A phase transition in block-weighted random maps

Journées Combinatoires de Bordeaux
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LIGM, Université Gustave Eiffel
Planar maps

Planar map $\mathfrak{m}$ = embedding on the sphere of a connected planar graph, considered up to homeomorphisms.

- **Rooted** planar map = map endowed with a marked oriented edge (represented by an arrow);
- **Size** $|\mathfrak{m}|$ = number of edges;
- **Corner** (does not exist for graphs !) = space between an oriented edge and the next one for the trigonometric order.

Planar map = planar graph + cyclic order on neighbours.
Universality results for planar maps

• Enumeration: $\kappa \rho^{-n} n^{-5/2}$ [Tutte 1963];

• Distance between vertices: $n^{1/4}$ [Chassaing, Schaeffer 2004];

• Scaling limit: Brownian sphere for quadrangulations [Le Gall 2013, Miermont 2013] and general maps [Bettinelli, Jacob, Miermont 2014];
Universality results for planar maps

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Universality:
- Same enumeration [Drmota, Noy, Yu 2020];
- Same scaling limit, e.g. for triangulations & $2q$-angulations [Le Gall 2013], simple quadrangulations [Addario-Berry, Albenque 2017].
Universality results for plane trees

- Enumeration: $\kappa \rho^{-n} n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];
Universality results for plane trees

- Enumeration: $\kappa \rho^{-n} n^{-3/2}$;
- Distance between vertices: $n^{1/2}$ [Flajolet, Odlyzko 1982];
- Scaling limit: Brownian tree [Aldous 1993, Le Gall 2006];

- Universality:
  - Same enumeration,
  - Same scaling limit, even for some classes of maps; e.g. outerplanar maps [Caraceni 2016], maps with a boundary of size $>> n^{1/2}$ [Bettinelli 2015].

Models with (very) constrained boundaries
Motivation

Inspired by [Bonzom 2016].

Two rich situations with universality results:

Planar maps

Plane trees

Interpolating model?

Brownian Sphere $S_e$

Brownian Tree $T_e$
Model definition

2-connected = two vertices must be removed to disconnect.
Block = maximal (for inclusion) 2-connected submap.
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Condensation phenomenon: a large block concentrates a macroscopic part of the mass [Banderier, Flajolet, Schaeffer, Soria 2001; Jonsson, Stefánsson 2011].
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Interpolating model using blocks!
Outline of the talk

A phase transition in block-weighted random maps

I. Model
II. Block tree of a map and its applications
   *Interlude.* Quadrangulations
III. Scaling limits
IV. Extension to other families of maps
V. Extension to tree-rooted maps
VI. Perspectives

with William Fleurat

with Marie Albenque & Éric Fusy
I. Model
Model

Inspired by [Bonzom 2016].

Goal: parameter that affects the typical number of blocks.

We choose: \( P_{n,u}(m) = \frac{u^{\#blocks(m)}}{Z_{n,u}} \)

where

- \( u > 0 \),
- \( \mathcal{M}_n = \{ \text{maps of size } n \} \),
- \( m \in \mathcal{M}_n \),
- \( Z_{n,u} = \text{normalisation.} \)

- \( u = 1 \): uniform distribution on maps of size \( n \);
- \( u \to 0 \): minimising the number of blocks (=2-connected maps);
- \( u \to \infty \): maximising the number of blocks (= trees!).

Given \( u \), asymptotic behaviour when \( n \to \infty \)?
\( n \approx 55,000 \)

\( u = \frac{8}{5} \)
$n \approx 80,000$

$u = 9/5$
\[ u = \frac{5}{2} \]

\[ n \approx 75\,000 \]
$n \approx 50\,000$

$u = 5$
Theorem [Fleurat, S. 23] Model exhibits a phase transition at $u = 9/5$. When $n \to \infty$:

- Subcritical phase $u < 9/5$: “general map phase” one huge block;
- Critical phase $u = 9/5$: a few large blocks;
- Supercritical phase $u > 9/5$: “tree phase” only small blocks.

