Solving combinatorial equations via computer algebra

Journées de combinatoire de Bordeaux, 6 February 2024

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Based on joint works with:

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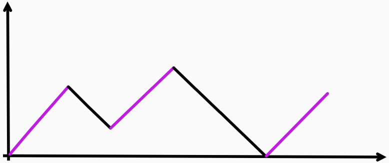
Sergey Yurkevich (University of Vienna)

What kind of objects are we considering?

walks in $\mathbb{N}$



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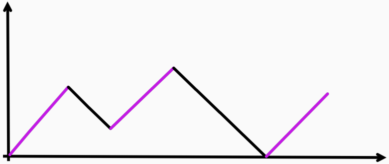
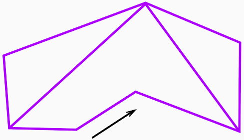


rooted planar maps



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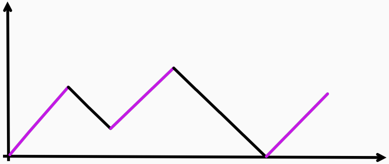

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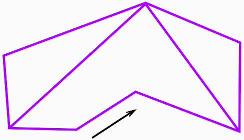

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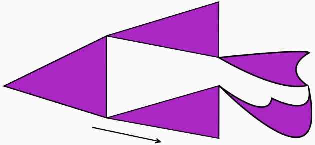

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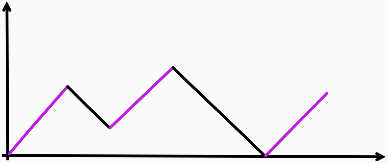

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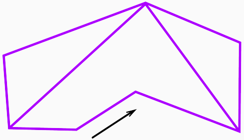

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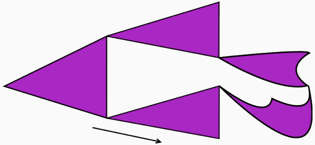


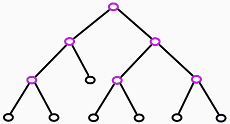
$\mathcal{A}$ set of **discrete** objects, s : $\mathcal{A} \to \mathbb{N}$ **size** function s.t.

$$\#\{a \in \mathcal{A} \mid s(a) = n\} < +\infty, \text{ for all } n \in \mathbb{N}.$$

Define the **generating function**

$$F(t) := \sum_{a \in \mathcal{A}} t^{s(a)} \qquad \in \mathbb{Q}[[t]]$$

Toy example: algebraic equation for binary trees



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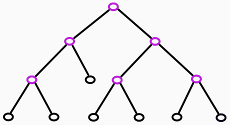

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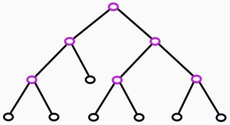

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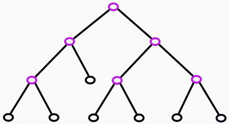

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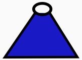
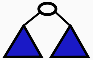
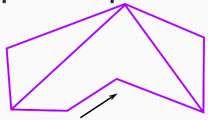
 $= \{\circ\} \cup$ 

$$F_{BT}(t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n \geq 0} \frac{1}{n+1}\binom{2n}{n} t^n$$

Catalan number

What about more sophisticated enumerations?

rooted planar maps



$$F(t,u) = 1 + tu\left(uF(t,u)^2 + \frac{uF(t,u) - F(t,1)}{u-1}\right)$$

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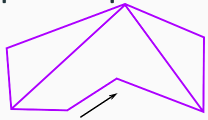

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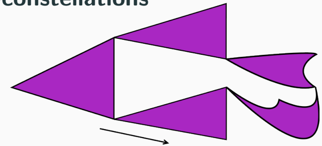

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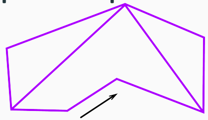

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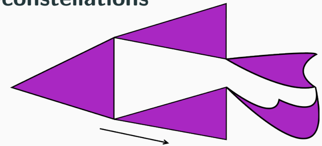

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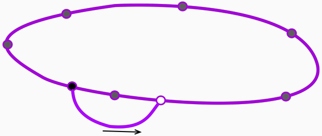
hard particles on planar maps



$$\begin{cases} F(t,u) = x - y + G(t,u) + tu\left(uF(t,u)^2 + \frac{uF(t,u)-F(t,1)}{u-1}\right) \\ G(t,u) = y + tsu\left(F(t,u)G(t,u) + \frac{G(t,u)-G(t,1)}{u-1}\right) \end{cases}$$

How to relate these combinatorial objects to such equations?

rooted planar maps

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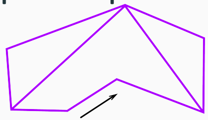

