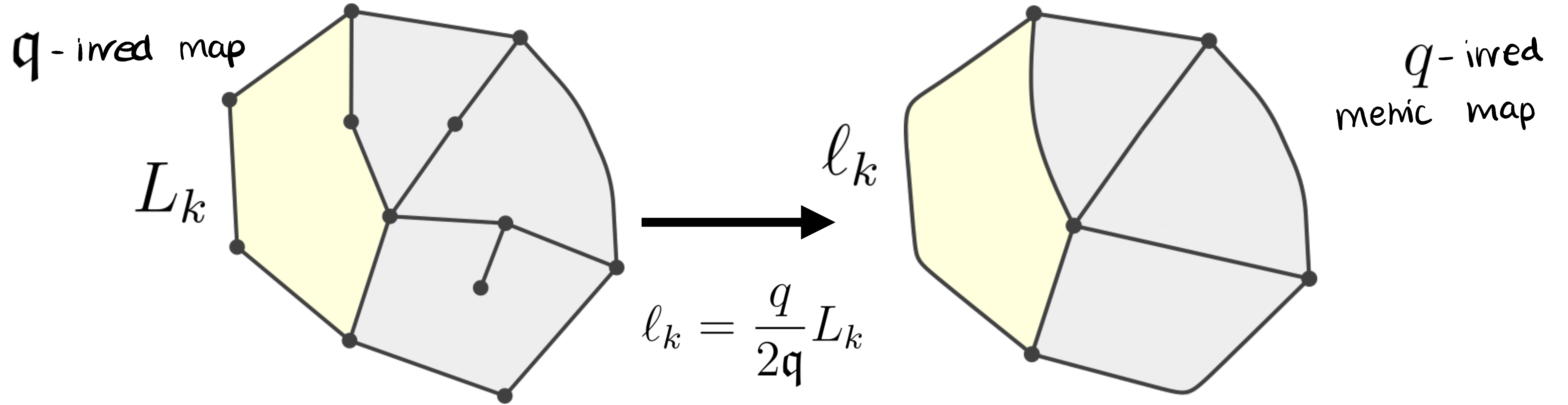


$q$ -irred metric  
map

# Enumeration

## IRREDUCIBLE METRIC MAPS

From maps (discrete)

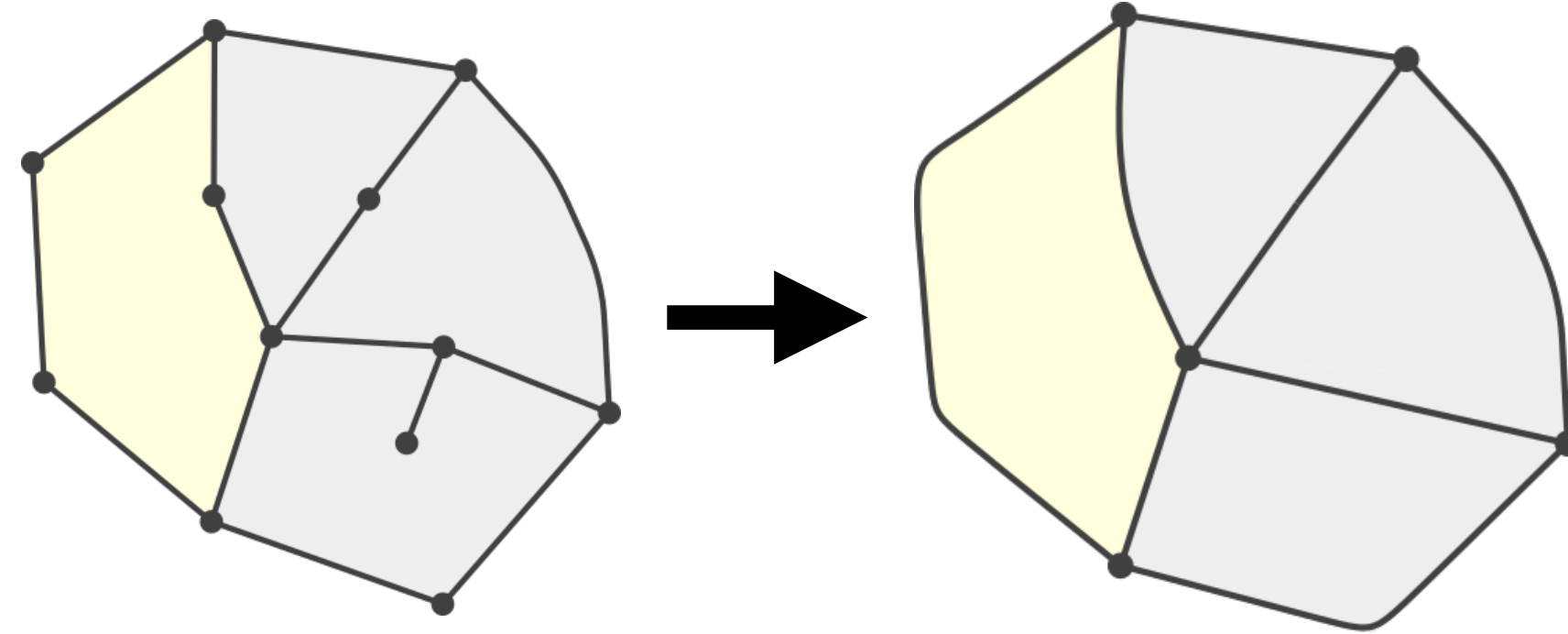


# Enumeration

## IRREDUCIBLE METRIC MAPS

From maps (discrete)

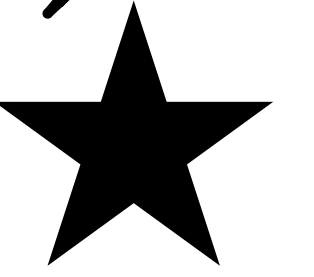
$q$ -irred. map



$q$ -irred metric map

$$V_{g,k}^{(q)}(\ell_1, \dots, \ell_k) = \frac{1}{2} [t^{6g-6+2k}] \hat{N}_{g,k}^{(q=\frac{1}{2}qt)} \left( \frac{1}{2} L_1 t, \dots, \frac{1}{2} L_k t \right)$$

T. Budd, A. Castro (to appear)

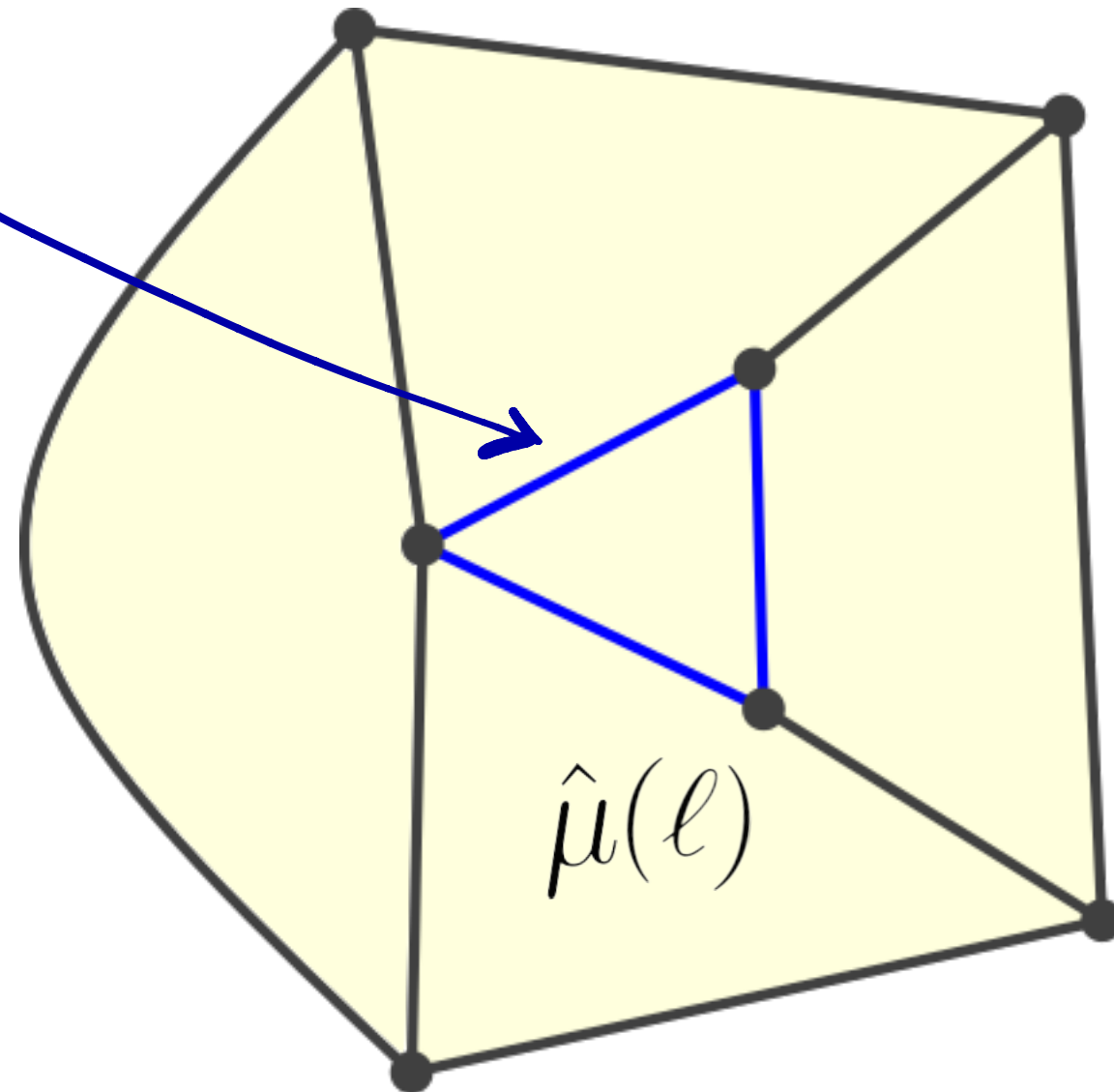


# Enumeration

## IRREDUCIBLE METRIC MAPS

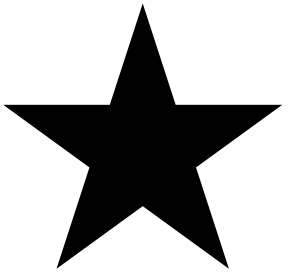
From metric maps (continuum)

$q$ -irred metric map



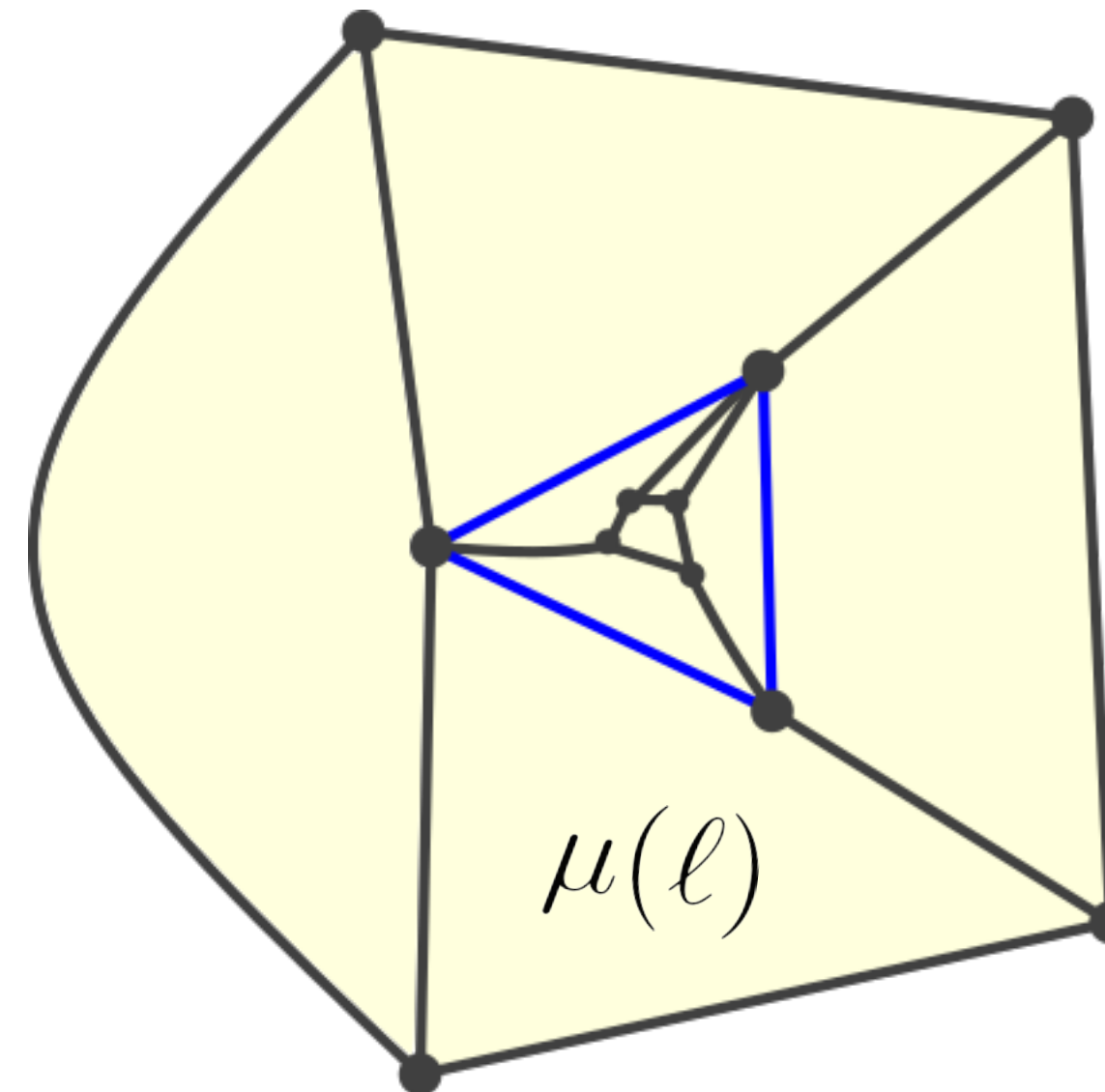
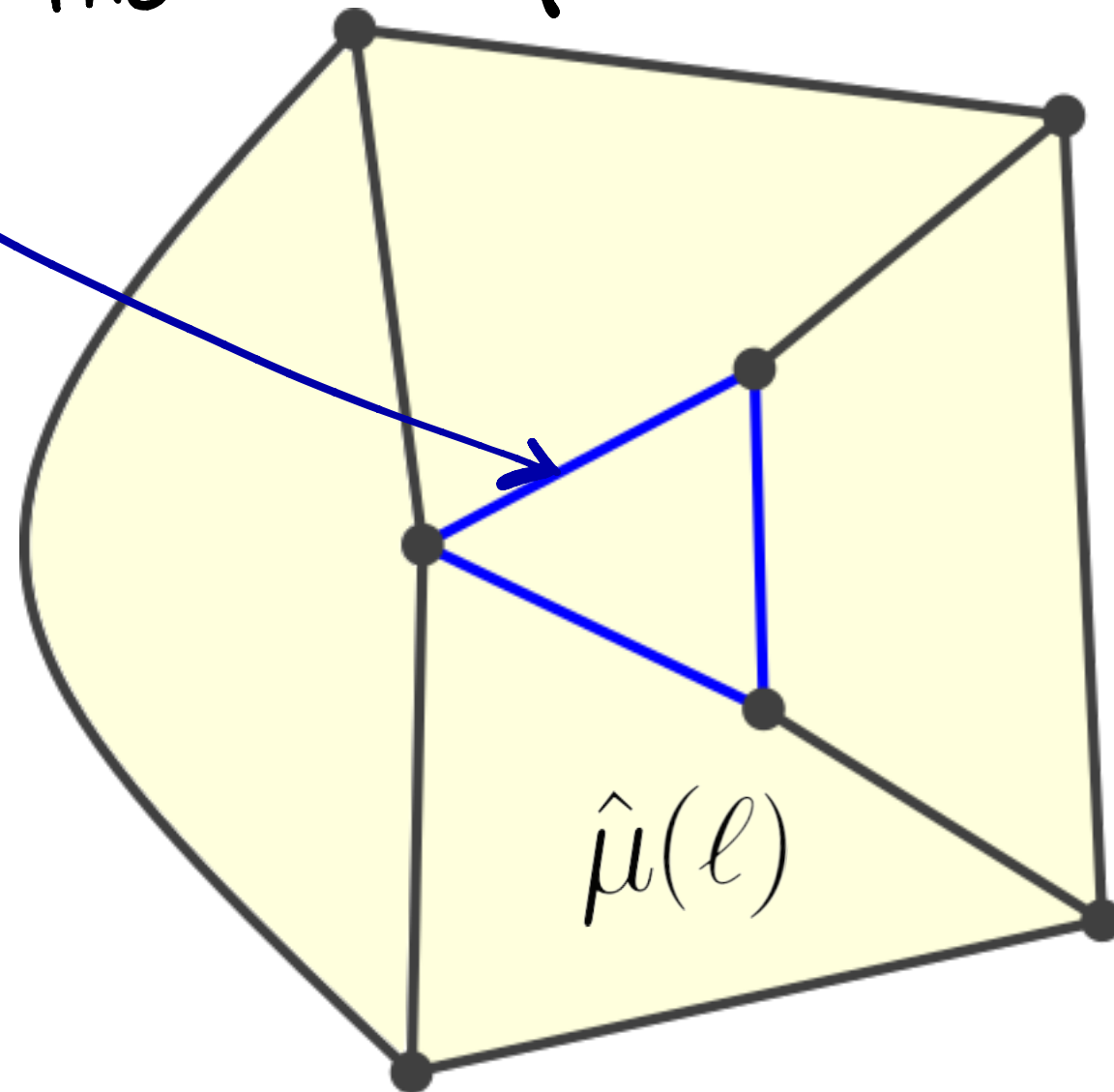
# Enumeration

