

Numerical Semigroups

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Laboratoire Bordelais de Recherche en Informatique
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1 Basic notions

- Genus, Frobenius number, multiplicity, gapsets
- Gapsets

2 Classical problems

- Frobenius' coin problem
- Wilf's conjecture

3 Counting by genus

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree
- Infinite chains in the tree
- Number of children vs number of siblings and Fibonacci bounds

4 Other trees

- Ordinarization transform and ordinarization tree
- Quasi-ordinarization transform and quasi-ordinarization forest

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Multiplicity: $\min(\Lambda^*)$

The (minimal) **generators** of Λ are

$$\Lambda^* \setminus (\Lambda^* + \Lambda^*)$$

$e(\Lambda)$ = number of minimal generators

Definition

[Eliahou-Fromentin] A **gapset** is a finite subset G of \mathbb{N}_0 satisfying

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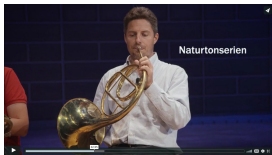
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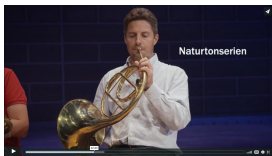
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G gapset $\iff \mathbb{N}_0 \setminus G$ numerical semigroup.

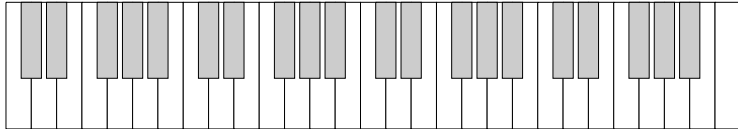
Harmonics: a sound example



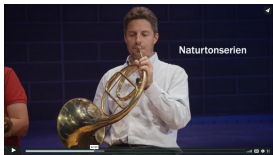
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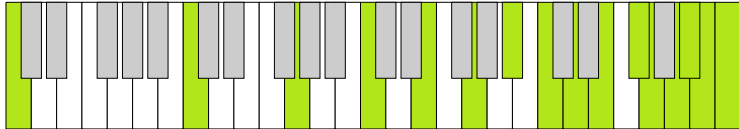
Divide the octave into 12 equal semitones.



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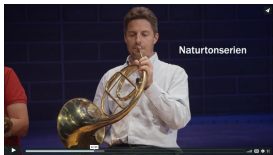


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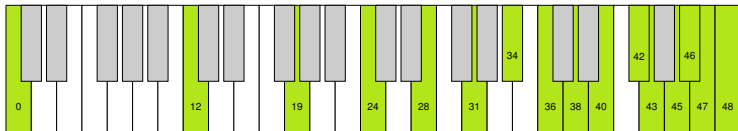


What semitone interval corresponds to each harmonic?

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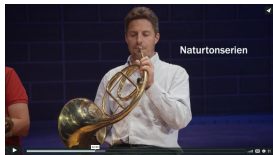


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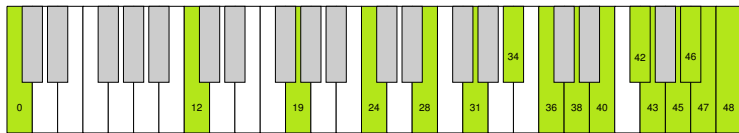


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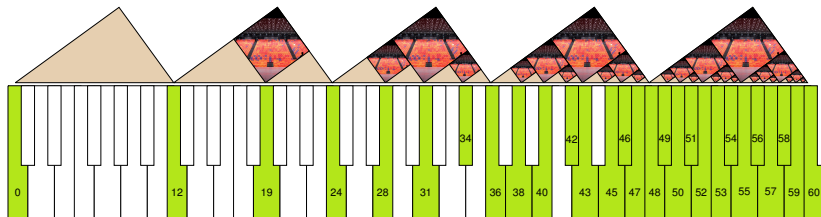


What semitone interval corresponds to each harmonic?

$$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, 49, 50, \rightarrow\}.$$

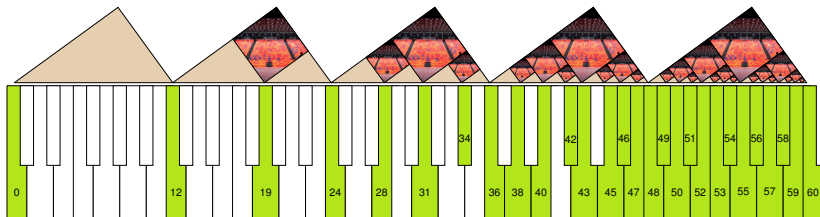
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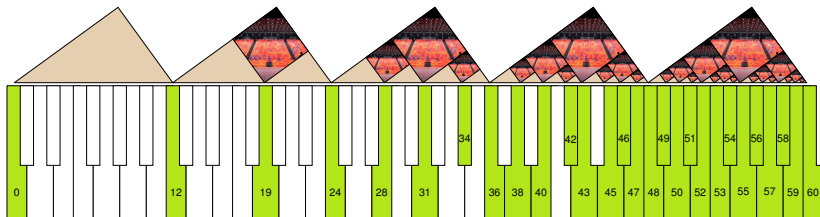
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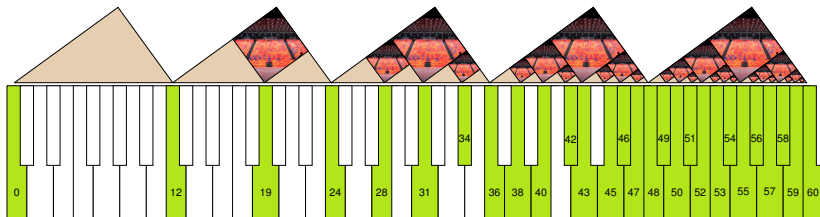


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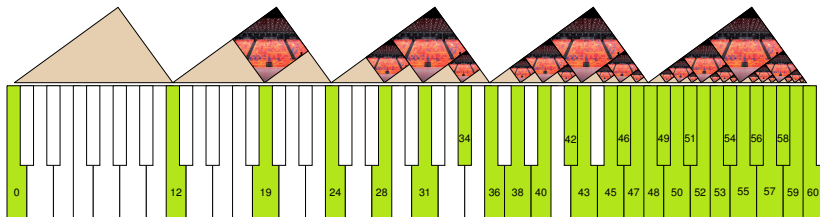


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12 is the maximum multiplicity under 3 conditions:

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- H is odd-filterable (H_1, H_3, H_5, \dots is closed under $+$ up to \dots)

B.: Tempered Monoids of Real Numbers, the Golden Fractal Monoid, and the Well-Tempered Harmonic Semigroup, Semigroup Forum, Springer, 2019.