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$a_{n,d} := \#\{\text{planar maps with } n \text{ edges},$
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$\sum_{n=0}^{\infty} a_n t^n$ generating function

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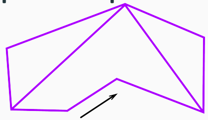

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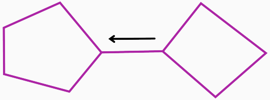

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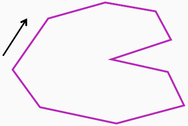

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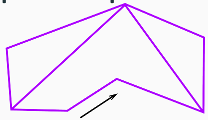

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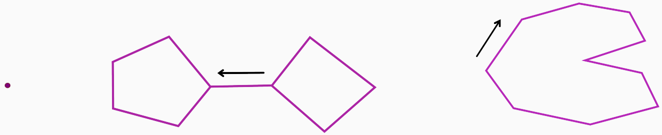

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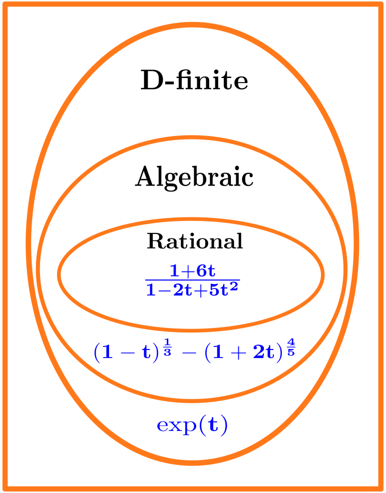
Solving functional equations



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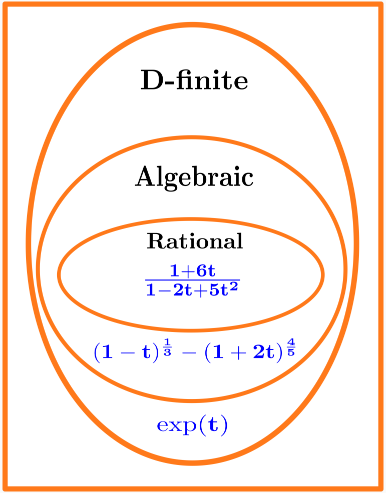

In this talk

Solving $=$ Classifying the initial series $F(t,1)$

$+$ Computing a witness of this classification

(e.g. $R \in \mathbb{Q}[z,t]$ s.t. $R(F(t,1),t) = 0$)

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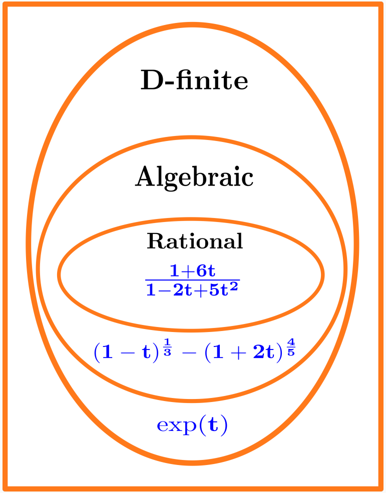

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Going back to our planar maps...

$F(t,1) = 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \cdots \qquad \in \mathbb{Q}[[t]]$

annihilated by $R = 27t^2z^2 + (1-18t)z + 16t - 1 \in \mathbb{Q}[z,t]$

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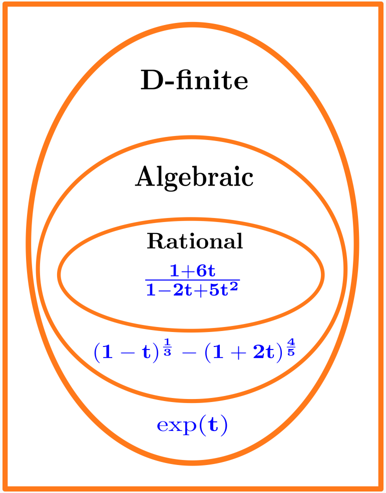

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From R:

- **(Recurrence)** $a_0 = 1$ and $(n+3)a_{n+1} - 6(2n+1)a_n = 0$,
- **(Closed-form)** $a_n = 2\frac{3^n(2n)!}{n(n+2)!}$,
- **(Asymptotics)** $a_n \sim 2\frac{12^n}{\sqrt{\pi n^5}}$, when $n \to +\infty$.

Content of the talk

Objectives

- **Introduce** so-called Discrete Differential Equations (DDEs),
- **Determine the nature** of the solutions of DDEs,
- **Decidability result**: algorithms for computing a witness,
- **Complexity analysis**: **quantitative estimates**.