## IRREDUCIBLE METRIC MAPS



From metric maps (continuum)

$q$ -irred metric map



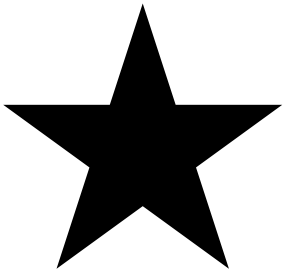
metric map  
( $q=0$ )

$$W^{(q)}(\ell)[\hat{\mu}] = W^{(0)}(\ell)[\mu]$$

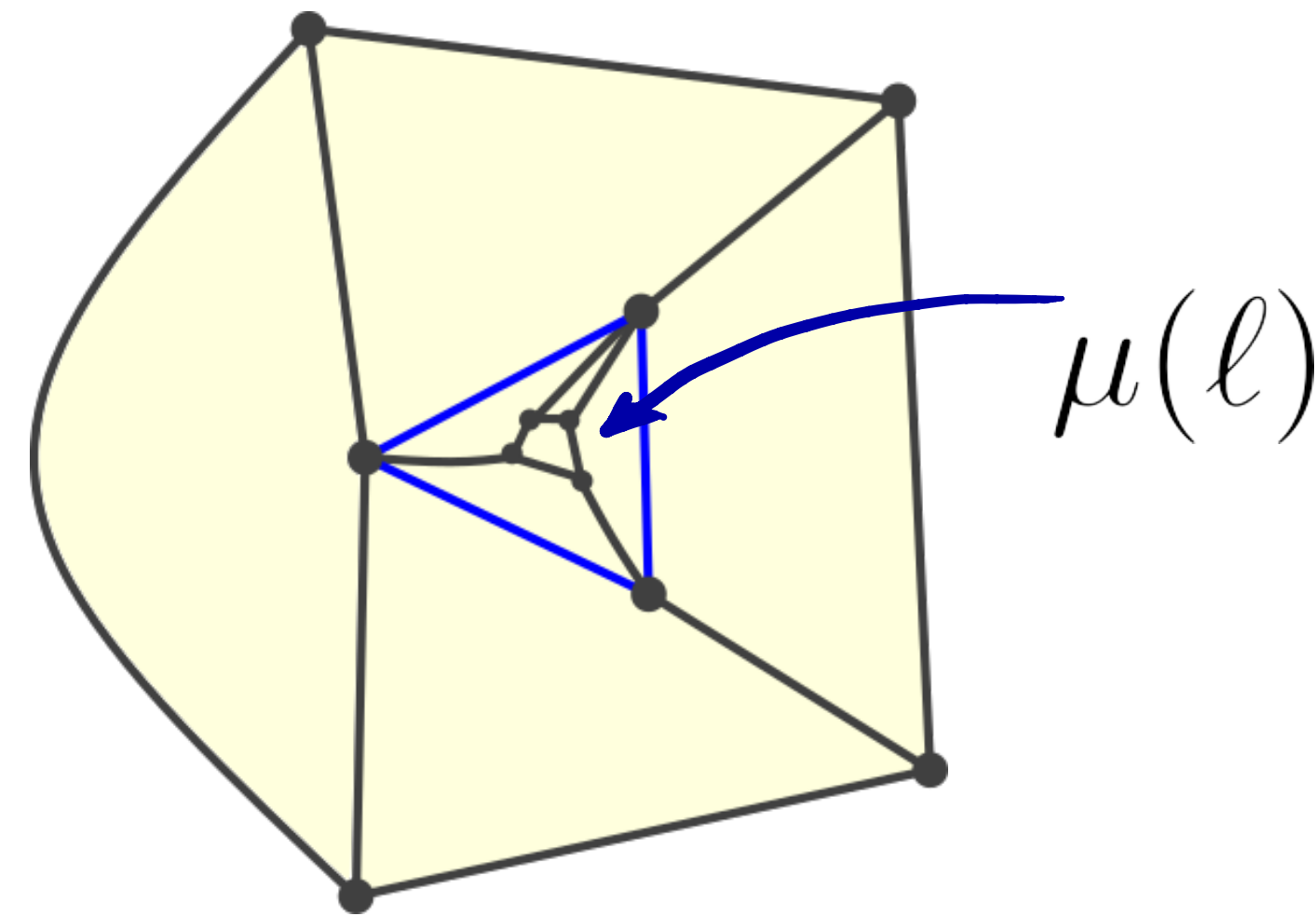
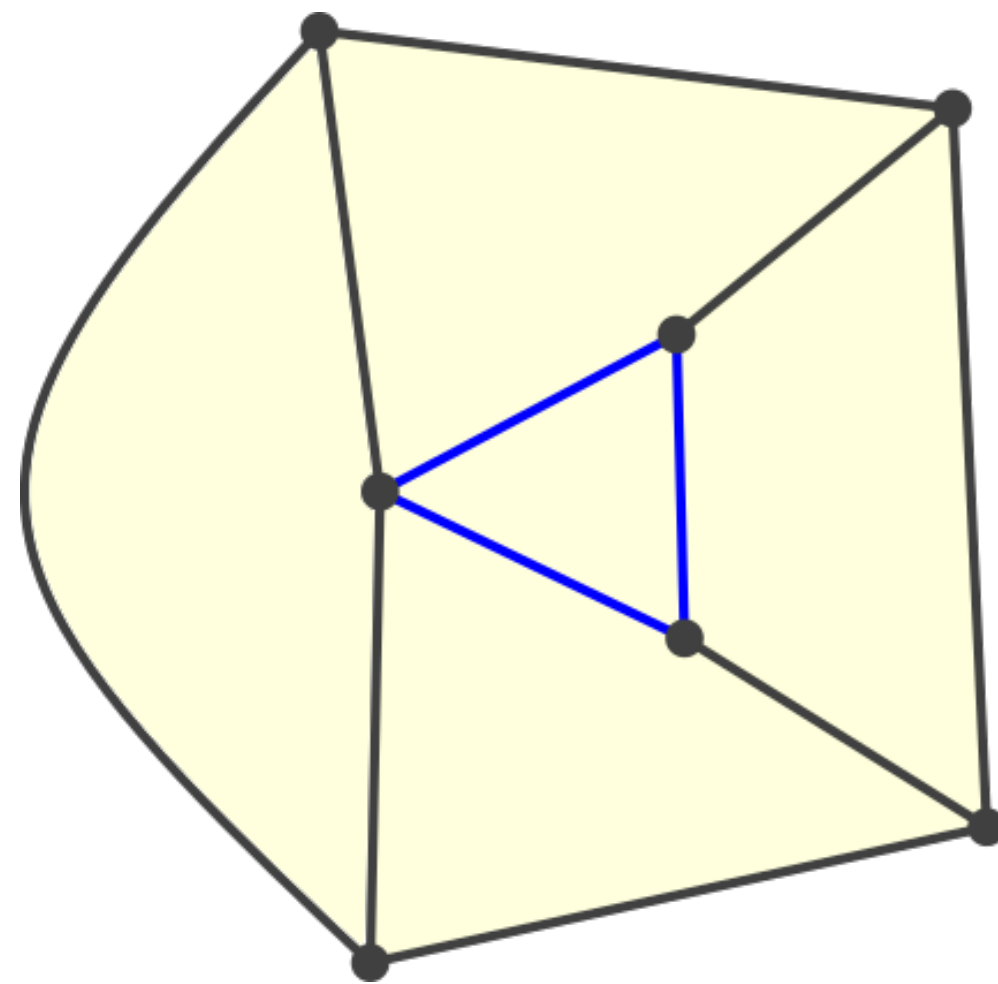
for all  $\ell \geq q$

# Enumeration

## IRREDUCIBLE METRIC MAPS

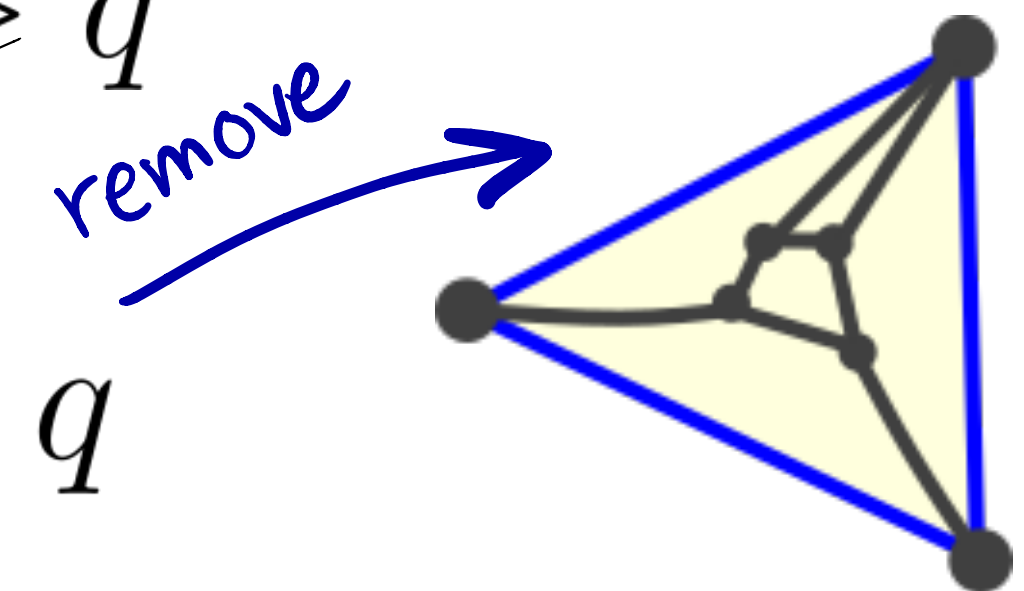


From metric maps (continuum)



$$W^{(q)}(\ell)[\hat{\mu}] = W^{(0)}(\ell)[\mu] \text{ for all } \ell \geq q$$

$$W^{(0)}(\ell)[\mu] + \frac{\mu(\ell)}{\ell} = 0 \text{ for all } \ell < q$$



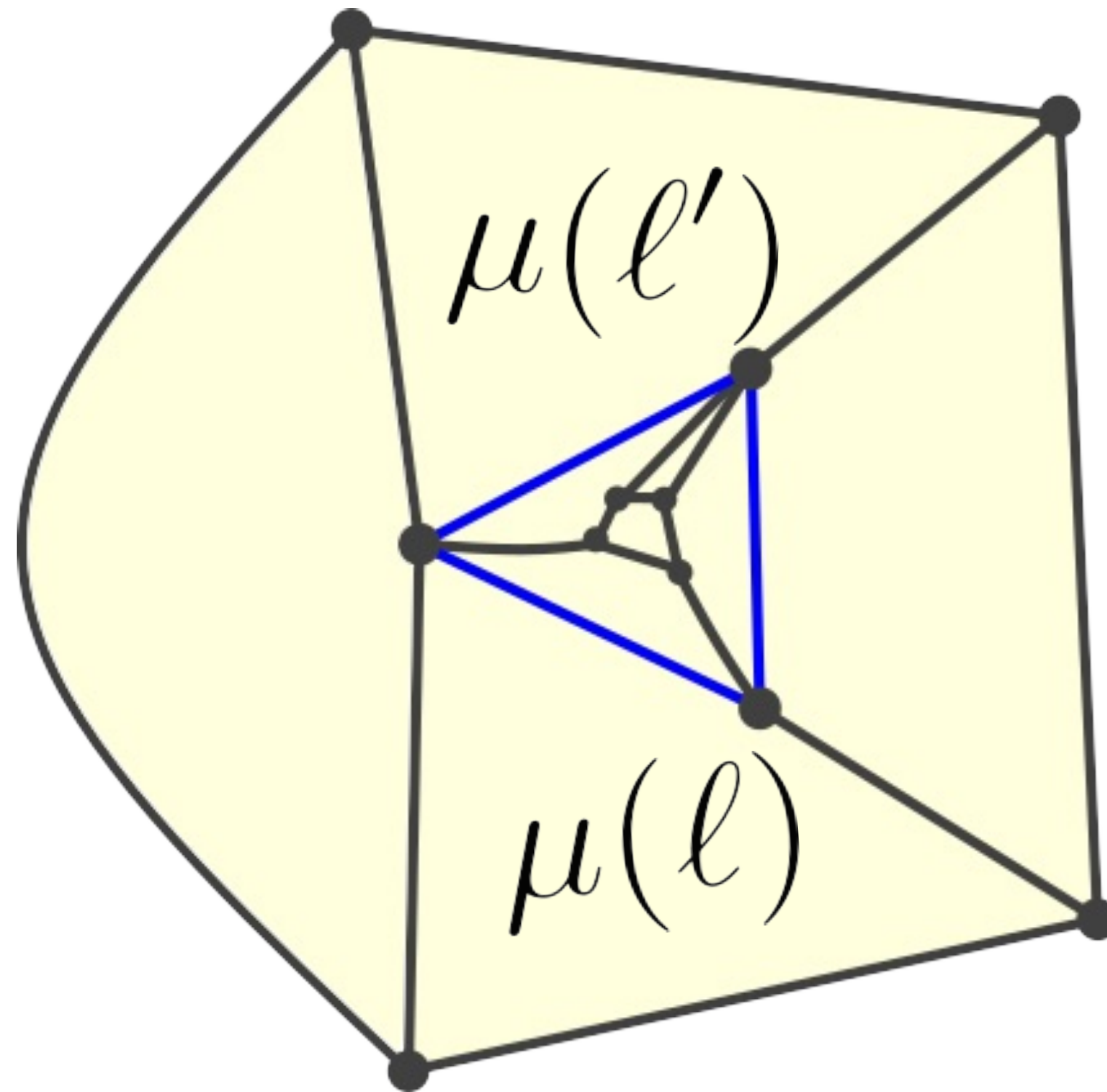


# DISK FUNCTION

metric maps

C. Itzykson, J.B. Zuber (1992)

$$F_g^{(0)}[\mu] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^{\infty} dl_1 \mu(l_1) \cdots \int_0^{\infty} dl_k \mu(l_k) V_{g,n}^{(0)}(l_1, \dots, l_k)$$