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Frobenius' problem

Suppose we have coins of denominations a_1, \dots, a_n



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Wilf's conjecture (1978)

$$c(\Lambda) \leq e(\Lambda) \cdot \#L(\Lambda)$$

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- $n_3 = 4$
- $n_4 = 7$
- $n_5 = 12$
- $n_6 = 23$
- $n_7 = 39$
- $n_8 = 67$
- \vdots

Conjecture

[B. 2008]

- ① $n_g \geq n_{g-1} + n_{g-2}$
- ②
 - $\lim_{g \rightarrow \infty} \frac{n_{g-1} + n_{g-2}}{n_g} = 1$
 - $\lim_{g \rightarrow \infty} \frac{n_g}{n_{g-1}} = \phi$

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Weaker unsolved conjecture

[B. 2007] $n_g \leq n_{g+1}$

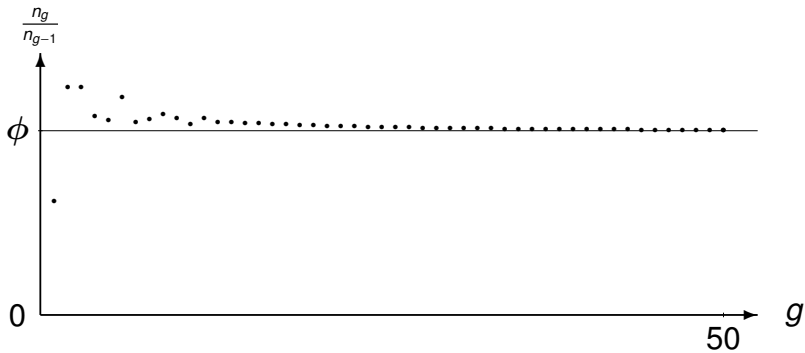


Counting semigroups by genus

g	n_g	$n_{g-1} + n_{g-2}$	$\frac{n_{g-1} + n_{g-2}}{n_g}$	$\frac{n_g}{n_{g-1}}$
0	1			
1	1			1
2	2	2	1	2
3	4	3	0.75	2
4	7	6	0.857143	1.75
5	12	11	0.916667	1.71429
6	23	19	0.826087	1.91667
7	39	35	0.897436	1.69565
8	67	62	0.925373	1.71795
9	118	106	0.898305	1.76119
10	204	185	0.906863	1.72881
11	343	322	0.938776	1.68137
12	592	547	0.923986	1.72595
13	1001	935	0.934066	1.69088
14	1693	1593	0.940933	1.69131
15	2857	2694	0.942947	1.68754
16	4806	4550	0.946733	1.68218
17	8045	7663	0.952517	1.67395
18	13467	12851	0.954259	1.67396
19	22464	21512	0.957621	1.66808
20	37396	35931	0.960825	1.66471
21	62194	59860	0.962472	1.66312
22	103246	99590	0.964589	1.66006
23	170963	165440	0.967695	1.65588
24	282828	274209	0.969526	1.65432
25	467224	453791	0.971249	1.65197
26	770832	750052	0.973042	1.64981
27	1270267	1238056	0.974642	1.64792
28	2091030	2041099	0.976121	1.64613
29	3437839	3361297	0.977735	1.64409
30	5646773	5528869	0.979120	1.64254
31	9266788	9084612	0.980341	1.64108
32	15195070	14913561	0.981474	1.63973
33	24896206	24461858	0.982554	1.63844
34	40761087	40091276	0.983567	1.63724
35	66687201	65657293	0.984556	1.63605
36	109032500	107448288	0.985470	1.63498
37	178158289	175719701	0.986312	1.63399
38	290939807	287190789	0.987114	1.63304
39	474851445	469098096	0.987884	1.63213
40	774614284	765791252	0.988610	1.63128

Counting semigroups by genus

Behavior of $\frac{n_g}{n_{g-1}}$



What is known

- All terms up to $n_{75} = 17924213336425401 \simeq 10^{16}$
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Dyck paths and Catalan bounds (w. de Mier), semigroup tree and Fibonacci bounds, Elizalde's improvements, and others
- $\lim_{g \rightarrow \infty} \frac{n_g}{n_{g-1}} = \phi$
Alex Zhai (2013) with important contributions of Nathan Kaplan and Yufei Zhao.

Counting semigroups by genus

- Many efforts to compute n_g for large g :
 - N. Medeiros, B. (2008), M. Delgado: Brute approach
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genus	40	42	44	46	48	50	52	54	56	58	60
FH	1	2	7	19	53	145	372	978	2760	7398	21880
RGD	1	3	6	18	45	121	291	799	2101	5292	13785
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On-line encyclopedia of integer sequences: [A007323](#)

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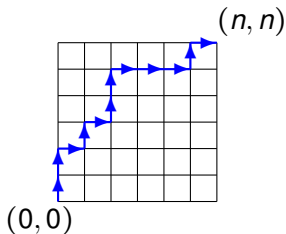
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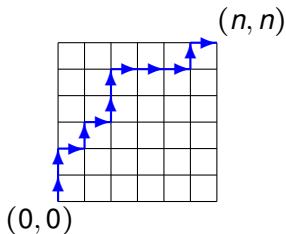
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Example



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The number of Dyck paths of order n is given by the **Catalan number**