We obtain explicit results on enumeration, size of blocks and scaling limits in each case.

→ A phase transition in block-weighted random maps
<table>
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<tr>
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II. Block tree of a map and its applications
Decomposition of a map into blocks

Inspiration from [Tutte 1963]
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\[ M(z) = B(zM^2(z)) \]

GS of 2-connected maps
Inspiration from [Tutte 1963]

Decomposition of a map into blocks

\[ M(z, u) = \sum_{m \in \mathcal{M}} z^{\| m \|} u^{\# \text{blocks}(m)} \]

GS of 2-connected maps

\[ M(z) = B(zM^2(z)) \]

With a weight \( u \) on blocks: \[ M(z, u) = uB(zM^2(z, u)) + 1 - u \]
## Results

For $M_n \leftrightarrow \mathbb{P}_{n,u}$

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For $M_n \mapsto \mathbb{P}_{n,u}$
Decomposition of a map into blocks

Inspiration from [Tutte 1963]
Decomposition of a map into blocks

With a weight $u$ on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$

Inspiration from [Tutte 1963]

$M(z, u) = \sum_{m \in M} z^{|m|} u^{\# \text{blocks}(m)}$

GS of 2-connected maps
Inspiration from [Tutte 1963]

Decomposition of a map into blocks

\[
M(z, u) = \sum_{m \in \mathcal{M}} z^{|m|} u^{\# \text{blocks}(m)}
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⇒ Underlying block tree structure, made explicit by [Addario-Berry 2019].

With a weight \(u\) on blocks: \(M(z, u) = uB(zM^2(z, u)) + 1 - u\)
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Inspiration from [Tutte 1963]

With a weight $u$ on blocks: $M(z, u) = uB(zM^2(z, u)) + 1 - u$
Properties of the block tree

- $\mathbf{m}$ is entirely determined by $T_m$ and $(b_v, v \in T_m)$ where $b_v$ is the block of $\mathbf{m}$ represented by $v$ in $T_m$;
- Internal node (with $2k$ children) of $T_m \leftrightarrow$ block of $\mathbf{m}$ of size $k$.

$T_{M_n}$ gives the block sizes of a random map $M_n$. 
Galton-Watson trees for map blocks

$\mu$-Galton-Watson tree: random tree where the number of children of each node is given by $\mu$ independently, with $\mu = \text{probability law on } \mathbb{N}$. 
Galton-Watson trees for map blocks

**μ-Galton-Watson tree:** random tree where the number of children of each node is given by \( \mu \) independently, with \( \mu = \) probability law on \( \mathbb{N} \).

**Theorem [Fleurat, S. 23]**

If \( M_n \hookrightarrow \mathbb{P}_{n,u} \), then there exists an (explicit) \( y = y(u) \) s.t. \( T_{M_n} \) has the law of a Galton-Watson tree of reproduction law \( \mu^{y,u} \) conditioned to be of size \( 2n \), with

\[
\mu^{y,u}(\{2k\}) = \frac{B_k y^k u^1_{k \neq 0}}{u B(y) + 1 - u}.
\]

\( u > 0 \)
Phase transition

When is $\mu^{y,u}$ critical? (= $\mathbb{E}(\mu) = 1$?)

$$\mathbb{E}(\mu^{y,u}) = 1 \iff u = \frac{1}{2yB'(y) - B(y) + 1}$$

covers $[9/5, + \infty)$ when $y$ covers $(0, \rho_B = 4/27]$. 
Phase transition

\( u_C = \frac{9}{5} \)

\[ y(u) = \frac{4}{27} \]

\[ y(u) = y \text{ s.t. } \mathbb{E}(\mu^{y(u),u}) = 1 \]

"Map regime"  
"Tree regime"
Largest blocks?