# DISK FUNCTION

metric maps

C. Itzykson, J.B. Zuber (1992)

$$F_g^{(0)}[\mu] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^{\infty} dl_1 \mu(l_1) \cdots \int_0^{\infty} dl_k \mu(l_k) V_{g,n}^{(0)}(l_1, \dots, l_k)$$

$$F_0^{(0)}[\mu] = \frac{1}{4} \int_0^{E_\mu} dr (x(u))^2$$

$$x(u) = u - \int_0^{\infty} dl \mu(l) I_0(l\sqrt{u})$$

string equation

$$x(E_\mu) = 0$$

$E_\mu$ : solution to string equation

# DISK FUNCTION

metric maps

C. Itzykson, J.B. Zuber (1992)

$$F_g^{(0)}[\mu] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^{\infty} dl_1 \mu(l_1) \cdots \int_0^{\infty} dl_k \mu(l_k) V_{g,n}^{(0)}(l_1, \dots, l_k)$$

$$F_0^{(0)}[\mu] = \frac{1}{4} \int_0^{E_\mu} dr (x(u))^2$$

$$x(u) = u - \int_0^{\infty} dl \mu(l) I_0(l\sqrt{u})$$

$$W^{(0)}(l)[\mu] = \frac{\delta F_0^{(0)}[\mu]}{\delta \mu(l)}$$

Generating function of  
metric maps w/one  
marked boundary  
**DISK FUNCTION**

# DISK FUNCTION

## metric maps

$$W^{(0)}(\ell)[\mu] = \int_0^\infty \underbrace{W^{(0)}(\ell, p)}_{\text{cylinder generating function}} \mu(p) dp - 2 \underbrace{\frac{\partial^2 W^{(0)}(\ell, p)}{\partial p^2}}_{\text{generating function of m.m. w/}} \Big|_{p=0}$$

1 marked face ( $\ell$ )  
1 unmarked face ( $p$ )

1 marked face ( $\ell$ )  
1 marked vertex ( $p$ )

Euler-characteristic

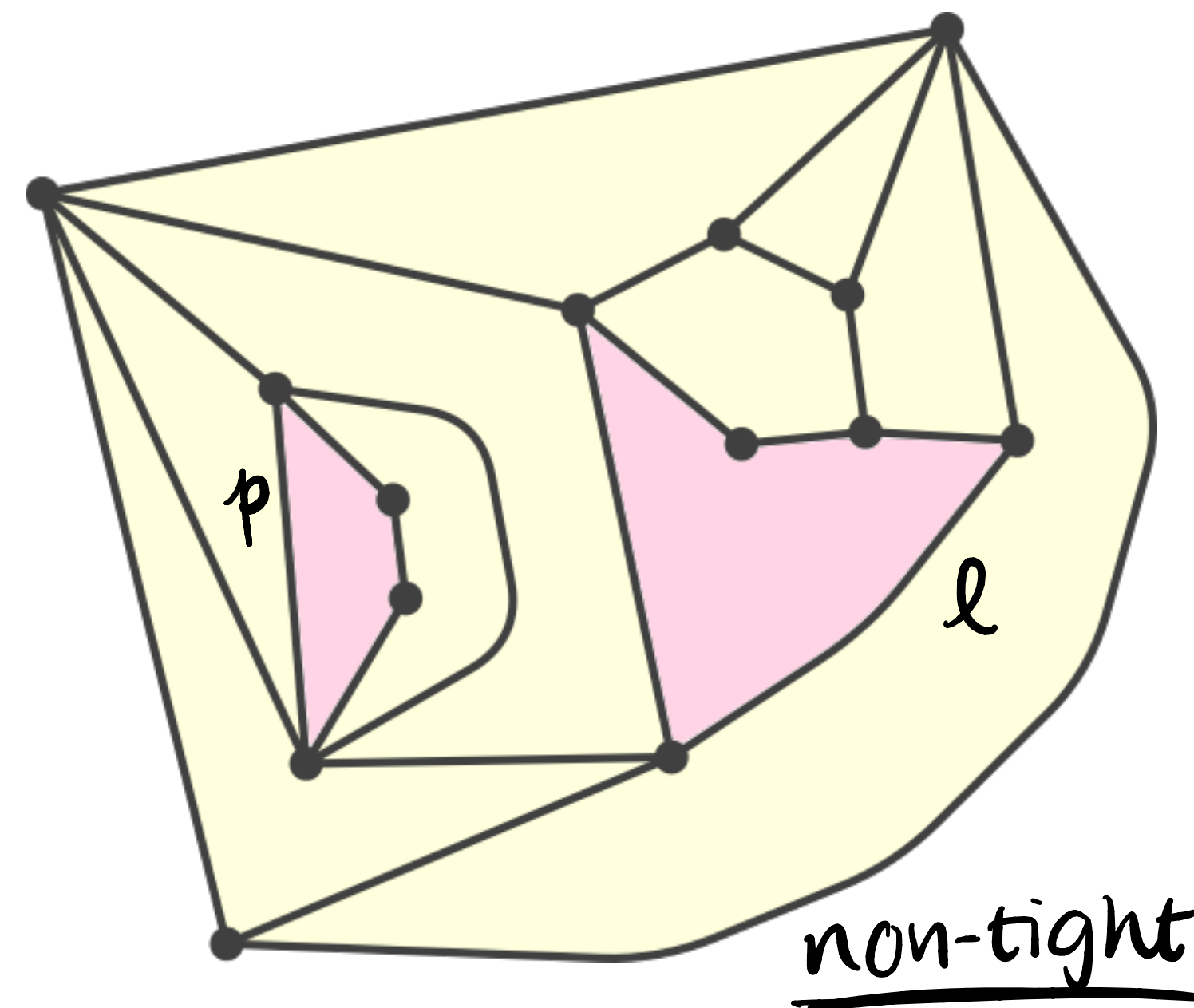
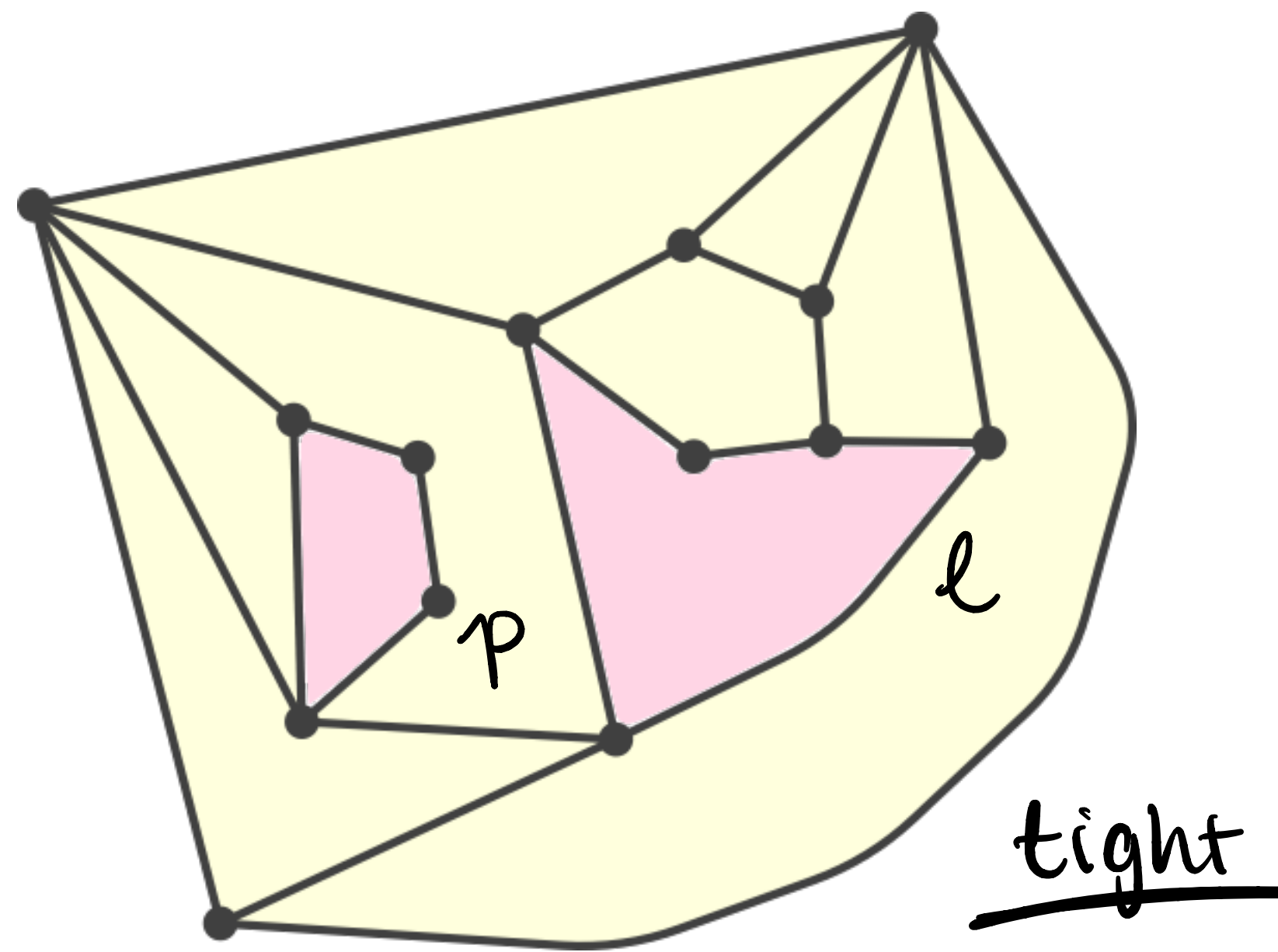
$$\left( 1 = |F(m)| - \frac{|V(m)|}{2} \right)$$



# HALF-TIGHT CYLINDERS

$$W^{(0)}(\ell)[\mu] = \int_0^\infty W^{(0)}(\ell, p) \mu(p) dp - 2 \left. \frac{\partial^2 W^{(0)}(\ell, p)}{\partial p^2} \right|_{p=0}$$

$$W^{(0)}(\ell, p)[\mu] = H(\ell, p) + \int_0^p dy H(\ell, y) y H(p, y) \quad (\ell > p)$$



J. Bouttier  
E. Quilley  
A. Miermont  
(2022)

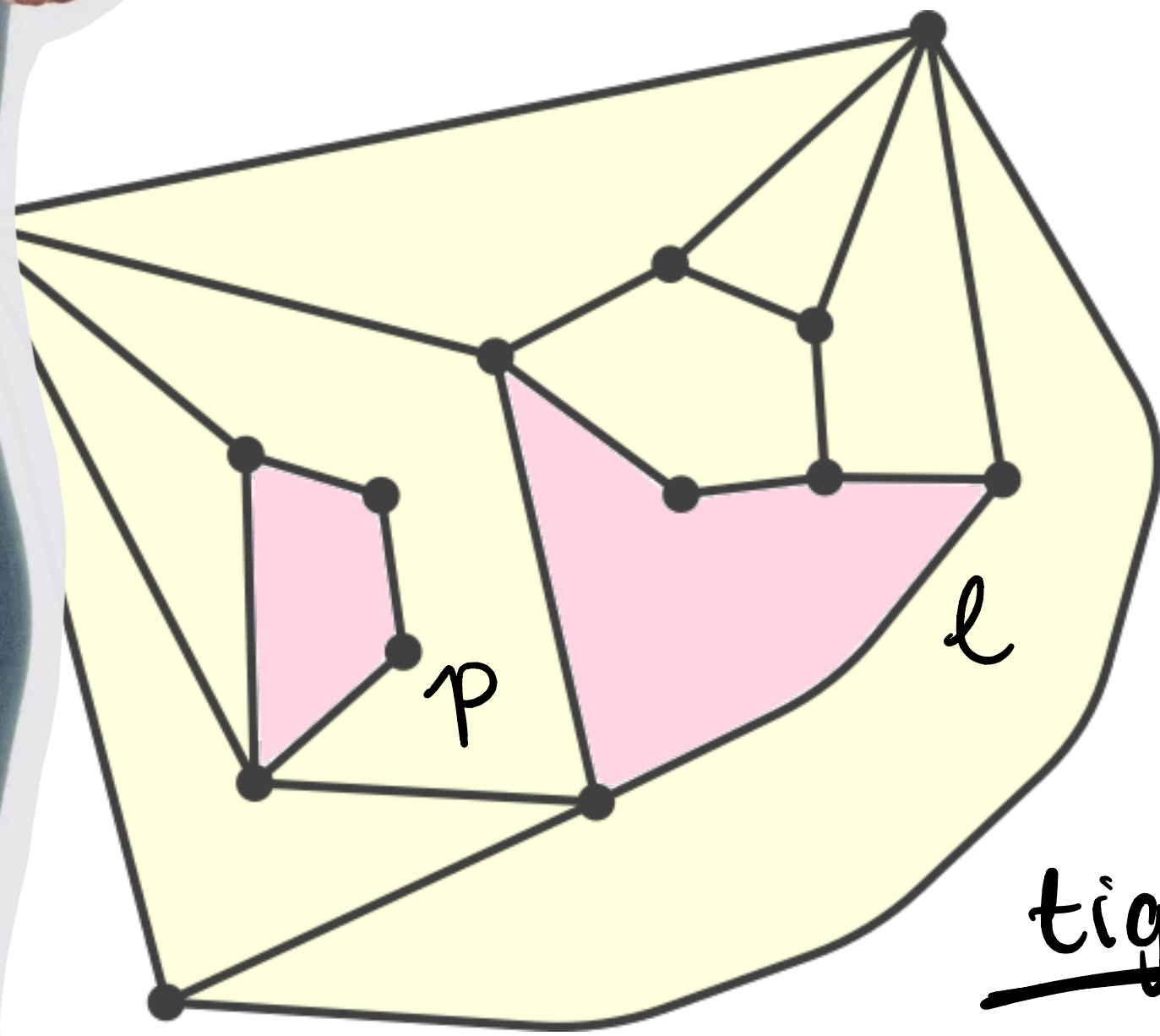
T. Budd.  
B. Zonneveld  
(2023)

$$H(\ell, p)[\mu] = -\frac{1}{p} \frac{\partial}{\partial p} I_0 \left( \sqrt{\ell^2 - p^2} \sqrt{E_\mu} \right) \mathbf{1}_{\{\ell \geq p\}}.$$

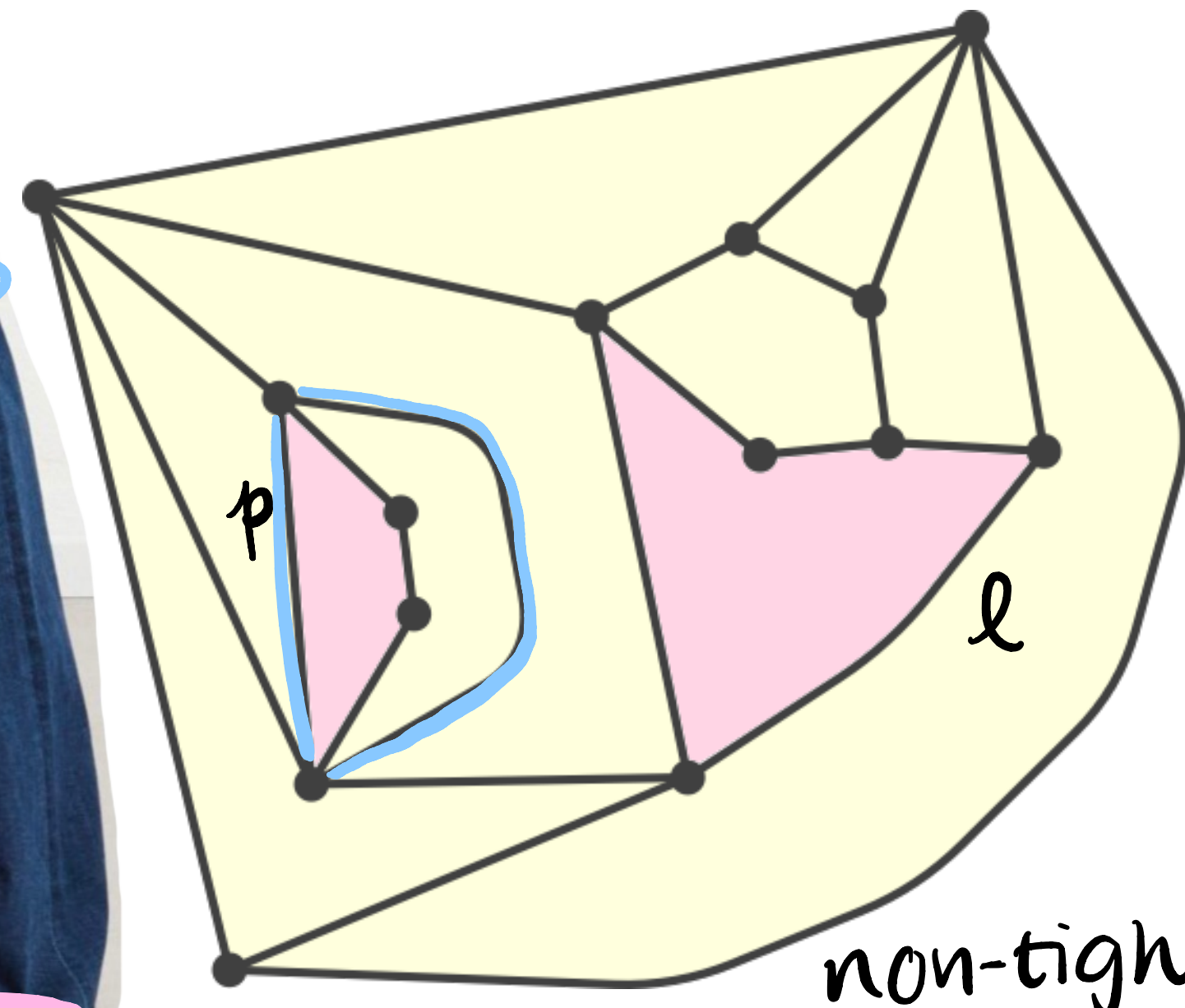
# HALF-TIGHT CYLINDERS

$$W^{(0)}(\ell)[\mu] = \int_0^\infty W^{(0)}(\ell, p) dp - 2 \left. \frac{\partial^2 W^{(0)}(\ell, p)}{\partial p^2} \right|_{p=0}$$

$$W^{(0)}(\ell, p)[\mu] = H(\ell, p) + \int_0^p H(\ell, y) y H(p, y) dy \quad (\ell > p)$$



tight



non-tight

$$H(\ell, p)[\mu] = -\frac{1}{p} \frac{\partial}{\partial p} I_0 \left( \sqrt{\ell^2 - p^2} \sqrt{E_\mu} \right) \mathbf{1}_{\{\ell \geq p\}}.$$