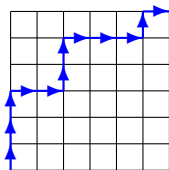
$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

Definition

The **square diagram** of a numerical semigroup is the path

$$e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \notin \Lambda, \end{cases} \quad \text{for } 1 \leq i \leq 2g.$$

Example

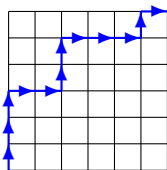


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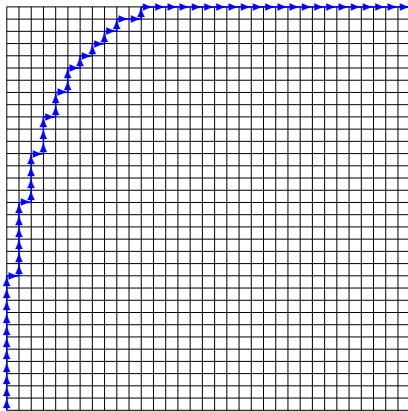
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It always goes from $(0, 0)$ to (g, g) .

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Lemma

[B., de Mier, 2007]

The square diagram of a numerical semigroup is a Dyck path.

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Corollary

$$n_g \leq C_g = \frac{1}{g+1} \binom{2g}{g}.$$

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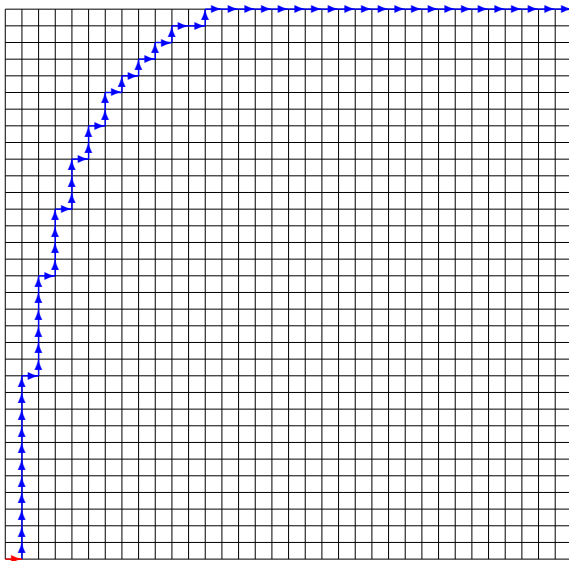
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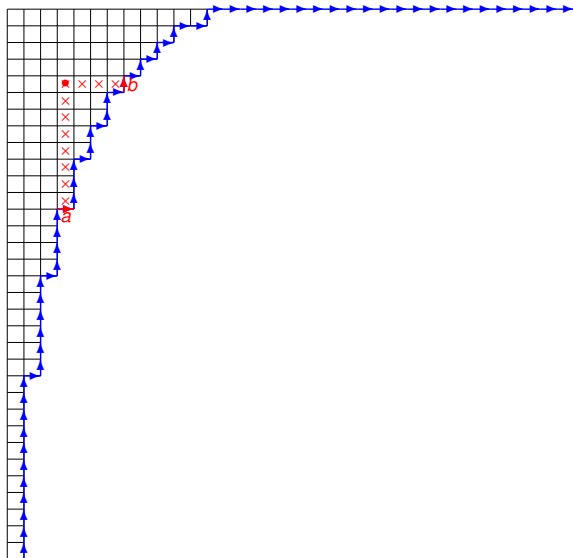
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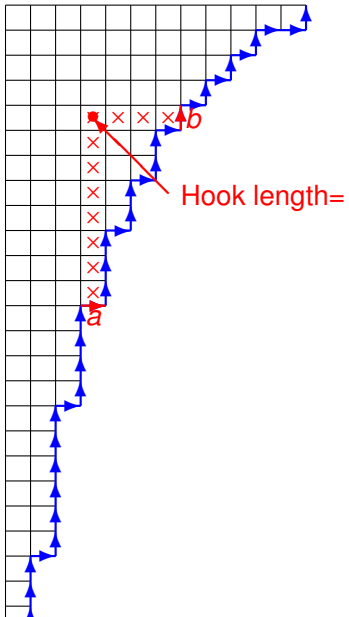
But... **not all Dyck paths correspond to numerical semigroups.**

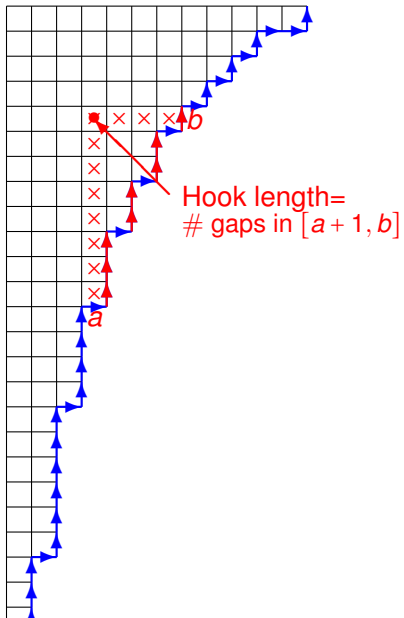
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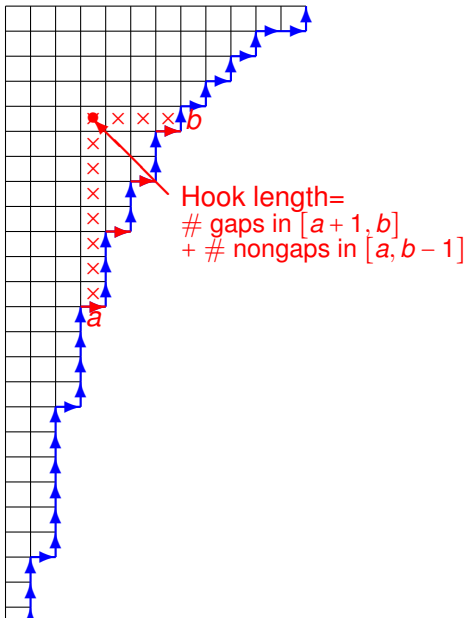