- Degrees of $T_{M_n}$ give the block sizes of the map $M_n$;
- Largest degrees of a Galton-Waston tree are well-known [Janson 2012].
## Rough intuition

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<td>$\mu_{y(u),u}({2k})$</td>
<td>$\sim c_u k^{-5/2}$</td>
<td></td>
<td>$\sim c_u \pi_u^k k^{-5/2}$</td>
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<tr>
<td>Galton-Watson tree</td>
<td>subcritical</td>
<td></td>
<td>critical</td>
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Dichotomy between situations:

- **Subcritical**: condensation, cf [Jonsson Stefánsson 2011];
- **Supercritical**: behaves as maximum of independent variables.
## Size $L_{n,k}$ of the $k$-th largest block

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<td>$L_{n,1}$</td>
<td>$\sim (1 - \mathbb{E}(\mu_{4/27,u}^{4/27,u}))n$</td>
<td>$\Theta(n^{2/3})$</td>
<td>$\frac{\ln(n)}{2 \ln \left( \frac{4}{27y} \right)} - \frac{5 \ln(\ln(n))}{4 \ln \left( \frac{4}{27y} \right)} + O(1)$</td>
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<td>$L_{n,2}$</td>
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[Stufler 2020]
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Interlude: quadrangulations
**Quadrangulations**

Def: map with all faces of degree 4.

Simple quadrangulation = no multiple edges.

Size $|q|$ = number of faces.

$$|V(q)| = |q| + 2, \quad |E(q)| = 2|q|.$$
Construction of a quadrangulation from a simple core
Construction of a quadrangulation from a simple core
Construction of a quadrangulation from a simple core
Block tree for a quadrangulation

With a weight $u$ on blocks: $Q(z, u) = uS(zQ^2(z, u)) + 1 - u$

Remember: $M(z, u) = uB(zM^2(z, u)) + 1 - u$
Tutte’s bijection

Map

Quadrangulation

[Tutte 1963]
Tutte’s bijection for 2-connected maps

Cut vertex => multiple edge

2-connected maps <= simple quadrangulations

[Brown 1965]
Block trees under Tutte’s bijection
Implications on results

We choose: \( P_{n,u}(q) = \frac{u^{\#blocks(q)}}{Z_{n,u}} \) where

\[ u > 0, \]

\[ Q_n = \{\text{quadrangulations of size } n\}, \]

\[ q \in Q_n, \]

\[ Z_{n,u} = \text{normalisation}. \]

Results on the size of (2-connected) blocks can be transferred immediately for quadrangulations and their simple blocks.
## Results

For $M_n \leftrightarrow \mathbb{P}_{n,u}$

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[Stufler 2020]
III. Scaling limits
Scaling limits

Convergence of the whole object considered as a metric space (with the graph distance), after renormalisation.

What is the limit of the sequence of metric spaces $(M_n, d/n^?)_{n \in \mathbb{N}}$?

(map or quadrangulation)

(Convergence for Gromov-Hausdorff topology)
Lemma For $M_n \hookrightarrow \mathbb{P}_{n,u'}$

- If $u > 9/5$,
  \[
  \frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e.
  \]
- If $u = 9/5$,
  \[
  \frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}.
  \]
Scaling limit of supercritical and critical maps

Lemma For $M_n \sim \mathcal{P}_{n, u}$

- If $u > 9/5$,
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  \frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e.
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- If $u = 9/5$,
  \[
  \frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}.
  \]

Proof Known scaling limits of critical Galton-Watson trees

- with finite variance [Aldous 1993, Le Gall 2006];
- infinite variance and polynomial tails [Duquesne 2003].
Scaling limit of supercritical and critical maps

Theorem For $M_n \hookrightarrow P_{n,u'}$

- [Stufler 2020] If $u > 9/5$,
  \[
  \frac{c_3(u)}{n^{1/2}} T_{M_n} \rightarrow \mathcal{T}_e.
  \]

- [Fleurat, S. 23] If $u = 9/5$,
  \[
  \frac{c_2}{n^{1/3}} T_{M_n} \rightarrow \mathcal{T}_{3/2}.
  \]
# Scaling limit of supercritical and critical maps

**Theorem** For $M_n \leftrightarrow \mathbb{P}_{n,u}$

- [Stufler 2020] If $u > 9/5$, 
  \[ \frac{c_3(u)}{n^{1/2}} T_{M_n} \to \mathcal{T}_e \]
  \[ \frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e. \]

- [Fleurat, S. 23] If $u = 9/5$, 
  \[ \frac{c_2}{n^{1/3}} T_{M_n} \to \mathcal{T}_{3/2}, \]
  \[ \frac{C_2}{n^{1/3}} M_n \to \mathcal{T}_{3/2}. \]

**Proof** Distances in $M_n$ behave like distances in $T_{M_n}$!
Difficult part = show that distances in $M_n$ behave like distances in $T_{M_n}$.