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Plan

I Perform the above points for DDEs [Bousquet-Mélou, Jehanne '06; Bostan, Chyzak, N., Safey El Din '22; Bostan, N., Safey El Din '23]

II Perform the above points for *systems* of DDEs [N., Yurkevich '23]

Objects of interest: Discrete Differential Equations

Definition

Given $f \in \mathbb{Q}[u]$, $k \geq 1$, and $Q \in \mathbb{Q}[y_0, \ldots, y_k, t, u]$,

$$F = f + t \cdot Q(F, \Delta F, \ldots, \Delta^k F, t, u) \qquad \textbf{(DDE)}$$

is a *Discrete Differential Equation*, where $\Delta : F \in \mathbb{Q}[u][[t]] \mapsto \frac{F(t,u) - F(t,1)}{u-1} \in \mathbb{Q}[u][[t]]$, and where for $\ell \geq 1$ we define $\Delta^{\ell+1} = \Delta^\ell \circ \Delta$.

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Theorem
[Bousquet-Mélou, Jehanne '06]

The unique solution in $\mathbb{Q}[u][[t]]$ *of* **(DDE)** *is algebraic over* $\mathbb{Q}(t,u)$.

$\rightsquigarrow$ **Constructive proof** $\Longrightarrow$ **algorithm**

Bousquet-Mélou and Jehanne's algorithm

Input: $F(t,u) = 1 + tu\left(F(t,u)^3 + (2F(t,u) + F(t,1))\frac{F(t,u)-F(t,1)}{u-1} + \frac{F(t,u)-F(t,1)-(u-1)\partial_u F(t,1)}{(u-1)^2}\right)$,

Output: $81t^2F(t,1)^3 - 9t(9t-2)F(t,1)^2 + (27t^2-66t+1)F(t,1) - 3t^2 + 47t - 1 = 0.$

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- **Show** that there exist distinct $U_1, U_2 \in \bigcup_{d \geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]]$ s.t. $\partial_x P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0$,

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Output: $81t^2F(t,1)^3 - 9t(9t-2)F(t,1)^2 + (27t^2-66t+1)F(t,1) - 3t^2 + 47t - 1 = 0.$

- **Compute** $P \in \mathbb{Q}(t)[x,u,z_0,z_1]$ such that $P(F(t,u),u,F(t,1),\partial_u F(t,1)) = 0$,
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$$\partial_u F(t,u) \cdot \partial_x P(F(t,u),u,F(t,1),\partial_u F(t,1)) + \partial_u P(F(t,u),u,F(t,1),\partial_u F(t,1)) = 0,$$

- **Show** that there exist distinct $U_1, U_2 \in \bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]]$ s.t. $\partial_x P(F(t,U_i),U_i,F(t,1),\partial_u F(t,1)) = 0$,
- **Gather** the 7 equations in 7 unknowns and 1 parameter

$$\text{For } 1 \leq i \leq 2, \begin{cases} P(F(t,U_i),U_i,F(t,1),\partial_u F(t,1)) = 0, \\ \partial_x P(F(t,U_i),U_i,F(t,1),\partial_u F(t,1)) = 0, \\ \partial_u P(F(t,U_i),U_i,F(t,1),\partial_u F(t,1)) = 0, \\ m \cdot (U_1 - U_2) - 1 = 0. \end{cases}$$

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Elimination theory

- Eliminate all series but $F(t,1)$

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$\rightarrow$ **Resultants**

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$\rightarrow$ **Resultants**

$\rightarrow$ **Gröbner bases**

$$\mathcal{S}: \qquad \text{For } 1 \le i \le 2, \begin{cases} P(F(t,U_i), F(t,1), \partial_u F(t,1), t, U_i) = 0, \\ \partial_x P(F(t,U_i), F(t,1), \partial_u F(t,1), t, U_i) = 0, \\ \partial_u P(F(t,U_i), F(t,1), \partial_u F(t,1), t, U_i) = 0, \end{cases} \qquad U_1 - U_2 \neq 0.$$

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Assumptions

- U_1, U_2 are **distinct series**,
- $\mathcal{S}$ has **finitely many solutions** in $\overline{\mathbb{Q}(t)}^6$,

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[Bostan, N., Safey El Din '23]

Under the above assumptions: $\delta := \deg(P)$

- There exists some nonzero polynomial $R \in \mathbb{Q}[z_0, t]$ whose partial degrees are upper bounded by $\boldsymbol{D := \delta^2(\delta - 1)^4}$, such that $R(F(t,1), t) = 0$.
- There exists an algorithm computing this R in $O_{\log}(D^3)$ ops. in $\mathbb{Q}$.

(We proved a **general version** of this result)

NEW!

ddesolver: package presentation [N. 24]

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- **Maple package** dedicated to solving discrete differential equations,

https://github.com/HNotarantonio/ddesolver

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ddesolver Public

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NEW!

- **Maple package** dedicated to solving discrete differential equations,
- **Available** in a **Git repository**,
- Relies on **evaluation–interpolation** and **fast multi-modular arithmetic**,
- **Can be coupled with libraries** for efficient Gröbner bases computations (e.g. **msolve**).