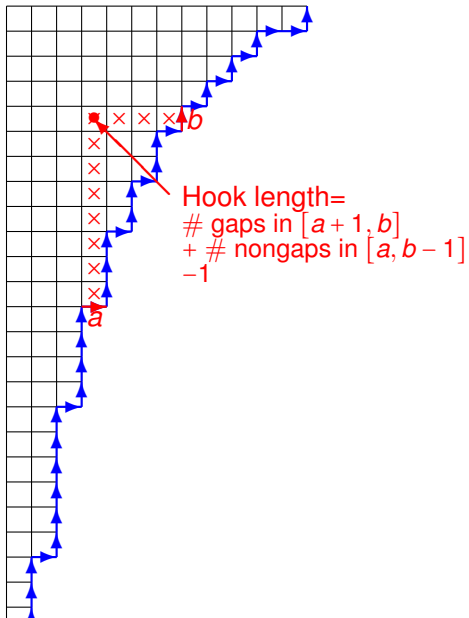
Use the *augmented Dyck path* (from 0) and compute Hook lengths.

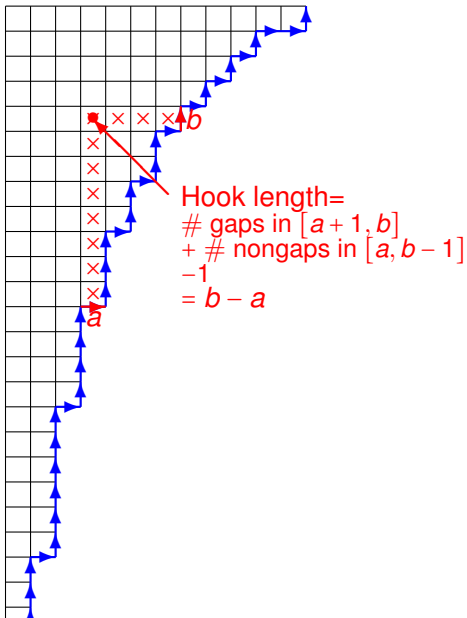


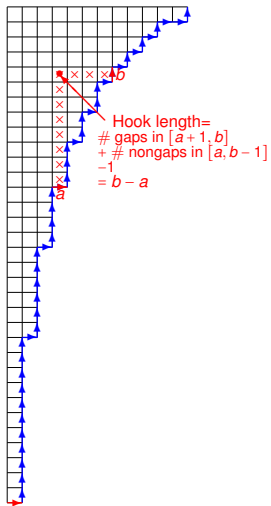






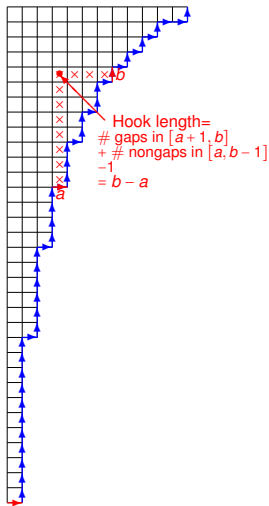






All Hook lengths

$$H(D) = \{b - a : b \text{ a gap}, a \text{ a nongap}\},$$

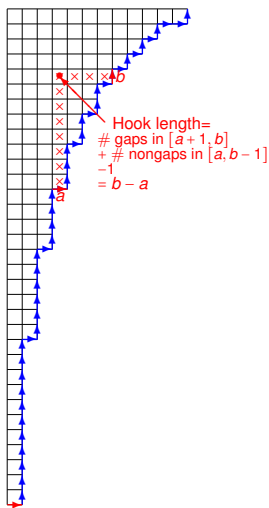


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By the gapset definition, an (augmented) Dyck path corresponds to a numerical semigroup if and only if

$$H(D) \subseteq h(D)$$

[Constantin, Houston-Edwards, Kaplan]

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Tree \mathcal{T} of numerical semigroups

Let $N_g = \{ \text{semigroups of genus } g \}$.

$$\begin{aligned} f : N_g &\longrightarrow N_{g-1} \\ \Lambda &\mapsto \Lambda \cup \{F(\Lambda)\} \end{aligned}$$

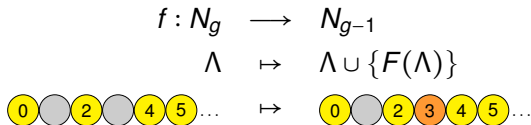
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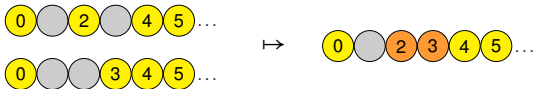
$$\begin{array}{rcl} f : N_g & \longrightarrow & N_{g-1} \\ \Lambda & \mapsto & \Lambda \cup \{F(\Lambda)\} \\ \begin{array}{c} \text{0} \text{ } \text{2} \text{ } \text{4} \text{ } \text{5} \dots \\ \text{ } \text{ } \text{ } \text{ } \end{array} & \mapsto & \begin{array}{c} \text{0} \text{ } \text{2} \text{ } \text{3} \text{ } \text{4} \text{ } \text{5} \dots \\ \text{ } \text{ } \text{ } \text{ } \end{array} \end{array}$$