Let $\kappa = \mathbb{E}("\text{diameter}\" \text{bipointed block})$. By a "law of large numbers"-type argument

$$d_{M_n}(e_1, e_2) \simeq \kappa d_{T_{M_n}}(e_1, e_2).$$

Difficult for the critical case => large deviation estimates
$u = \frac{9}{5}$

$n \approx 80000$
$u = 5/2$

$n \approx 75000$
$u = 5$

$n \approx 50,000$
Scaling limits of subcritical maps

Theorem [Fleurat, S. 23] If $u < 9/5$, for $M_n \leftrightarrow \mathbb{P}_{n,u}$ and denoting $B(M_n)$ its largest block:

$$d_{GHP} \left( \frac{C_1(u)}{n^{1/4}} M_n, \frac{1}{n^{1/4}} B(M_n) \right) \to 0.$$ 

So, if $cn^{-1/4} B_n \to \mathcal{S}_e$, then

$$\frac{C_1(u)}{cn^{1/4}} M_n \to \mathcal{S}_e.$$ 

See [Addario-Berry, Wen 2019] for a similar result and method.
Scaling limits of subcritical maps

Theorem [Fleurat, S. 23] If \( u < 9/5 \), for \( Q_n \leftrightarrow \mathcal{P}_{n,u} \) a quadrangulation:

\[
\frac{C_1(u)}{n^{1/4}} Q_n \rightarrow \mathcal{S}_e.
\]

Moreover, \( Q_n \) and its simple core converge jointly to the same Brownian sphere.

Proof

- Previous theorem;
- Scaling limit of uniform simple quad. rescaled by \( n^{1/4} \) = Brownian sphere [Addario-Berry Albenque 2017].
Diameter of a decoration ≤ blocks to cross × max diameter of blocks

\[ \leq \text{diam}(T_{M_n}) \times (O(n^{2/3}))^{1/4 + \delta} = \text{diam}(T_{M_n}) \times O(n^{1/6 + \delta}) \]

\[ = O(n^{1/6 + 2\delta}) = o(n^{1/4}). \]

\[ T_{M_n} \text{ is a subcritical Galton-Watson tree} \]

[Chapuy Fusy Giménez Noy 2015]
Subcritical case

Diameters of decorations = \( o(n^{1/4}) \).

Diameter of a decoration \( \leq \) blocks to cross \( \times \) max diameter of blocks

\[ \leq \text{diam}(T_{M_n}) \times (O(n^{2/3}))^{1/4+\delta} = \text{diam}(T_{M_n}) \times O(n^{1/6+\delta}) \]

\[ = O(n^{1/6+2\delta}) = o(n^{1/4}). \]

[Chapuy Fusy Giménez Noy 2015]
Subcritical case

Diameters of decorations = $\Theta(n^{1/4})$.

Decorations = groups of smaller blocks

Large block of size $\Theta(n)$

The scaling limit of $M_n$ (rescaled by $n^{1/4}$) is the scaling limit of uniform blocks!
\[ n \approx 55000 \]
\[ u = \frac{8}{5} \]

\[ n \approx 55000 \]
## Results

<table>
<thead>
<tr>
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<th>$u = 9/5$</th>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>Enumeration</strong> [Bonzom 2016]</td>
<td>$\rho(u)^{-n}n^{-5/2}$</td>
<td>$\rho(u)^{-n}n^{-5/3}$</td>
<td>$\rho(u)^{-n}n^{-3/2}$</td>
</tr>
<tr>
<td><strong>Size of</strong></td>
<td>$\sim (1 - \mathbb{E}(\mu^{4/27,u}))n$</td>
<td>$\Theta(n^{2/3})$</td>
<td>$\frac{\ln(n)}{2 \ln \left( \frac{4}{27y} \right)} - \frac{5 \ln(\ln(n))}{4 \ln \left( \frac{4}{27y} \right)} + O(1)$</td>
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**Scaling limit of $M_n$**