J. Bouttier  
E. Quilley  
A. Miermont  
(2022)

T. Budd.  
B. Zonneveld  
(2023)



# HALF-TIGHT CYLINDERS

$$W^{(0)}(\ell)[\mu] = \int_0^\infty W^{(0)}(\ell, p) \mu(p) dp - 2 \left. \frac{\partial^2 W^{(0)}(\ell, p)}{\partial p^2} \right|_{p=0}$$

substitute.

$$W^{(0)}(\ell, p)[\mu] = H(\ell, p) + \int_0^p dy H(\ell, y) y H(p, y) \quad (\ell > p)$$

$$H(\ell, p)[\mu] = -\frac{1}{p} \frac{\partial}{\partial p} I_0 \left( \sqrt{\ell^2 - p^2} \sqrt{E_\mu} \right) \mathbf{1}_{\{\ell \geq p\}}.$$

T. Budd.  
B. Zonneveld  
(2023)

## DISK SUBSTITUTION / SUBTRACTION

Use  $W^{(0)}(\ell)[\mu] = \int_0^\infty W^{(0)}(\ell, p)\mu(p)dp - 2 \left. \frac{\partial^2 W^{(0)}(\ell, p)}{\partial p^2} \right|_{p=0}$

to solve

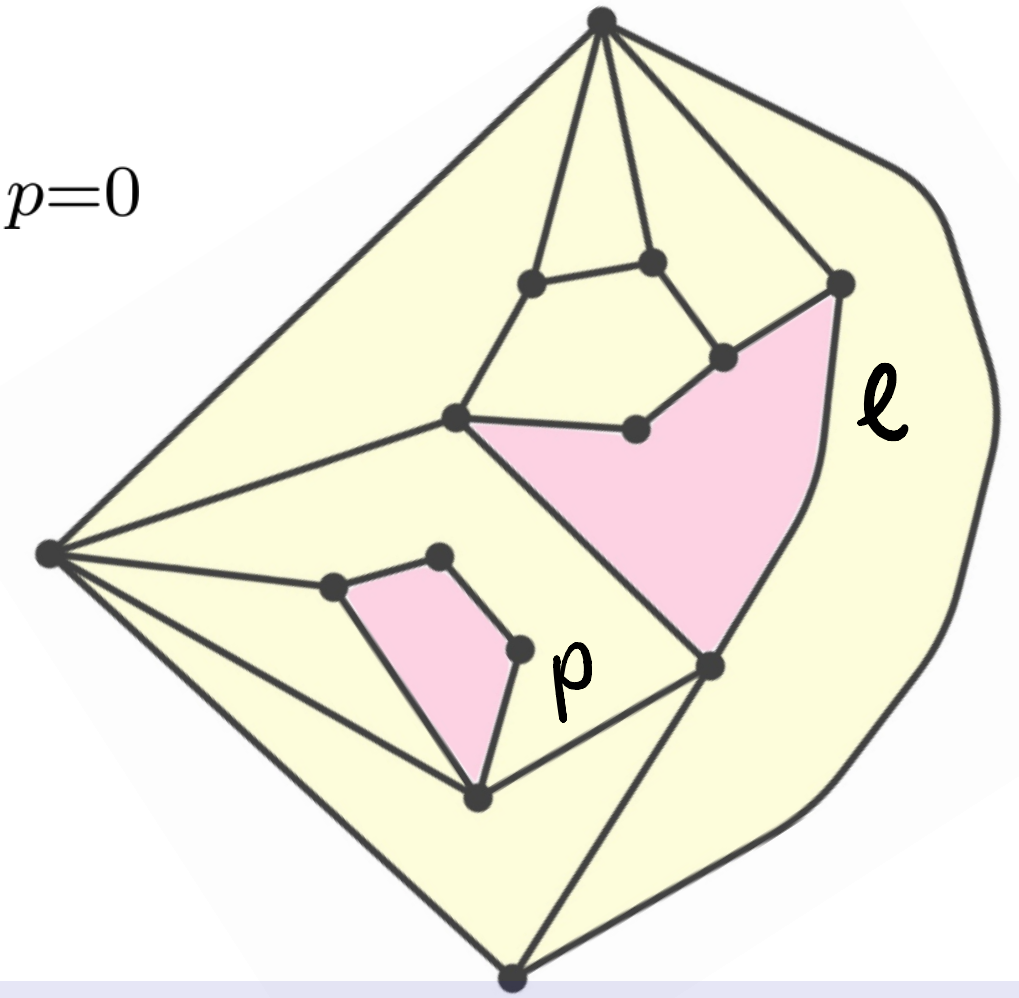
$$W^{(0)}(\ell)[\mu] + \frac{\mu(\ell)}{\ell} = 0$$

for  $\ell < q$



# DISK SUBSTITUTION / SUBTRACTION

$$W^{(0)}(\ell)[\mu] = \int_0^\infty W^{(0)}(\ell, p)\mu(p)dp - 2 \left. \frac{\partial^2 W^{(0)}(\ell, p)}{\partial p^2} \right|_{p=0}$$



$$W^{(0)}(\ell)[\mu] + \frac{\mu(\ell)}{\ell} =$$

$$\frac{\mu(\ell)}{\ell} + \int_0^\infty dp W^{(0)}(\ell, p)\mu(p) = \frac{U_+(\ell)}{\ell} + \int_0^\ell dy H(\ell, y)U_+(y)$$

with

$$U_+(y) = \mu(y) + y \int_y^\infty dp H(p, y)\mu(p)$$

# DISK SUBSTITUTION / SUBTRACTION

$$W^{(0)}(\ell)[\mu] + \frac{\mu(\ell)}{\ell} = \frac{\tilde{\mu}(\ell)}{\ell} + \int_0^\ell dy H(\ell, y) \tilde{\mu}(y)$$

with

$$\tilde{\mu}(y) = U_+(y) - U_-(y)$$

$$W^{(0)}(\ell)[\mu] = \int_0^\infty W^{(0)}(\ell, p) \mu(p) dp - 2 \left. \frac{\partial^2 W^{(0)}(\ell, p)}{\partial p^2} \right|_{p=0}$$

## DISK SUBSTITUTION / SUBTRACTION

$$\begin{aligned}\tilde{\mu}(y) &= U_+(y) - U_-(y) \\ &= \frac{\partial}{\partial y} \left( \frac{2\sqrt{E_\mu}}{y} J_1(y\sqrt{E_\mu}) - \int_y^\infty d\ell \mu(\ell) I_0 \left( \sqrt{(\ell^2 - y^2)E_\mu} \right) \right)\end{aligned}$$

to solve

$$0 = W^{(0)}(\ell)[\mu] + \frac{\mu(\ell)}{\ell} = \frac{\tilde{\mu}(\ell)}{\ell} + \int_0^\ell dy H(\ell, y) \tilde{\mu}(y) \quad \text{for } \ell < a$$

## DISK SUBSTITUTION / SUBTRACTION

$$\begin{aligned}\tilde{\mu}(y) &= U_+(y) - U_-(y) \quad x^{(y)}(E_\mu) \\ &= \frac{\partial}{\partial y} \left( \frac{2\sqrt{E_\mu}}{y} J_1(y\sqrt{E_\mu}) - \int_y^\infty d\ell \mu(\ell) I_0 \left( \sqrt{(\ell^2 - y^2)E_\mu} \right) \right)\end{aligned}$$

$$\tilde{\mu}(y) = 0 \quad \text{for all } y \leq q.$$

$$x^{(y)}(E_\mu) = 0 \quad \text{for all } y \leq q.$$

Since  $x^{(0)}(E_\mu) = 0$  and  $x' = 0$

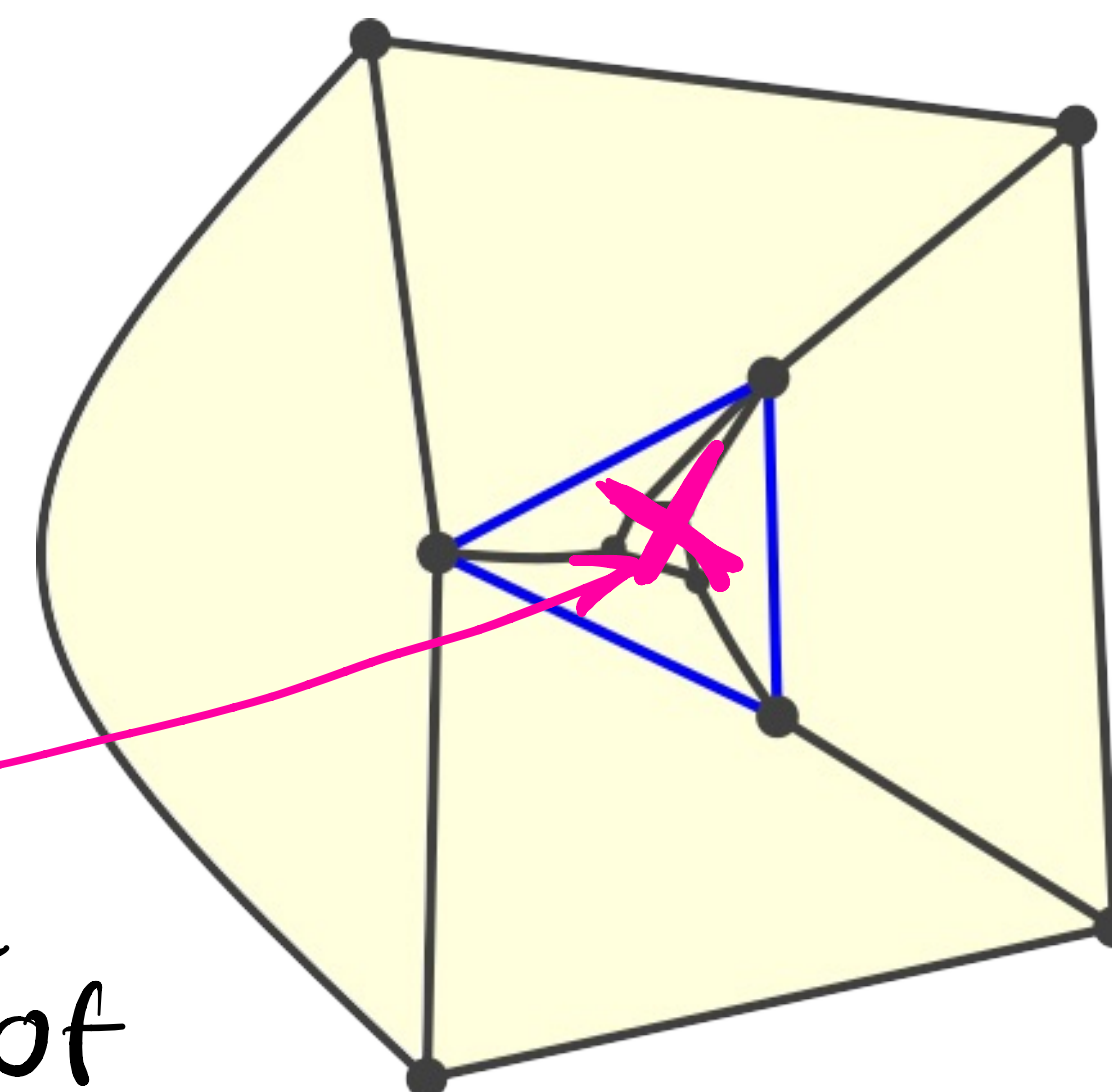


## DISK SUBSTITUTION / SUBTRACTION

$$\tilde{\mu}(y) = U_+(y) - U_-(y)$$

$$\tilde{\mu}(y) = 0 \quad \text{for all } y \leq q.$$

forbid  
disks of  
perimeter  $y \leq q$



$$U_+(y) = \begin{cases} U_-(y) & \text{for } y \leq q, \\ \mu(y) + y \int_y^\infty dp H(p, y) \mu(p) & \text{for } y > q. \end{cases}$$

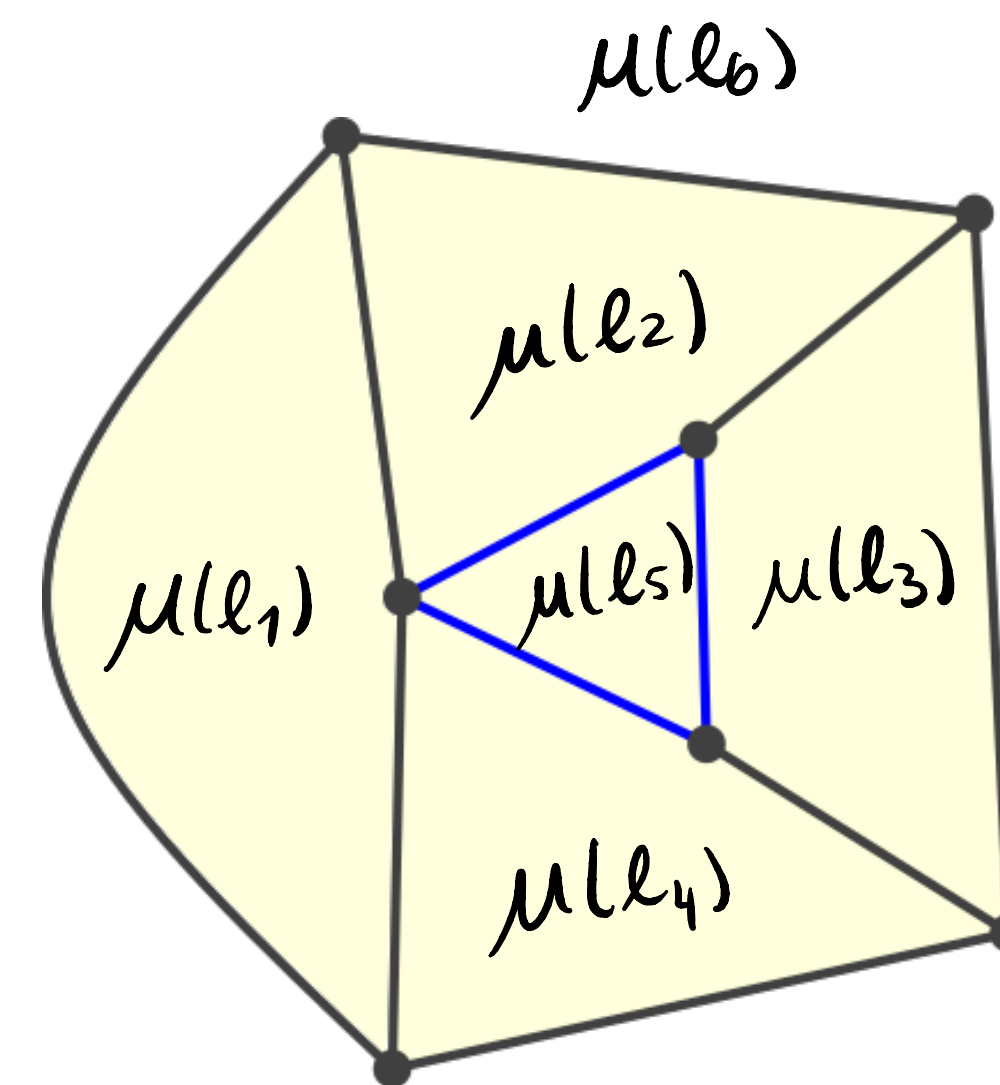
## EXAMPLE

Consider a  $q$ -irreducible metric map.