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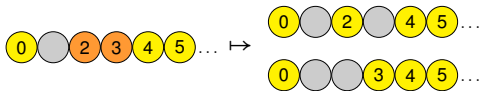
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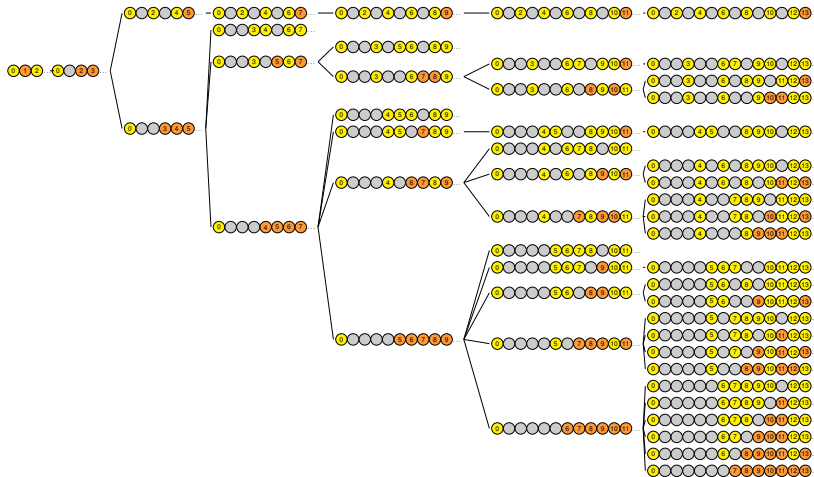
Not injective



f^{-1} : Take out one by one all generators of Λ larger than $F(\Lambda)$.

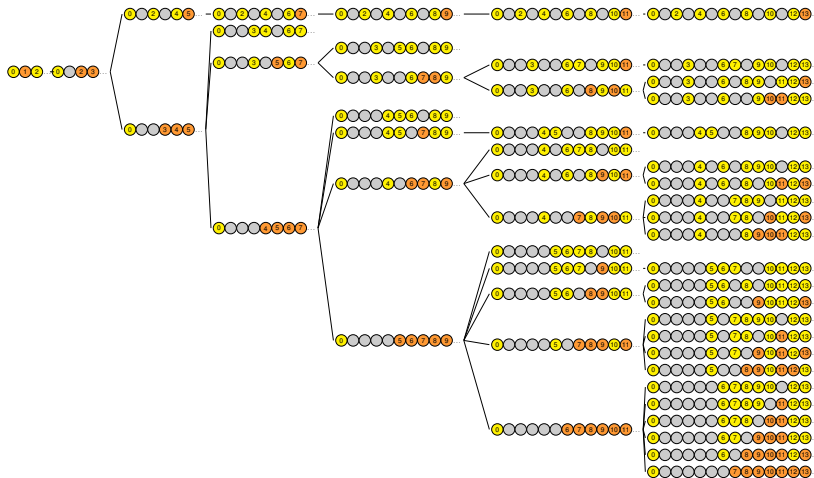


Tree \mathcal{T} of numerical semigroups



The parent of Λ is $f(\Lambda)$.

Tree \mathcal{T} of numerical semigroups



The **parent** of Λ is $f(\Lambda)$.

The **children** of Λ is the set $\text{children}(\Lambda) = f^{-1}(\Lambda)$.

Tree \mathcal{T} of numerical semigroups

If $\text{children}(\Lambda) \neq \emptyset$ for all Λ , then $n_g \leq n_{g+1}$ would follow immediately.

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Questions:

- 1 characterize infinite chains in the tree,
- 2 number of children of a node in terms of the number of siblings

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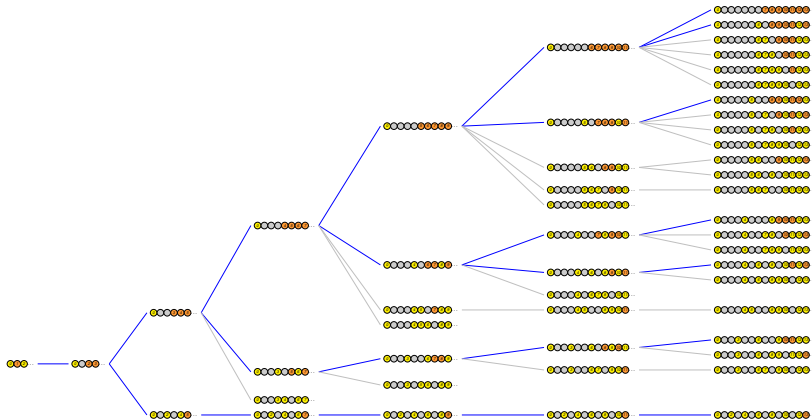
Theorem (B., Bulygin, 2009)

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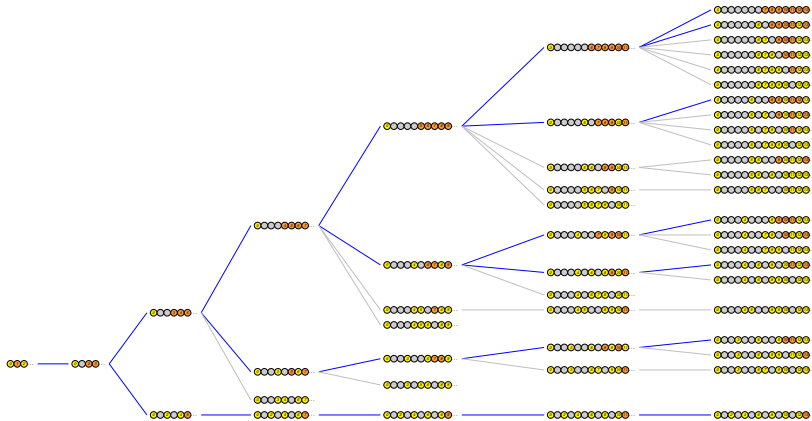
① Λ has ∞ descendants $\iff d \neq 1$.



Theorem (B., Bulygin, 2009)

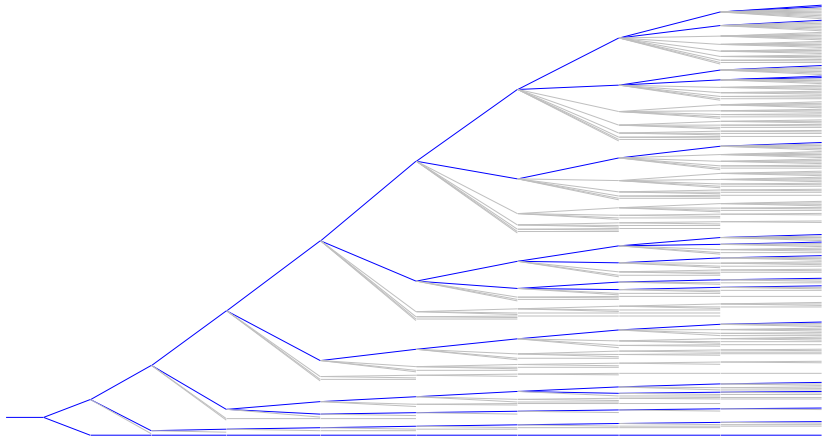
Let $d = \gcd(L(\Lambda))$. Then,

- 1 Λ has ∞ descendants $\iff d \neq 1$.
- 2 If $d \neq 1$ then Λ lies in infinitely many infinite chains if and only if d is not prime.