- $\frac{C_1(u)}{n^{1/4}}M_n \rightarrow \mathcal{S}_e$
- $\frac{C_2}{n^{1/3}}M_n \rightarrow \mathcal{T}_{3/2}$
- $\frac{C_3(u)}{n^{1/2}}M_n \rightarrow \mathcal{T}_e$ [Stufler 2020]

Assuming the convergence of 2-connected maps towards the Brownian sphere.
IV. Extension to other families of maps
**Extension to other models**

[Banderier, Flajolet, Schaeffer, Soria 2001]:

**Table 3.** Composition schemas, of the form $\mathcal{M} = \mathcal{C} \circ \mathcal{H} + \mathcal{D}$, except the last one where $\mathcal{M} = (1 + \mathcal{M}) \times (\mathcal{C} \circ \mathcal{H})$.

<table>
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<tr>
<th>maps, $M(z)$</th>
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<th>submaps, $H(z)$</th>
<th>coreless, $D(z)$</th>
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<td>$z/(1 - z(1 + M))^2$</td>
<td>$z(1 + M)^2$</td>
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Extension to other models

[Banderier, Flajolet, Schaeffer, Soria 2001]:

Table 3. Composition schemas, of the form $M = C \circ H + D$, except the last one where $M = (1 + M) \times (C \circ H)$.

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→ Unified study of the phase transition for block-weighted random planar maps Z. Salvy (Eurocomb’23)
Theorem [S. 23] Model of the preceding table without coreless maps exhibits a phase transition at some explicit $u_C$.

When $n \to \infty$:

- Subcritical phase $u < u_C$: "general map phase" one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: "tree phase" only small blocks.

We obtain explicit results on enumeration and size of blocks in each case.
V. Extension to tree-rooted maps
Decorated maps are interesting

From a theoretical physics point of view:

- Before: “pure gravity” case (nothing happens on the surface);
- Now: decorated map (things happen! new behaviours! excitement!).
Tree-rooted maps

= (rooted planar) maps endowed with a spanning tree.

\[ M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \]  

[Mullin 67]

We want to study block-weighted tree-rooted maps.
Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.

\[ M(z) = B(zM^2(z)) \]

GS of 2-connected tree-rooted maps
Tree-rooted maps are not so nice

\[ M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \] so
Tree-rooted maps are not so nice

\[ M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \] so

\[ [z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3}; \]
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\[ M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \] so

\[ [z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3}; \quad \rho_M = \frac{1}{16}; \]
Tree-rooted maps are not so nice

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\[ [z^n]M(z) \sim \frac{4}{\pi} \times 16^n \times n^{-3}; \quad \rho_M = \frac{1}{16}; \]

\[ M(\rho_M) = 8 - \frac{64}{3\pi} \approx 1.2 \text{ so } M \text{ is not algebraic...} \]

\[ P(z, M(z)) = 0 \]
Tree-rooted maps are not so nice

\[ M(z) = \sum_{n \geq 0} \text{Cat}_n \text{Cat}_{n+1} z^n \text{ so} \]

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• Fortunately, it is still \textit{D}-finite

\[ P_0(z) \frac{\partial^2 M}{\partial z^2}(z) + P_1(z) \frac{\partial M}{\partial z}(z) + P_2(z)M(z) + P_3(z) = 0. \]
2-connected tree-rooted maps are naughty

Using $M(z) = B(zM^2(z))$ and the properties of $M$, we show
2-connected tree-rooted maps are naughty