$$\mu(l) = c l e^{-\frac{l^2}{2t}} \quad \text{for } l \geq q,$$

string equation

$$\frac{2\sqrt{E_\mu}}{q} J_1 \left( q\sqrt{E_\mu} \right) = c t e^{-\frac{q^2}{2t} + \frac{tE_\mu}{2}}$$



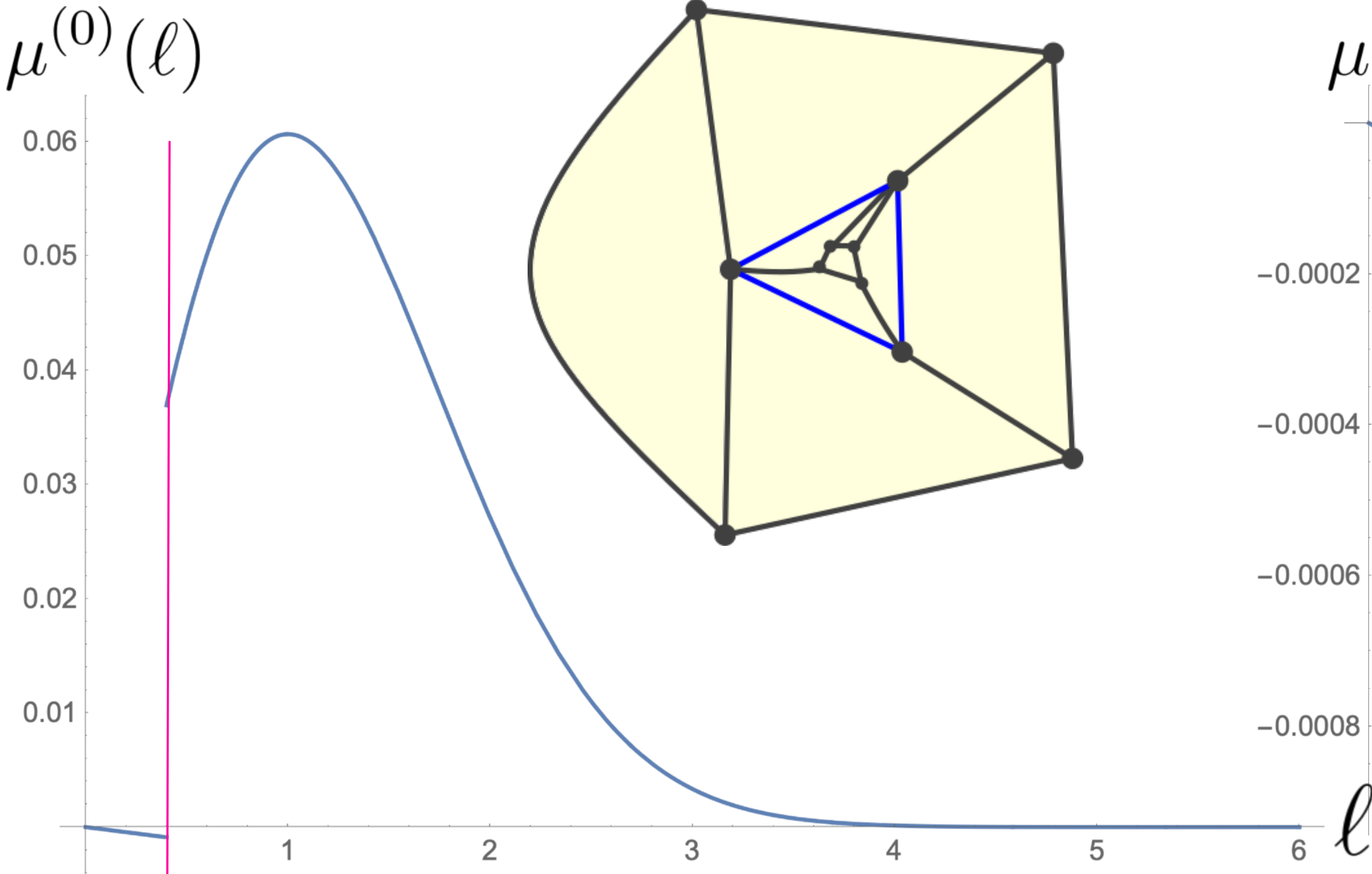
## EXAMPLE

$$\mu(\ell) = c\ell e^{-\frac{\ell^2}{2t}} \quad \text{for } \ell \geq q,$$

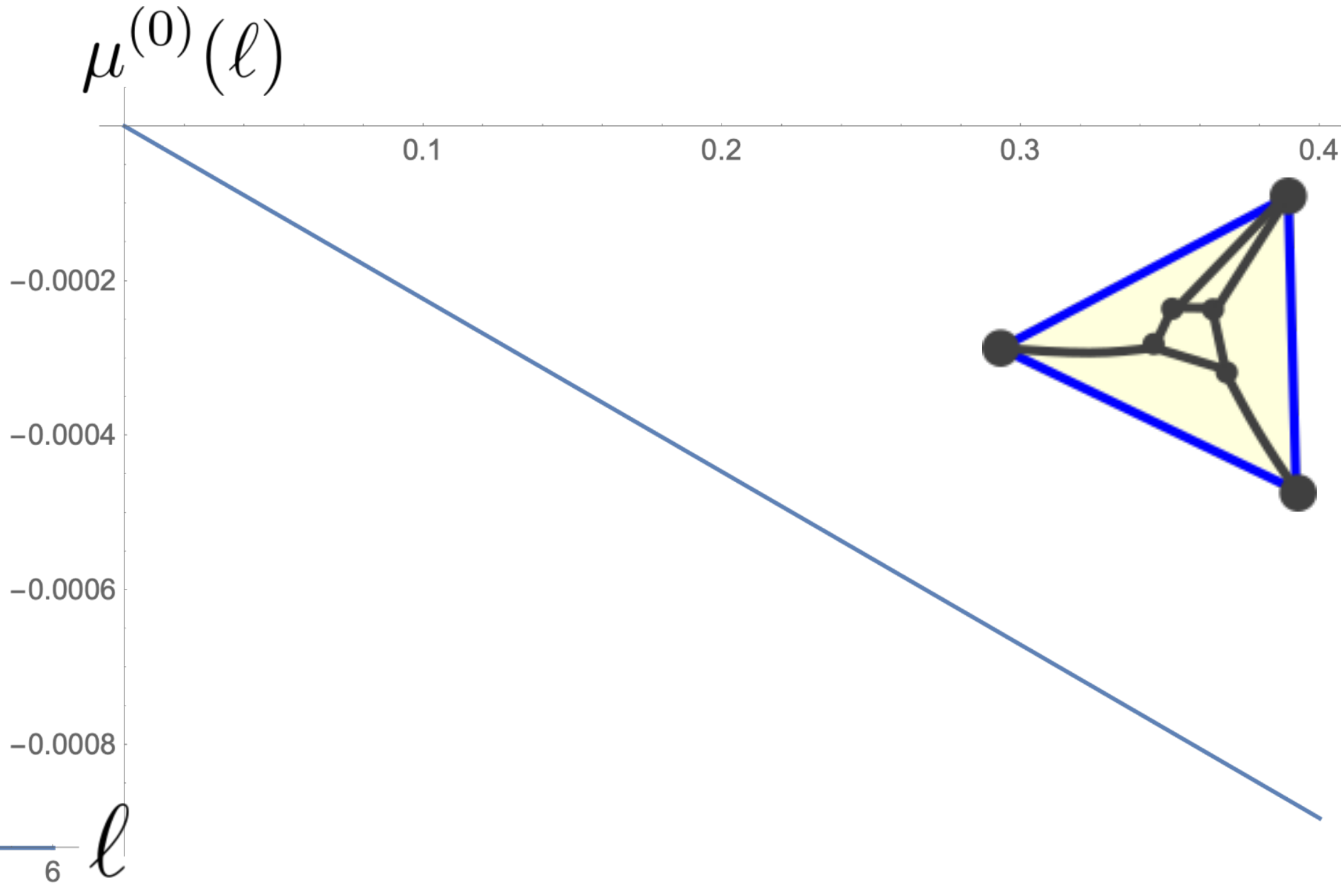
$$\begin{aligned} \mu^{(0)}(\ell) &= \ell c e^{-\frac{\ell^2}{2t}} (1 - e^{-\frac{tE_\mu}{2}}) + \frac{2E_\mu J_2(\ell\sqrt{E_\mu})}{\ell} \\ &+ \ell \int_\ell^q dy c y e^{-\frac{y^2}{2t} + \frac{tE_\mu}{2}} \sqrt{\frac{E_\mu}{y^2 - \ell^2}} J_1\left(\sqrt{y^2 - \ell^2} \sqrt{E_\mu}\right) \\ &- \ell \int_\ell^q dy \frac{2E_\mu J_2(y\sqrt{E_\mu})}{y} \sqrt{\frac{E_\mu}{y^2 - \ell^2}} J_1\left(\sqrt{y^2 - \ell^2} \sqrt{E_\mu}\right) \end{aligned}$$

# EXAMPLE

$$W^{(q)}(L)[\mu_q, \mu_{q+1}, \dots] = W^{(0)}(L)[\mu_1^{(q)}, \mu_2^{(q)}, \dots, \mu_q^{(q)}, \mu_{q+1}, \dots]$$



$q = 0.4, c = 0.1, t = 1$



$E_\mu = 0.097092$



# IRREDUCIBLE METRIC MAPS so far...

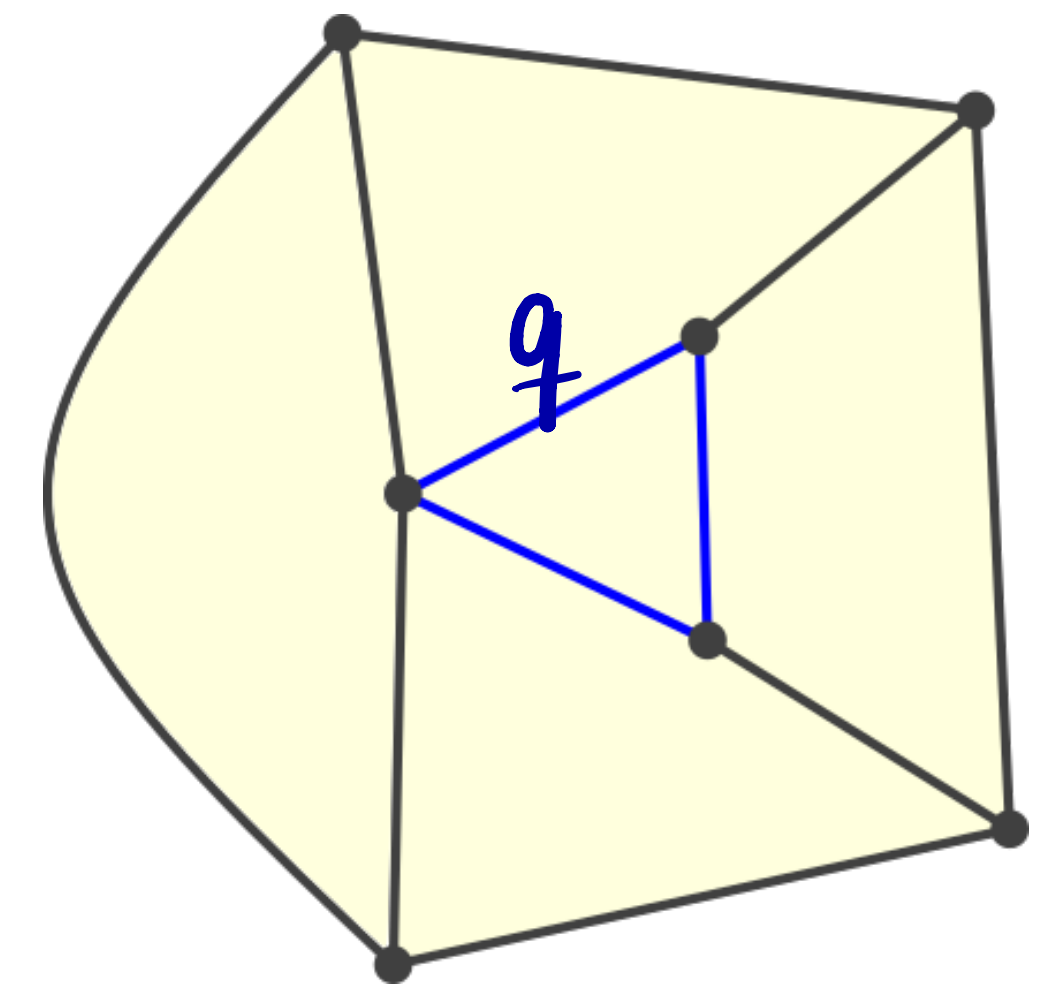
$$g = 0$$

String equation

$$x^{(q)}(u) = \frac{2\sqrt{u}}{q} J_1(q\sqrt{u}) - \int_q^\infty d\ell \mu(\ell) I_0\left(\sqrt{(\ell^2 - q^2)u}\right).$$

Generating function

$$F_0^{(q)}[\mu] = \frac{1}{4} \int_0^{E_\mu} du \left(x^{(q)}(u)\right)^2$$



# IRREDUCIBLE METRIC MAPS so far...

$$g = 0$$

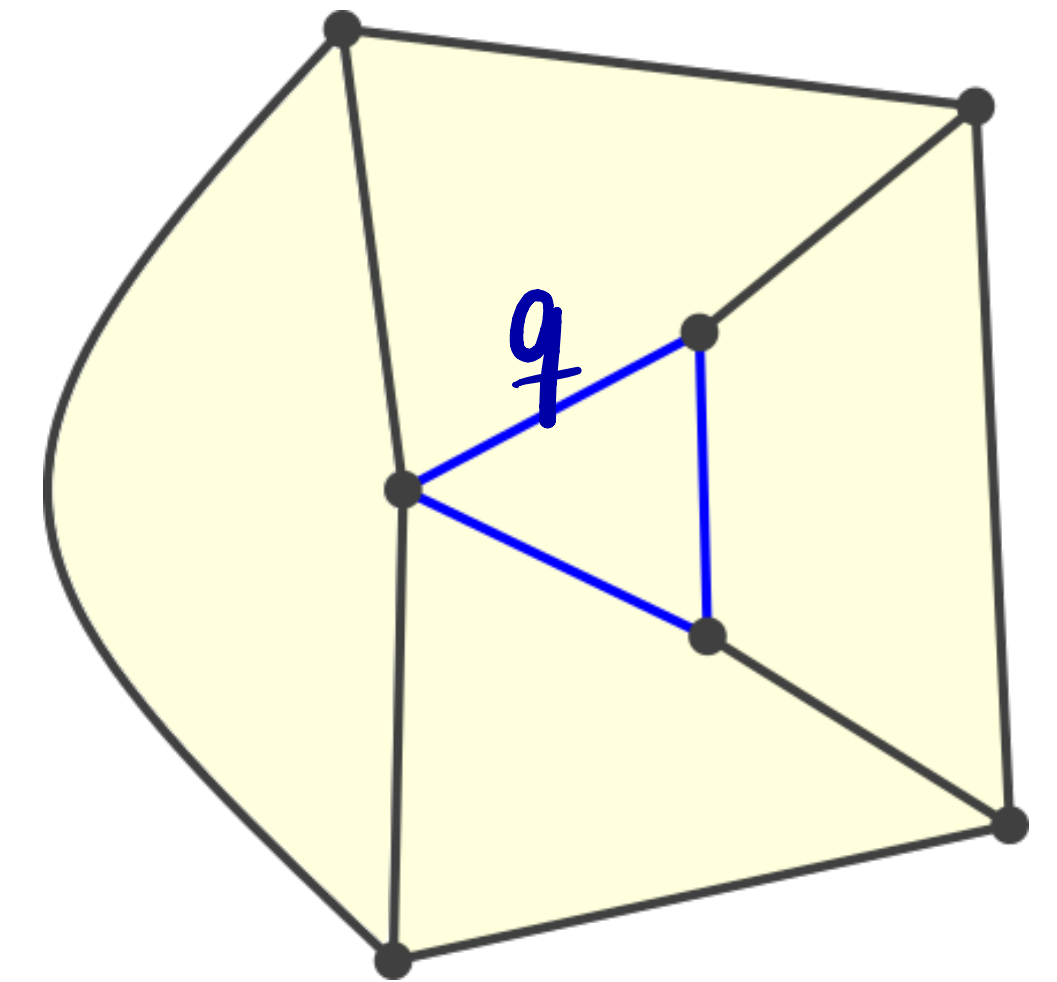
string equation

$$x^{(q)}(u) = \frac{2\sqrt{u}}{q} J_1(q\sqrt{u}) - \int_q^\infty d\ell \mu(\ell) I_0\left(\sqrt{(\ell^2 - q^2)u}\right).$$

Generating function

$$F_0^{(q)}[\mu] = \frac{1}{4} \int_0^{E_\mu} du \left(x^{(q)}(u)\right)^2$$

$$g > 0 \quad ?$$



# MOMENTS

$$g > 0$$

for meric maps.