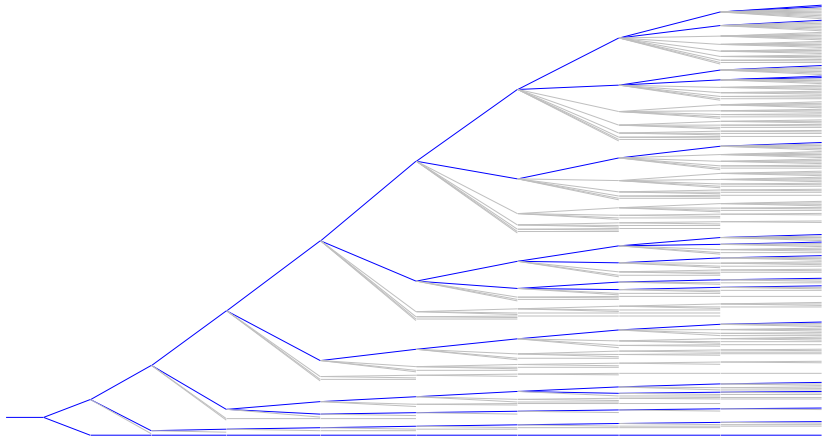
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$$\lim_{g \rightarrow \infty} \frac{\#\{\Lambda \in N_g : \Lambda \in \text{infinite chain}\}}{n_g} = 0.$$



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Most numerical semigroups have a **finite** number of descendants.

Tools in the proof:

- most numerical semigroups have multiplicity ($\min(\Lambda^*)$)

$$m \sim \frac{5+\sqrt{5}}{10}g$$

- most numerical semigroups have $F \sim 2m$.

(Kaplan, Ye, 2013, Singhal, 2022)

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Tree \mathcal{T} of numerical semigroups

A numerical semigroup is **ordinary** if all its gaps are consecutive.



Tree \mathcal{T} of numerical semigroups

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Lemma

Λ ordinary,
 $\#\text{children}(\Lambda) = k,$ } \Rightarrow its children have $0, 1, \dots, k-3, k-1, k+1$
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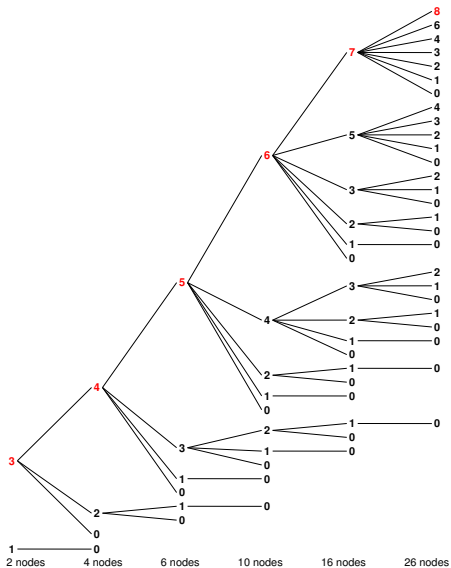
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Λ non-ordinary,
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- at least $0, \dots, k-1$ children, respectively,
- at most $1, \dots, k$ children, respectively

Lower bound for the number of descendants of semigroups of genus g

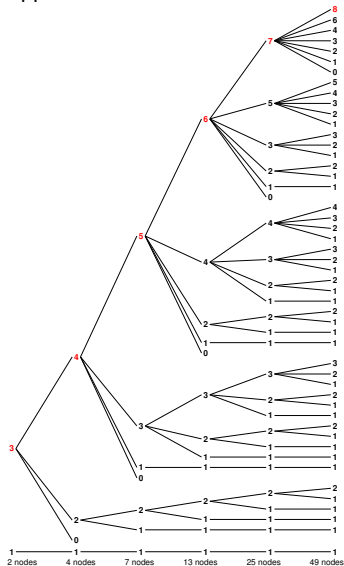


Lemma

For $g \geq 3$,

$$2F_g \leq n_g$$

Upper bound for the number of descendants of semigroups of genus g



$$1 + 3 \cdot 2^{g-3} \text{ nodes}$$

Lemma

For $g \geq 3$,

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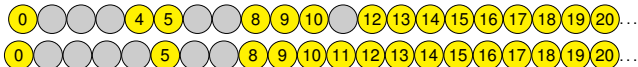


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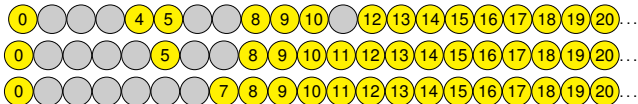


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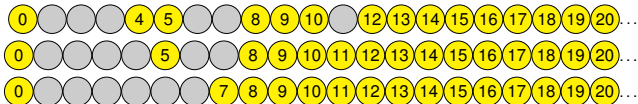