Using $M(z) = B(zM^2(z))$ and the properties of $M$, we show

- $\rho_B = \rho_M M^2(\rho_M) = \frac{4(3\pi - 8)^2}{9\pi^2} \approx 0.091$

is not algebraic so $B$ is not $D$-finite.
2-connected tree-rooted maps are naughty

Using \( M(z) = B(zM^2(z)) \) and the properties of \( M \), we show

- \( \rho_B = \rho_M M^2(\rho_M) = \frac{4(3\pi - 8)^2}{9\pi^2} \approx 0.091 \)

is not algebraic so \( B \) is not \( D \)-finite

- \( B \) is \( D \)-algebraic

\[
P \left( \frac{\partial^2 B}{\partial y^2}(y), \frac{\partial B}{\partial y}(y), B(y), y \right) = 0.
\]
Enumeration of 2-connected tree-rooted maps

Using $M(z) = B(zM^2(z))$ and the properties of $M$, we show

**Theorem [Albenque, Fusy, S. 24+]**

$$[y^n]B(y) \sim \frac{4(3\pi - 8)^3}{27\pi(4 - \pi)^3} \times \rho_B^{-n} \times n^{-3}.$$
Block decomposition of tree-rooted maps

The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.

\[ M(z) = B(zM^2(z)) \]

GS of 2-connected tree-rooted maps
The decomposition of maps into blocks extends into a decomposition of tree-rooted maps into tree-rooted blocks.

$$M(z, u) = uB(zM^2(z, u)) + 1 - u$$
Phase transition

Theorem [Albenque, Fusy, S. 24+] Model exhibits a phase transition at $u_C = \frac{9\pi(4 - \pi)}{420\pi - 81\pi^2 - 512} \approx 3.02$.

When $n \to \infty$:

- Subcritical phase $u < u_C$: “general tree-rooted map phase” one huge block;
- Critical phase $u = u_C$: a few large blocks;
- Supercritical phase $u > u_C$: “tree phase” only small blocks.

We obtain results on enumeration, size of blocks and scaling limits in each case.
### Results

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<tr>
<th>For $M_n \leftrightarrow \mathcal{P}_{n,u}$</th>
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<tr>
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<td>Size of</td>
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### Results

For $M_n \leftrightarrow P_{n,u}$

<table>
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- **Size of**
  - the largest block
  - the second one

  $\sim (1 - \mathbb{E}(\mu^{\rho_B,u}))n$
  $\Theta(n^{1/2})$

- **Scaling limit of $M_n$**

  $\frac{C_2}{n^{1/2} \ln(n)^{1/2}} M_n \to \mathcal{T}_e$
  $\frac{C_3(u)}{n^{1/2}} M_n \to \mathcal{T}_e$
Interlude: tree-rooted quadrangulations
Interlude: tree-rooted quadrangulations

$M(z) = Q(z)$ does not hold!
VI. Perspectives
Extension to more involved decompositions

• For maps: maps into loopless blocks, 2-connected maps into 3-connected blocks;
Extension to more involved decompositions

- For maps: maps into loopless blocks, 2-connected maps into 3-connected blocks;

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Extension to more involved decompositions

• For maps: maps into loopless blocks, 2-connected maps into 3-connected blocks;
• For decorated maps: tree-rooted quadrangulations into simple blocks, Schnyder woods / 3-orientations / 2-orientations into irreducible blocks.
Critical window?

Phase transition very sharp => what if $u = 9/5 \pm \varepsilon(n)$?

- Block size results still hold if $u_n = 9/5 - \varepsilon(n), \varepsilon^3 n \to \infty$;
- For $u_n = 9/5 + \varepsilon(n)$, this is the case as well: when $\varepsilon^3 n \to \infty$

$$ L_{n,1} \sim 2.7648 \varepsilon^{-2} \ln(\varepsilon^3 n) $$

(analogous to [Bollobás 1984]’s result for Erdős-Rényi graphs!);

- Results exist for scaling limits in ER graphs [Addario-Berry, Broutin, Goldschmidt 2010], open question in our case.

Is there a critical window? If so, what is its width?
Thank you!