C. Itzykson,  
J.B. Zuber  
(1992)

$$F_1^{(0)}[\mu] = -\frac{1}{24} \log M_0,$$

$$F_g^{(0)}[\mu] = \left( \frac{2}{(M_0)^2} \right)^{g-1} \mathcal{P}_g \left( \frac{M_1}{M_0}, \dots, \frac{M_{3g-3}}{M_0} \right) \quad \text{for } g \geq 2$$

$$M_p = \frac{\partial^{p+1} x}{\partial u^{p+1}} (E_\mu)$$

topological  
recursion

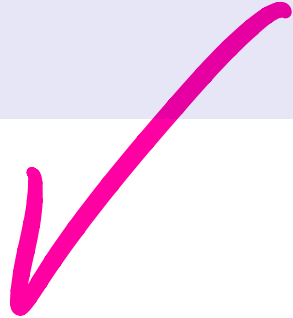
$$x(u) = u - \int_0^\infty dl \mu(l) I_0(l\sqrt{u})$$

# MOMENTS

$$g > 0$$

$$M_p^{(0)} = \delta_{p=0} - \frac{4^{-p}}{2p!} \int_0^\infty \ell d\ell \bar{U}_+(\ell, E_\mu) \ell^{2p}.$$

$$\int_0^\infty \ell d\ell \bar{U}_+(\ell, E_\mu) \ell^{2p} = \int_0^q \ell d\ell \bar{U}_-(\ell, E_\mu) \ell^{2p} + \int_q^\infty \ell d\ell \bar{U}_+(\ell, E_\mu) \ell^{2p}$$

$$M_p^{(0)}|_{\text{irred}} = \sum_{k=0}^p \frac{1}{(p-k)!} \left(\frac{q}{2}\right)^{2(p-k)} \left( \frac{\partial^{k+1} x^{(q)}}{\partial u^{k+1}} (E_\mu) \right)$$




# IRREDUCIBLE METRIC MAPS

Generating functions  
 $\forall g$

$$F_0^{(q)}[\mu] = \frac{1}{4} \int_0^{E_\mu} du \left(x^{(q)}(u)\right)^2$$

$$F_1^{(0)}[\mu] = -\frac{1}{24} \log M_0,$$

$$F_g^{(0)}[\mu] = \left(\frac{2}{(M_0)^2}\right)^{g-1} \mathcal{P}_g \left(\frac{M_1}{M_0}, \dots, \frac{M_{3g-3}}{M_0}\right) \quad \text{for } g \geq 2$$

$$x^{(q)}(u) = \frac{2\sqrt{u}}{q} J_1(q\sqrt{u}) - \int_q^\infty d\ell \mu(\ell) I_0\left(\sqrt{(\ell^2 - q^2)r}\right).$$

$$M_p^{(q)} = \sum_{k=0}^p \frac{1}{(p-k)!} \left(\frac{q}{2}\right)^{2(p-k)} \left(\frac{\partial^{k+1} x^{(q)}}{\partial u^{k+1}}(E_\mu)\right).$$

T. Budd (2022)

T. Budd  
A. Castro (2025)

# IRREDUCIBLE METRIC MAPS

T. Budd (2022)

T. Budd

A. Castro (2025)

$$F_g^{(q)}[\mu] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_q^{\infty} dl_1 \mu(l_1) \cdots \int_q^{\infty} dl_k \mu(l_k) V_{g,k}^{(q)}(l_1, \dots, l_k) \quad \checkmark$$

$$F_0^{(q)}[\mu] = \frac{1}{4} \int_0^{E_\mu} du (x^{(q)}(u))^2$$

$$F_1^{(0)}[\mu] = -\frac{1}{24} \log M_0,$$

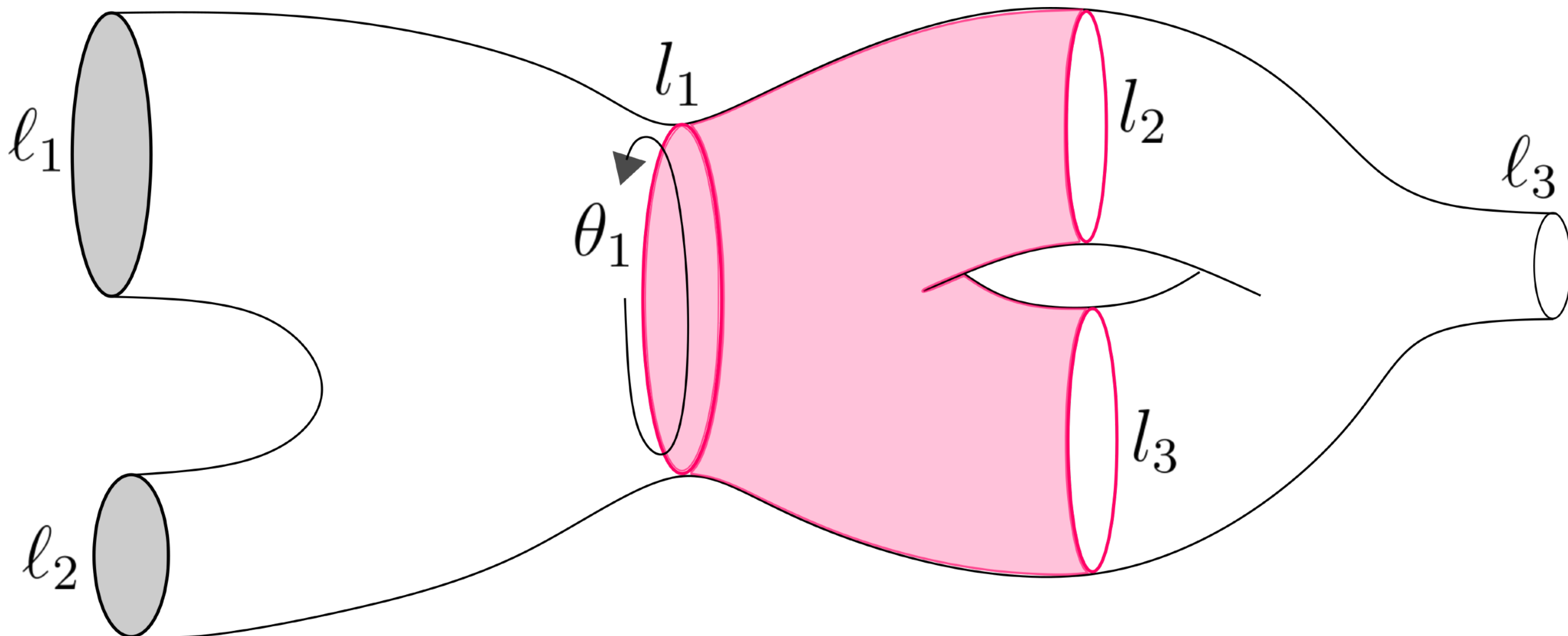
$$F_g^{(0)}[\mu] = \left( \frac{2}{(M_0)^2} \right)^{g-1} \mathcal{P}_g \left( \frac{M_1}{M_0}, \dots, \frac{M_{3g-3}}{M_0} \right) \quad \text{for } g \geq 2$$

# Relation to hyperbolic surfaces

## Weil-Petersson volumes

Mirzakhani (2005)

$V_{g,k}^{WP}(l_1, \dots, l_k)$  Polynomials in  $\{l_i^2\}_{i=1}^k$

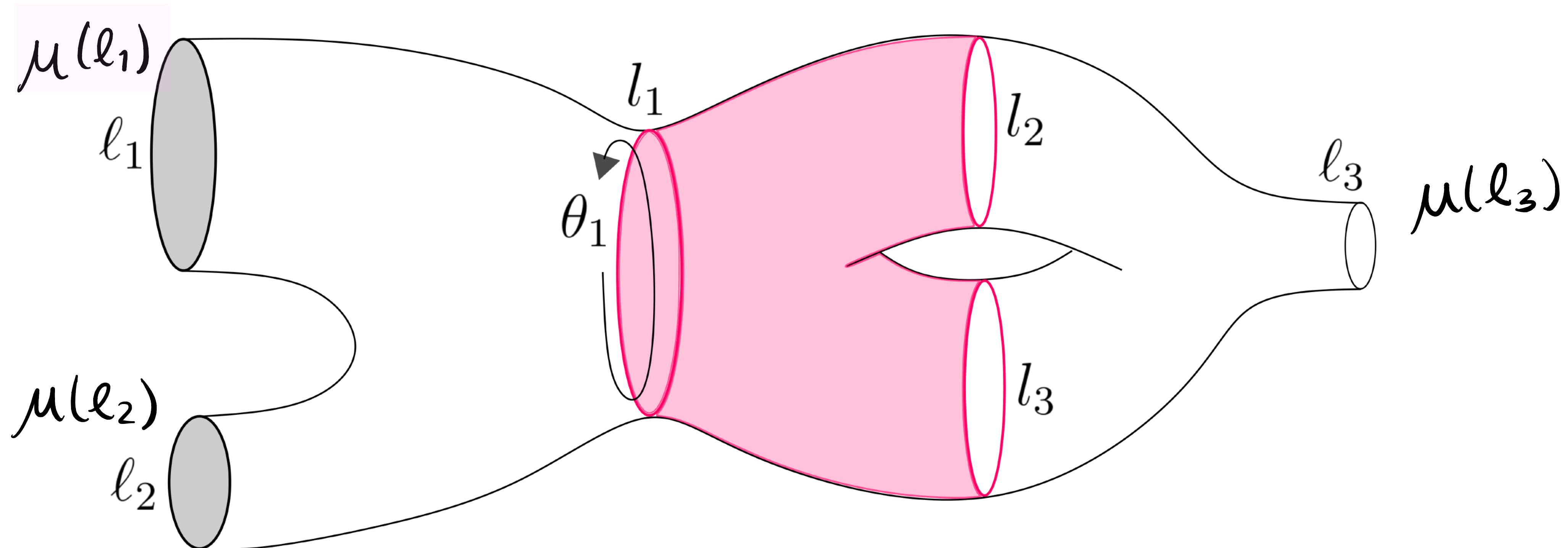


# Relation to hyperbolic surfaces

Kaufmann, Manin  
Zagier (1996)

## Weil-Petersson volumes

$$F_g^{WPP}[\mu] = \sum_{k=0}^{\infty} \frac{2^{2-2g-k}}{k!} \int_0^{\infty} d\alpha_1 \mu(l_1) \cdots \int_0^{\infty} dl_k \mu(l_k) V_{g,k}^{WPP}(l_1, \dots, l_k),$$





# Relation to hyperbolic surfaces

## Weil-Petersson volumes

Kaufmann, Manin  
Zagier (1996)

$$F_g^{WP}[\mu] = \sum_{k=0}^{\infty} \frac{2^{2-2g-k}}{k!} \int_0^{\infty} d\alpha_1 \mu(\alpha_1) \cdots \int_0^{\infty} d\alpha_k \mu(\alpha_k) V_{g,k}^{WP}(\alpha_1, \dots, \alpha_k),$$

Generating function

$$F_0^{WP}[\mu] = \frac{1}{4} \int_0^{E_\mu} du x(u)^2,$$

String equation

$$x(u) = \frac{\sqrt{u}}{\pi} J_1(2\pi\sqrt{u}) - \int_0^{\infty} dl \mu(l) I_0(l\sqrt{u})$$

Disk function

$$W^{WP}(b)[\mu] = \frac{\delta F_0^{WP}}{\delta \mu(b)} = -\frac{1}{2} \int_0^{E_\mu} du x(u) I_0(l\sqrt{u}).$$

# Relation to hyperbolic surfaces

## Weil-Petersson volumes

Hyperbolic surface

$$W^{WP}(b)[\mu] = \frac{\delta F_0^{WP}}{\delta \mu(b)} = -\frac{1}{2} \int_0^{E_\mu} du x(u) I_0(b\sqrt{u}).$$

$$x(u) = \frac{\sqrt{u}}{\pi} J_1(2\pi\sqrt{u}) - \int_0^\infty dl \mu(l) I_0(l\sqrt{u})$$

$$q = 2\pi$$
$$b \rightarrow \sqrt{b^2 + 4\pi^2}$$

q-irred. metric map

$$W^{(q)}(b)[\mu] = \frac{\delta F_0^{(q)}}{\delta \mu(b)} = -\frac{1}{2} \int_0^{E_\mu} du x^{(q)}(u) I_0(\sqrt{(b^2 - q^2)u}).$$

$$x^{(q)}(u) = \frac{2\sqrt{u}}{q} J_1(q\sqrt{u}) - \int_q^\infty dl \mu(l) I_0(\sqrt{(l^2 - q^2)u}).$$

# Relation to hyperbolic surfaces

## Weil-Petersson volumes

$$W_{\mu}^{(2\pi)}(b) = W_{\hat{\mu}}^{WP} \left( \sqrt{b^2 - 4\pi^2} \right)$$

$$\hat{\mu}(y) = \frac{y}{\sqrt{y^2 + 4\pi^2}} \mu \left( \sqrt{y^2 + 4\pi^2} \right)$$

$$W^{(q)}(\ell)[\mu] = \frac{\delta F_0^{(q)}}{\delta \mu(\ell)} = \sum_{k=2}^{\infty} \frac{1}{k!} \int_q^{\infty} dl_1 \mu(l_1) \cdots \int_q^{\infty} dl_k \mu(l_k) V_{0,k+1}^{(q)}(\ell, l_1, \dots, l_k)$$

# Relation to hyperbolic surfaces

## Weil-Petersson volumes

$$W_{\mu}^{(2\pi)}(b) = W_{\hat{\mu}}^{WP} \left( \sqrt{b^2 - 4\pi^2} \right) \quad \hat{\mu}(y) = \frac{y}{\sqrt{y^2 + 4\pi^2}} \mu \left( \sqrt{y^2 + 4\pi^2} \right)$$

$$W^{(q)}(\ell)[\mu] = \frac{\delta F_0^{(q)}}{\delta \mu(\ell)} = \sum_{k=2}^{\infty} \frac{1}{k!} \int_q^{\infty} d\ell_1 \mu(\ell_1) \cdots \int_q^{\infty} d\ell_k \mu(\ell_k) V_{0,k+1}^{(q)}(\ell, \ell_1, \dots, \ell_k)$$