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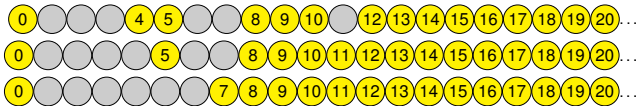
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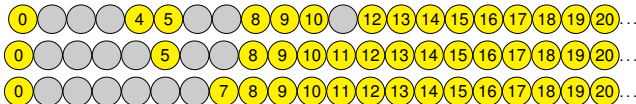
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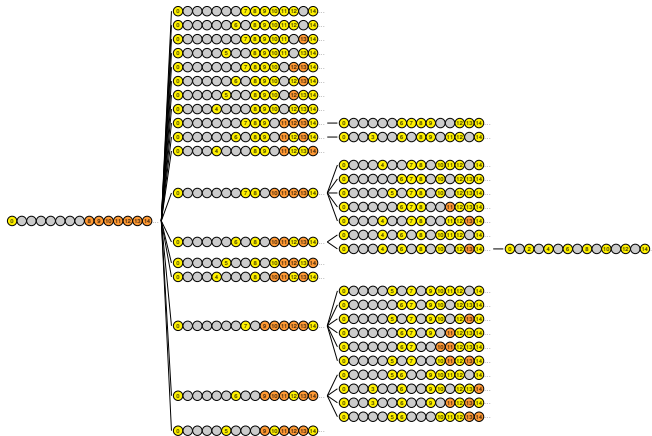
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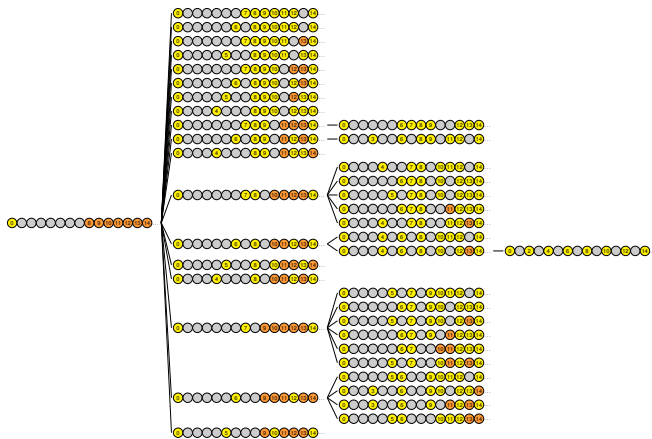
Tree \mathcal{T}_g of numerical semigroups of genus g

- **nodes** correspond to semigroups of genus g
- **edges** connect each semigroup to its ordinarization



Tree \mathcal{T}_g of numerical semigroups of genus g

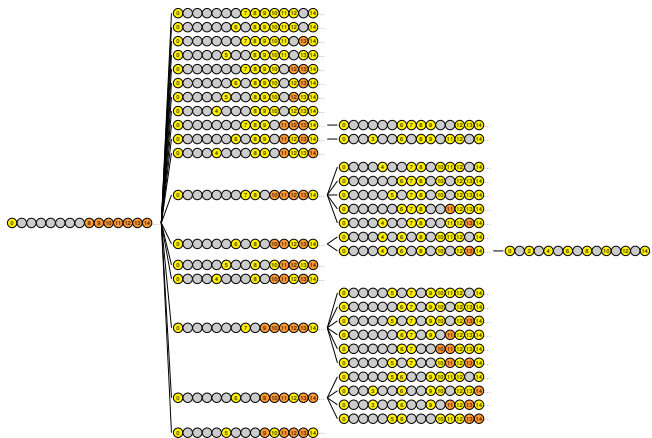
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\mathcal{T}_g is a tree rooted at the unique ordinary semigroup of genus g .

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\mathcal{T}_g is a tree rooted at the unique ordinary semigroup of genus g .

Contrary to \mathcal{T} , \mathcal{T}_g has only a **finite number of nodes** (indeed, n_g).

$n_{g,r}$: number of semigroups of genus g and ordinarization number r .

Conjecture

- $n_{g,r} \leq n_{g+1,r}$
- Equivalently, the number of semigroups in \mathcal{T}_g at a given depth is at most the number of semigroups in \mathcal{T}_{g+1} at the same depth.

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This conjecture would prove $n_g \leq n_{g+1}$.

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This result is proved for the lowest and largest depths.

The proof uses Freiman's theorem.

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Quasi-ordinary numerical semigroups

A non-ordinary semigroup Λ is a **quasi-ordinary** semigroup if $\Lambda \cup F$ is ordinary.



Quasi-ordinarization of semigroups

Quasi-ordinarization transform of a non-ordinary semigroup:

- Remove the multiplicity
- Add the second largest gap

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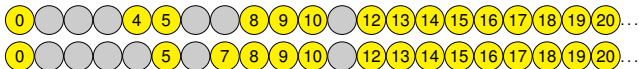
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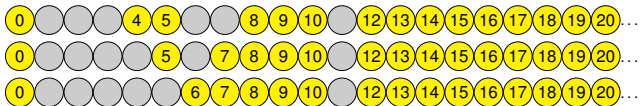
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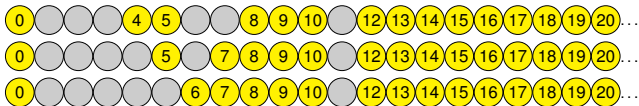
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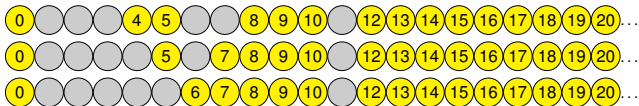


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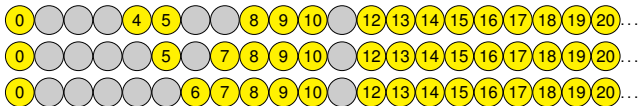


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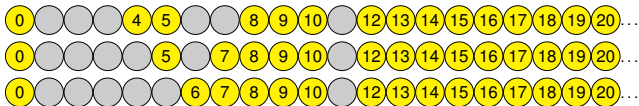


- The result is another numerical semigroup.
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- Repeating several times (:= **quasi-ordinarization number**) we obtain a quasi-ordinary semigroup.