T. Budd (2022)

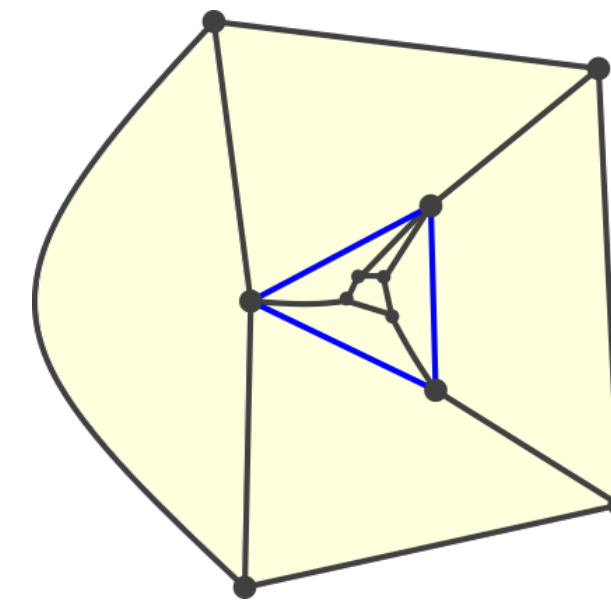
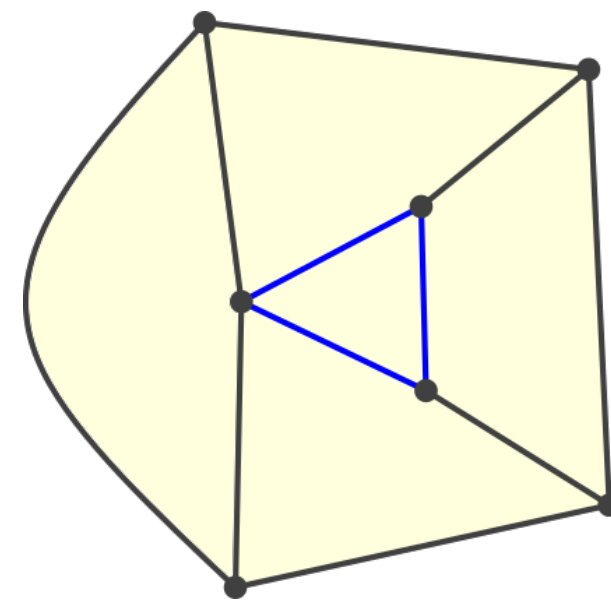
T. Budd  
A. Castro (2025)

$$V_{g,n}^{(2\pi)}(\ell_1, \dots, \ell_n) = 2^{2-2g-n} V_{g,n}^{WP} \left( \sqrt{\ell_1^2 - 4\pi^2}, \dots, \sqrt{\ell_n^2 - 4\pi^2} \right)$$



# SUMMARY

$$W^{(q)}(\ell)[\hat{\mu}] = W^{(0)}(\ell)[\mu] \quad \text{for all } \ell \geq q$$
$$W^{(0)}(\ell)[\mu] = -\frac{\mu(\ell)}{\ell} \quad \text{for all } \ell < q$$



Irreducible  
metric maps

Hyperbolic  
surfaces

Metric maps

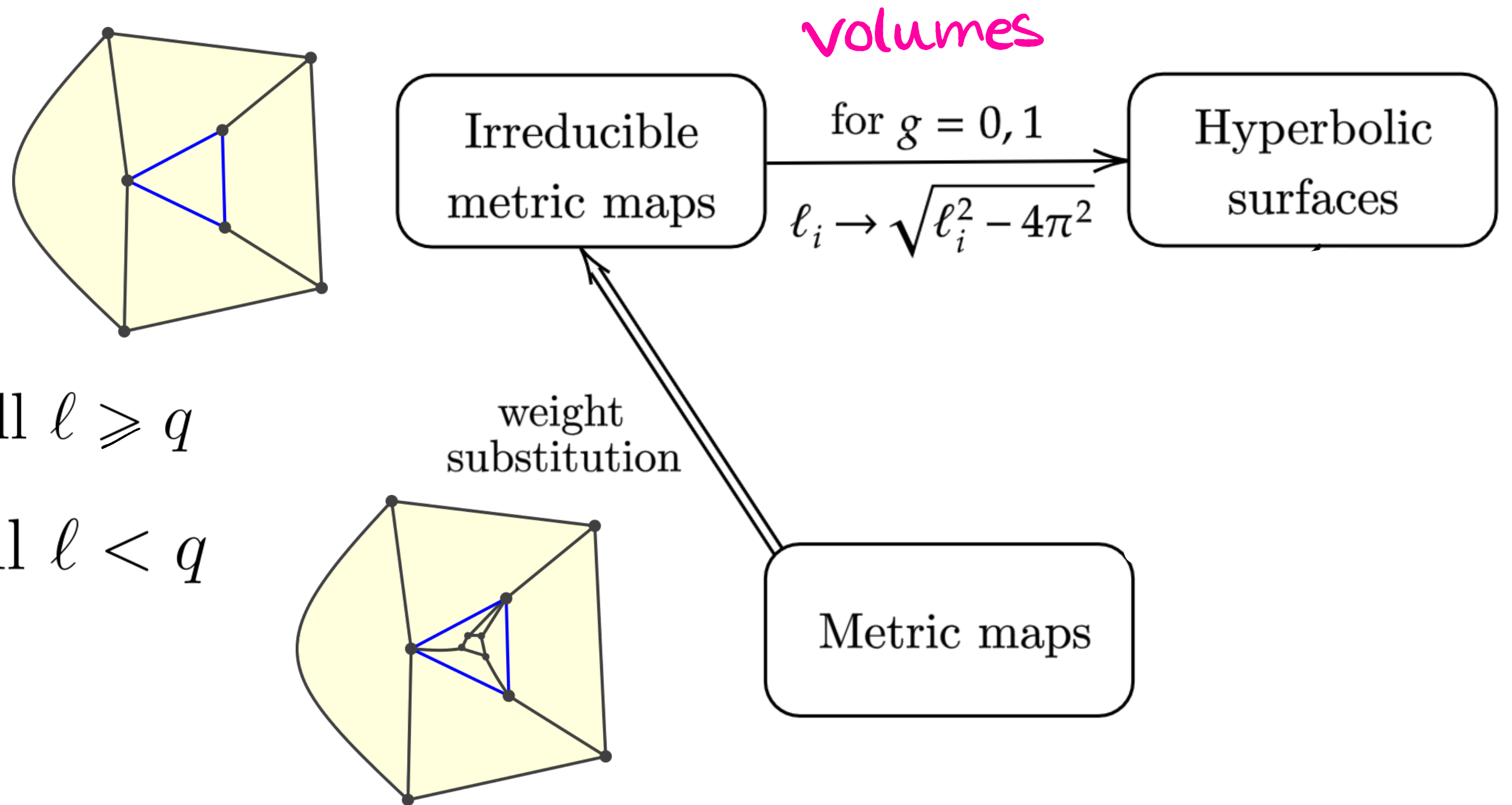
weight  
substitution

*combinatorial/  
geometric*

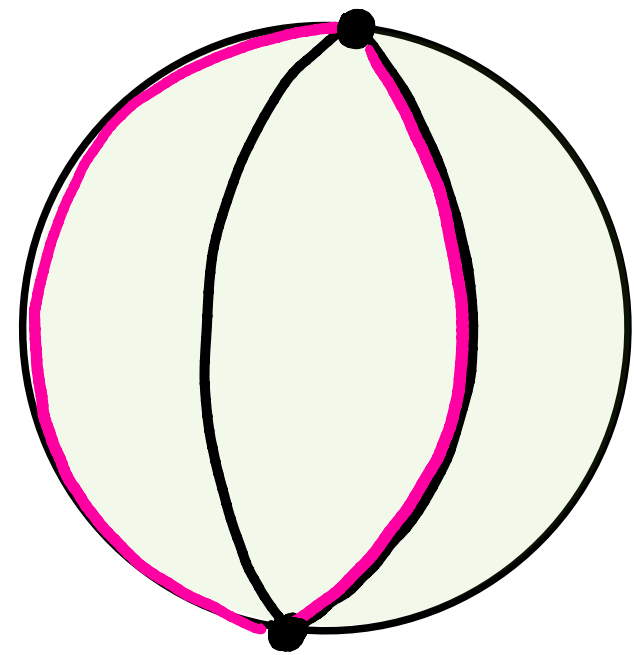
# SUMMARY

$$W^{(q)}(\ell)[\hat{\mu}] = W^{(0)}(\ell)[\mu] \quad \text{for all } \ell \geq q$$

$$W^{(0)}(\ell)[\mu] = -\frac{\mu(\ell)}{\ell} \quad \text{for all } \ell < q$$



# TO-DO in Math



T. Budd.

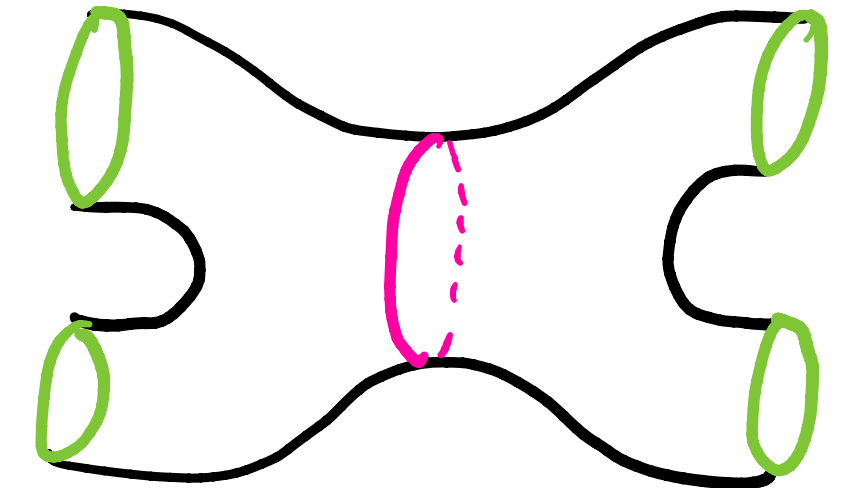
volumes

Irreducible  
metric maps

for  $g = 0, 1$

$$\ell_i \rightarrow \sqrt{\ell_i^2 - 4\pi^2}$$

Hyperbolic  
surfaces



T. Budd. &  
A. Castro

weight  
substitution

combinatorial  
/geometric

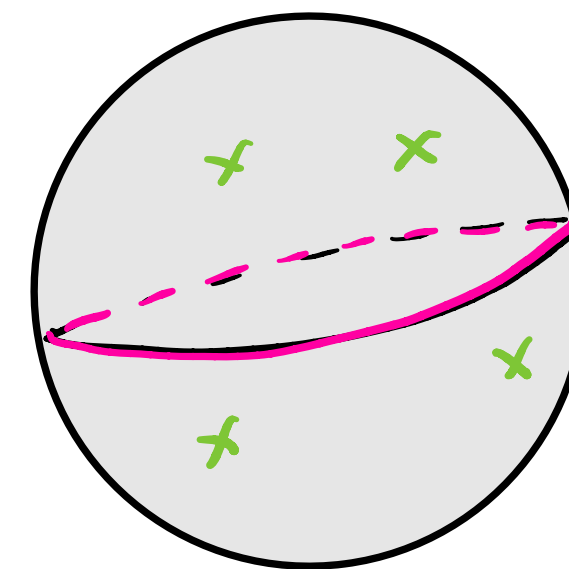
algebraic  
topology

$$\kappa_1 \rightarrow \psi$$

class

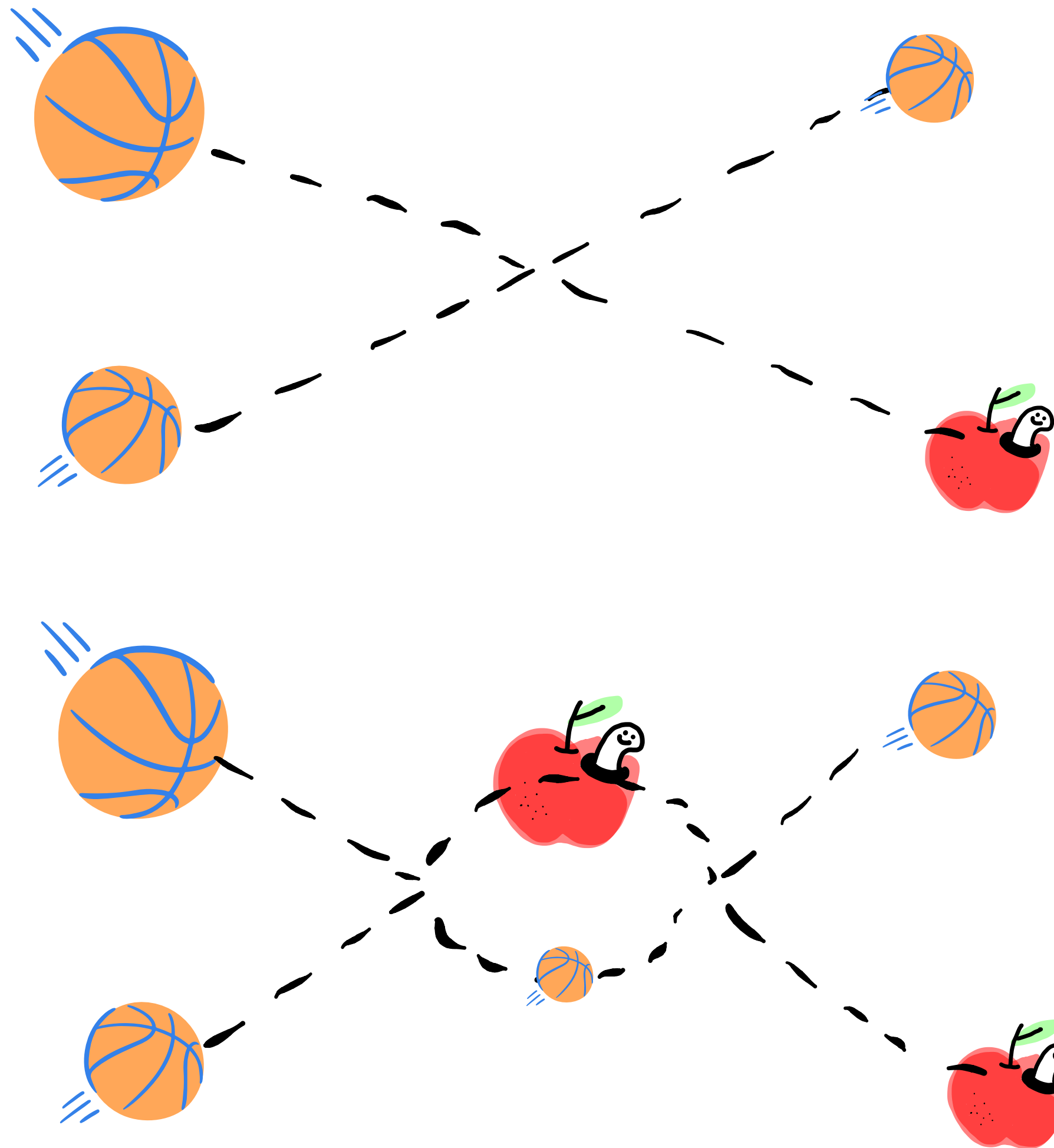
Kontsevich  
Witten.

Metric maps



# TO-DO in Physics

quantum field theory

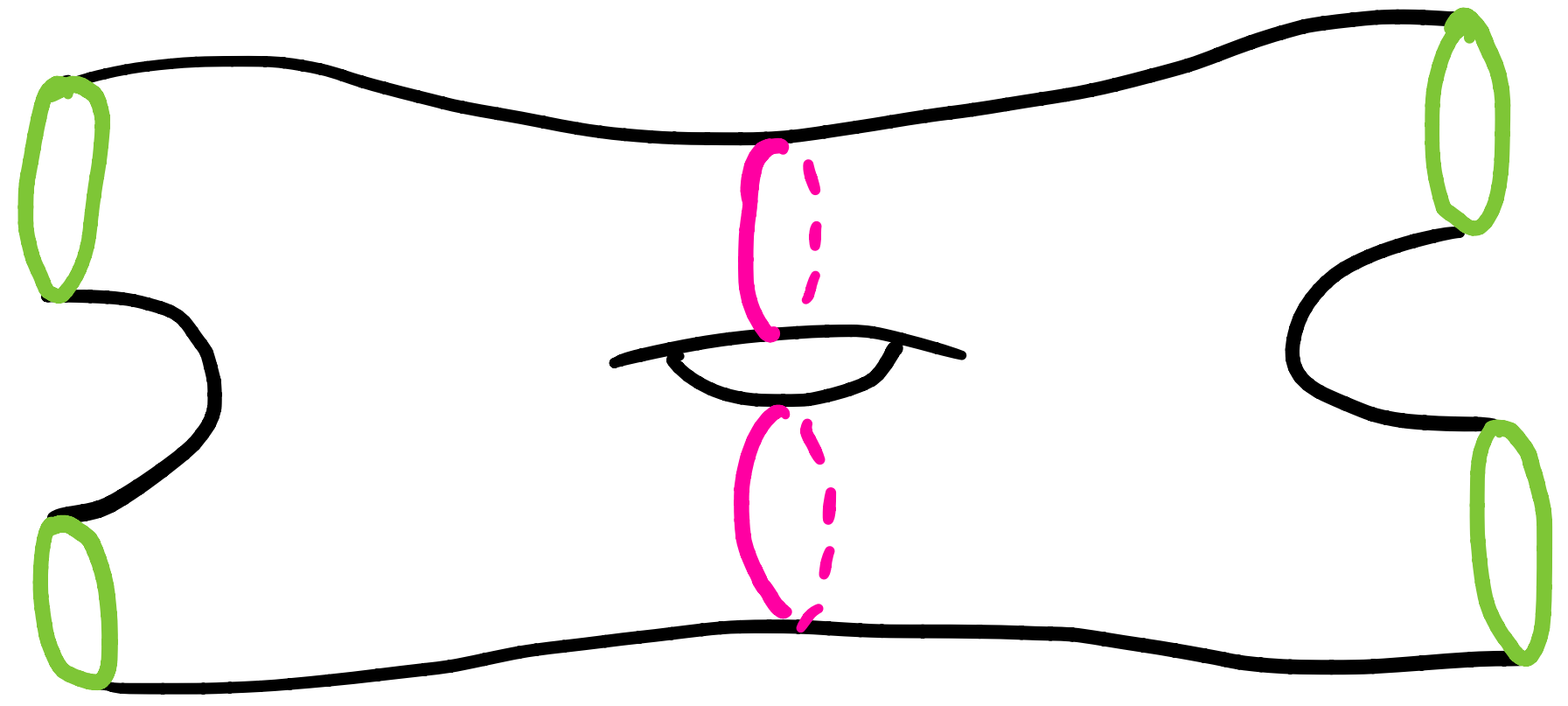
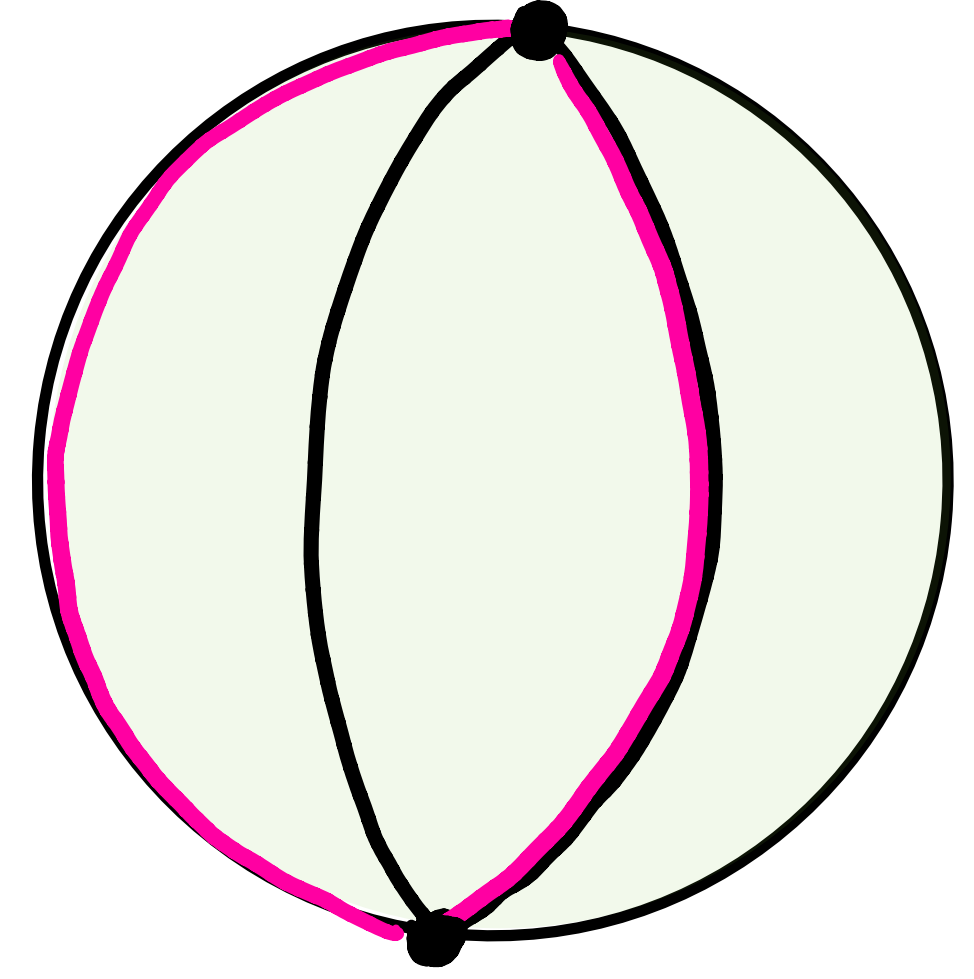
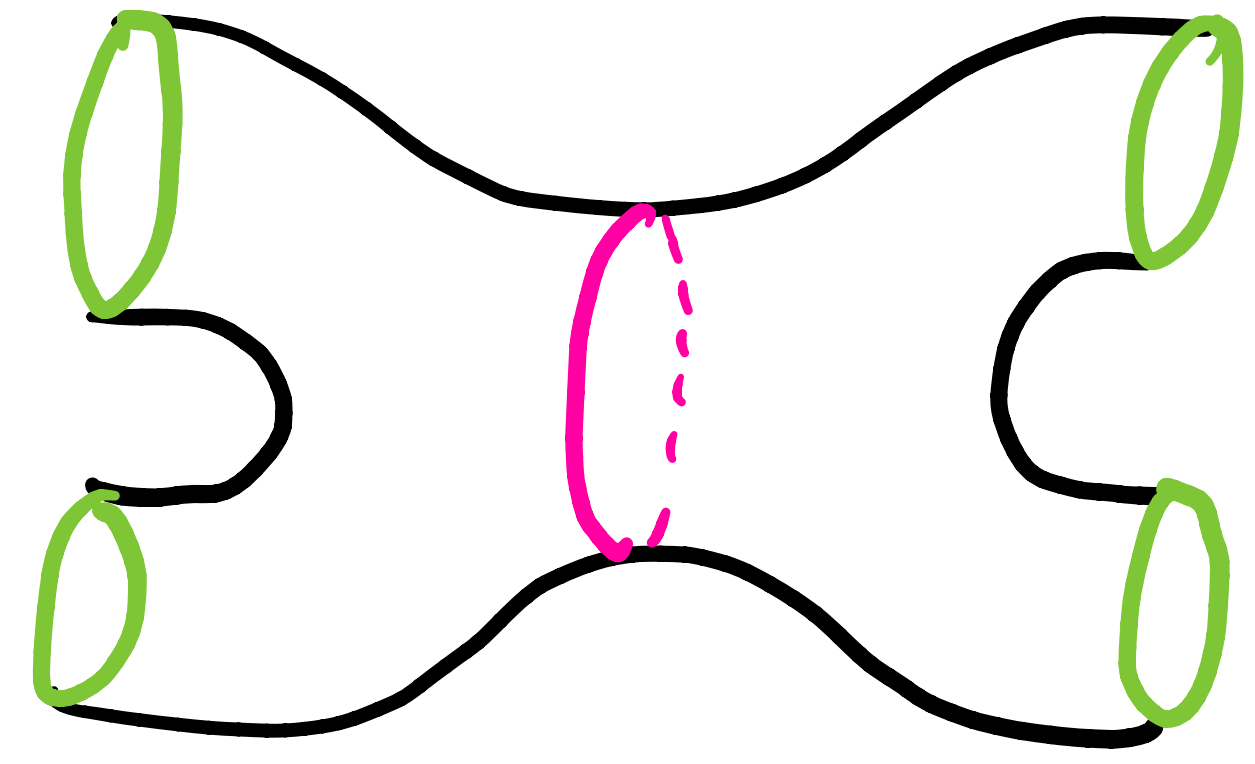




# TO-DO in Physics

string field theory

Zwiebach  
(1991)

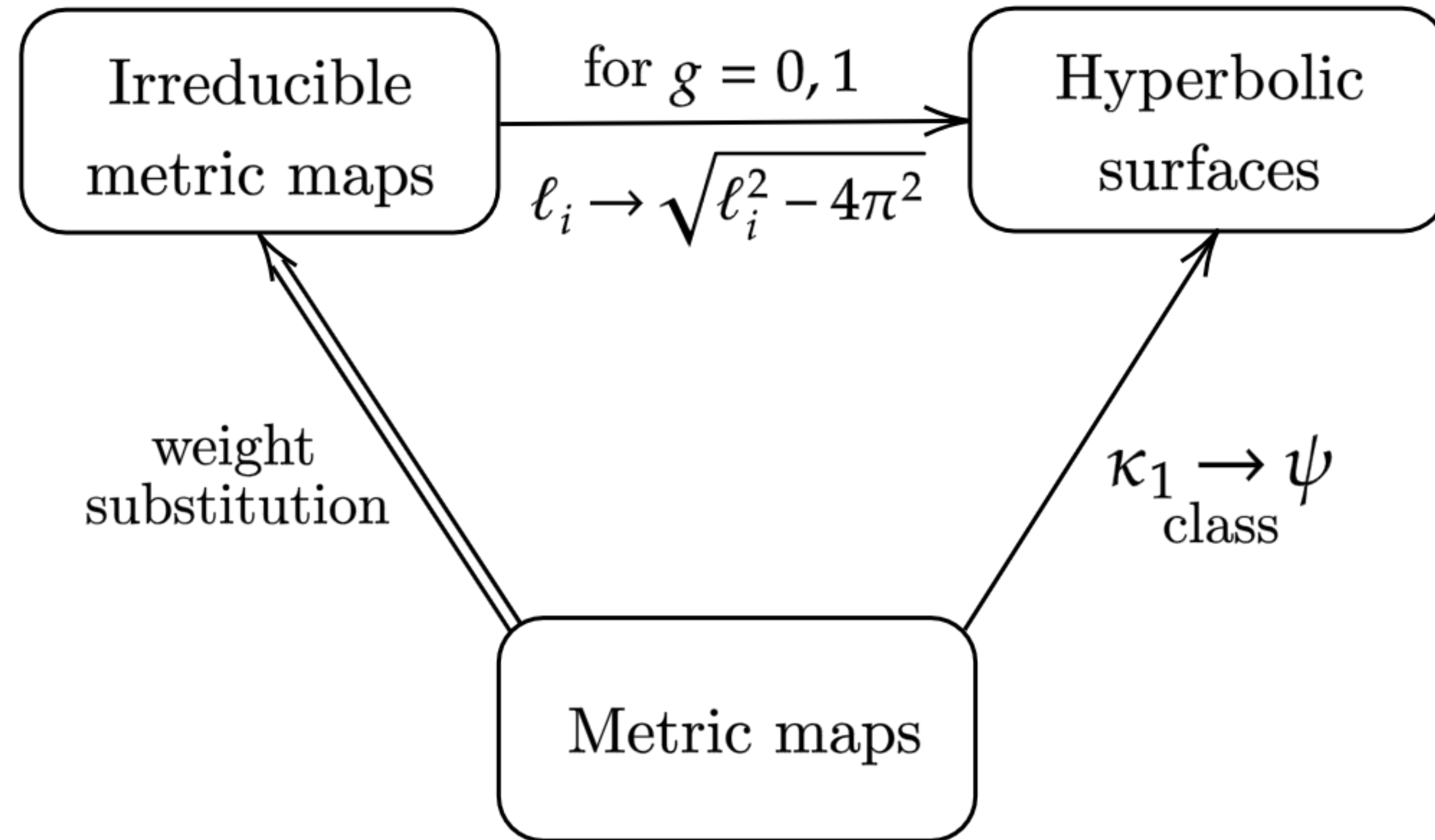


non-contractible  
cycles.

$\neq$  irreducibility  
in this talk

# TO-DO

*merci !*



# MOMENTS

$$g > 0$$

Does substitution work here?

$$M_p = \frac{\partial^{p+1} x}{\partial u^{p+1}} (E_\mu)$$

$$M_p^{(0)} = \delta_{p=0} - \frac{4^{-p}}{2p!} \int_0^\infty \ell d\ell \bar{U}_+(\ell, E_\mu) \ell^{2p}.$$

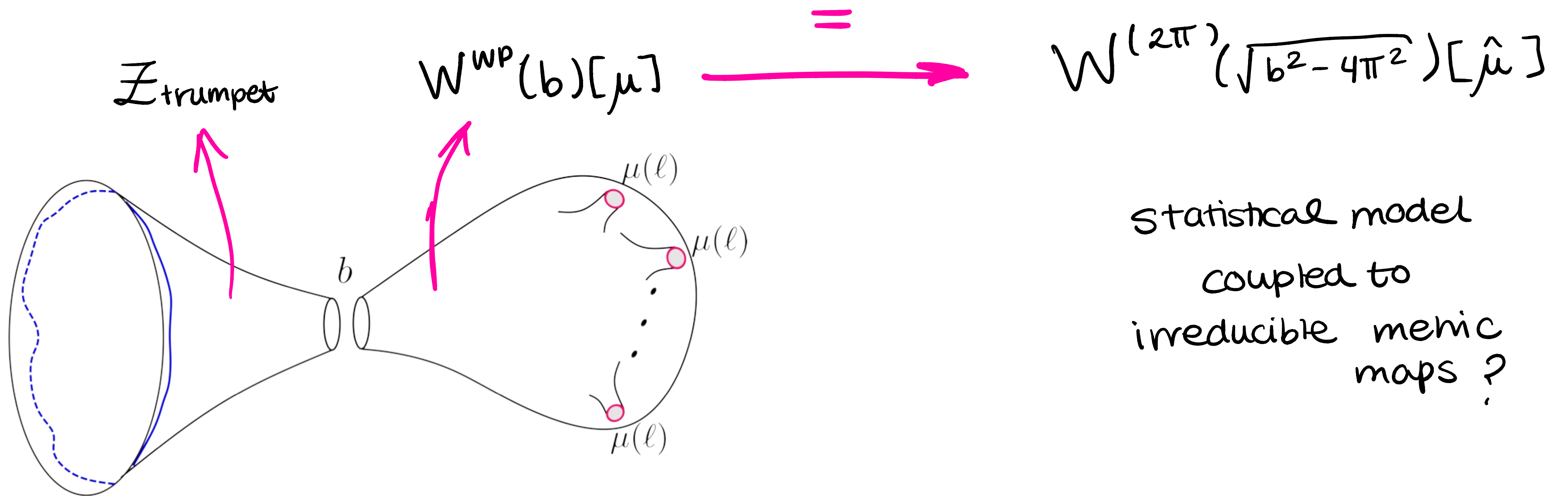
$$\tilde{\mu}(y) = U_+(y) - U_-(y)$$

$$= \frac{\partial}{\partial y} \left( \underbrace{\frac{2\sqrt{E_\mu}}{y} J_1(y\sqrt{E_\mu})}_{\bar{U}_-(\ell, E_\mu)} - \underbrace{\int_y^\infty d\ell \mu(\ell) I_0\left(\sqrt{(\ell^2 - y^2)E_\mu}\right)}_{\bar{U}_+(\ell, E_\mu)} \right)$$

# TO-DO in Physics

JT gravity

Saad  
Shenker  
Stanford  
(2019)





# Topological gravity

(Or Witten's conjecture)

$$V_{g,n}^{(0)}(\ell_1, \dots, \ell_n) = 2^{5-5g-2n} \sum_{d_0+d_1+\dots+d_n=d_{g,n}/2} \prod_{i=1}^n \frac{\ell_i^{2d_i}}{d_i!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$$