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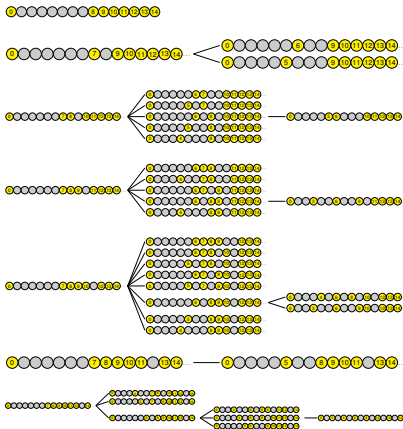


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Quasi-ordinarization transform of an ordinary semigroup is defined to be itself.

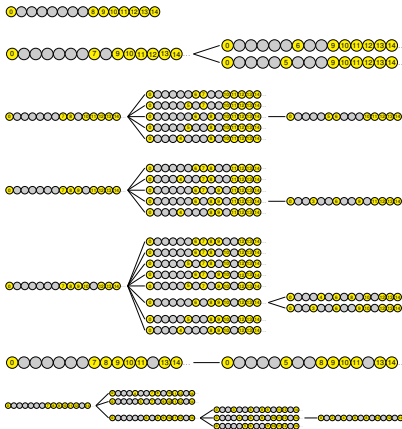
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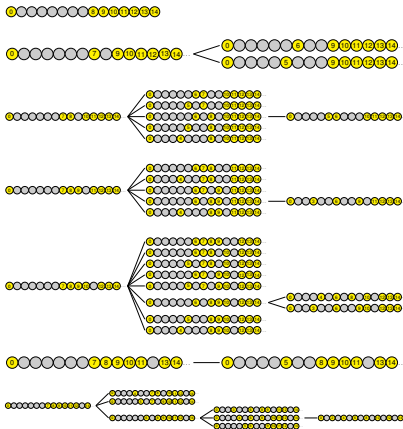
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\mathcal{F}_g is a forest with roots at the ordinary and quasi-ordinary semigroups

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\mathcal{F}_g is a forest with roots at the ordinary and quasi-ordinary semigroups

Contrary to \mathcal{T}_g , \mathcal{F}_g is a forest.

$n_{g,q}$: # of semigroups of genus g and quasi-ordinarization number q .

Conjecture

- $n_{g,q} \leq n_{g+1,q}$
- Equivalently, the number of semigroups in \mathcal{F}_g at a given depth is at most the number of semigroups in \mathcal{F}_{g+1} at the same depth.

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This conjecture would prove $n_g \leq n_{g+1}$.

Combinatorial Object Server++ Maintained by Torsten Mütze.

The screenshot shows the web interface of the Combinatorial Object Server++ (COS++). The browser address bar shows `combos.org/sgroup.html`. The page title is "The Combinatorial Object Server".

Generate numerical semigroups

Generate all numerical semigroups with a given genus.

Genus: (max. 15)

Multiplicity:

Conductor:

Embedding dimension:

Output format:

Output: numbering graphics

The output area displays a list of 23 numerical semigroups, each with its genus, multiplicity, conductor, and embedding dimension. The list is as follows:

```
1: 0 7
2: 0 6 8
3: 0 6 7 9
4: 0 5 7 10
5: 0 5 7 9 11
6: 0 5 7 9 10 12
7: 0 5 8
8: 0 5 7
9: 0 5 7 10
10: 0 5 7 9 10 12
11: 0 5 6 9
12: 0 5 6 9 10
13: 0 5 6 7 10
14: 0 4 8
15: 0 4 7 9 10
16: 0 4 7 9 11
17: 0 4 8 10
18: 0 4 6 9 10 12
19: 0 4 5 9 10 12
20: 0 3 9
21: 0 3 6 9 11
22: 0 3 6 7 9 10 12
23: 0 2 4 6 9 10 12
```

Total: 23

Object info

A **numerical semigroup** Λ is a subset of the non-negative integers $\{0, 1, \dots\}$ satisfying the following three conditions: The integer 0 must be contained in Λ ; the set Λ may miss only a finite number of integers; and $a, b \in \Lambda$ implies that $a + b \in \Lambda$, i.e., Λ is closed under addition. A missing element of Λ is called a **gap**, and the number of gaps is referred to as the **genus** of Λ . The **multiplicity** of Λ is the smallest non-zero element in Λ . The **conductor** is the smallest element in Λ that is larger than all gaps. The **embedding dimension** is the cardinality of a minimal system of generators of Λ .

The **characteristic vector** of a semigroup is the infinite bitstring that has a 1-bit at position i if and only if the integer i is contained in the group. We visualize the characteristic vector by colored boxes with 0=gray (gap) and 1=red (non-gap). For instance, the semigroup $\Lambda = \{0, 5, 8, 9, 10, 12, 13, 14, 15, 16, \dots\}$ has characteristic vector 0000100111011111... which is visualized as `0000100111011111...`

The algorithm running on this website is due to Bras-Amorós and Fernández-González (see [BA08], [Kap17], [BAFG20]), and it generates numerical semigroups in lexicographic order of their characteristic vectors by a tree-based approach.

- Enumeration (OEIS)
- Download source code
- References

Drawsgtree Maintained at Universitat Rovira i Virgili.

← → ↻ 🏠

Drawsgtree

A tool for visualizing properties in the semigroup tree 📄

By [Maria Bras Amoros](#)

📄 Please, cite as "Maria Bras Amoros. Drawsgtree. GitHub repository, 2022, 2023, 2024. <https://github.com/mbraseros/drawsgtree>"

Maximum genus *

Root at the semigroup
(default: 0 < 0 < 0 < 10 < 10)

Multiplicity

Node representation

List:

- Minimal generator set
- Gaps
- Gap bitstream and seed bitstream

Frame each tree node

Visit nodes by lexicographic order of the nongaps
(default: visit nodes in lexicographic order of the gaps)

Edge distinction

Discard the non-distinguished edges together with all its descendants

Alternative tree

Incremental with genus

Vertical tree growing down

Plain representation of objects using less memory

Black and white

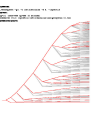
Rotated 90 degrees

Enlarge distance between generations by the specified factor

Enlarge distance between siblings by the specified factor

Scale the image by the specified factor

📄 1/1/2024/10/11



Thanks!