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Systems of Discrete Differential Equations

What about systems?

Systems of Discrete Differential Equations

What about systems?

Modelling **special Eulerian planar orientations**:

$$\begin{cases} F(t,u) = 1 + t \cdot \left(u + 2uF(t,u)^2 + 2uG(t,1) + u\frac{F(t,u)-uF(t,1)}{u-1} \right), \\ G(t,u) = t \cdot \left(2uF(t,u)G(t,u) + uF(t,u) + uG(t,1) + u\frac{G(t,u)-uG(t,1)}{u-1} \right). \end{cases}$$

[Bonichon, Bousquet-Mélou, Dorbec, Pennarun '17]

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[Bonichon, Bousquet-Mélou, Dorbec, Pennarun '17]

Modelling **hard particles on planar maps**:

$$\begin{cases} F(t,u) = x - y + G(t,u) + tu\left(uF(t,u)^2 + \frac{uF(t,u)-F(t,1)}{u-1}\right) \\ G(t,u) = y + tsu\left(F(t,u)G(t,u) + \frac{G(t,u)-G(t,1)}{u-1}\right) \end{cases}$$

[Bousquet-Mélou, Jehanne '06]

Previous works: case of systems of equations

- **Nature** of $\mathbf{F}(\mathbf{t}, \mathbf{1})$: [Popescu '86] $\implies$ $\mathbf{F}(\mathbf{t}, \mathbf{1})$ is **algebraic** over $\mathbb{K}(t)$

  BUT: the proof seems non constructive...

- **Algorithm** for computing $R \in \mathbb{K}[t, z]$ s.t. $R(t, \mathbf{F}(\mathbf{t}, \mathbf{1})) = 0$: [$\varnothing$]

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What about systems of **linear** DDEs?

- **Effective proof**: [Buchacher, Kauers '20] $\to$ **Talk at FPSAC '19!**

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What about systems of **linear** DDEs?

- **Effective proof**: [Buchacher, Kauers '20] $\rightarrow$ **Talk at FPSAC '19!**

- **Still, there are papers dealing with systems of DDEs!**

  Asinowski, Bacher, Banderier, Beaton, Bonichon, Bousquet-Mélou, Bouvel, Buchacher, Dorbec, Gittenberger, Guerrini, Jehanne, Kauers, Pennarun, Rinaldi, ...

$\rightarrow$ **There should be things to say for any system of DDEs**

Popescu's theorem yielding algebraicity of the solutions

[Popescu '86, Swan '98]

(1.4) THEOREM. *Let k be a field, $k\langle X\rangle$ the algebraic power series ring in $X = (X_1, \cdots, X_r)$ over k, f a finite system of polynomial equations over $k\langle X\rangle$ and $\hat{y} = (\hat{y}_1, \cdots, \hat{y}_n) \in k[\![X]\!]^n$ a formal solution of f such that $\hat{y}_i \in k[\![X_1, \cdots, X_{s_i}]\!]$, $1 \leqslant i \leqslant n$ for some positive integers $s_i \leqslant r$. Then there exists a solution $y = (y_1, \cdots, y_n)$ of f in $k\langle X\rangle$ such that $y_i \in k\langle X_1, \cdots, X_{s_i}\rangle$, $1 \leqslant i \leqslant n$.*

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- Solutions of systems of DDEs are **unique** with components in $\mathbb{Q}[u][[t]] \implies$ they are **algebraic**!

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[planar maps]
$$H(t,u)=1+t\left(u^2H(t,u)^2+u\,\frac{uH(t,u)-G(t,u)}{u-1}\right)$$

- There exists a *(unique!)* solution $(H,G)=(F,F(t,1))$, where $F\in\mathbb{Q}[u][[t]]$,
- The involved series are $F(t,1)$ and $F(t,u)$, and $\{\boldsymbol{t}\}\subset\{\boldsymbol{t},\boldsymbol{u}\}$.

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The proof seems **non constructive**... How to compute witnesses?

Constructive algebraicity theorem for solutions of systems of DDEs (FPSAC'23)

[N., Yurkevich '23]

Let $n, k \geq 1$ be integers and $f_1, \dots, f_n \in \mathbb{Q}[u]$, $Q_1, \dots, Q_n \in \mathbb{Q}[y_1, \dots, y_{n(k+1)}, t, u]$ be polynomials. Denote $\nabla^k F := F, \Delta F, \dots, \Delta^k F$. Then the system of DDEs

$$\begin{cases} \mathbf{(E_{F_1})}: \quad F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ \mathbf{(E_{F_n})}: \quad F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u). \end{cases} \tag{SDDEs}$$

admits a unique vector of solutions $(F_1, \dots, F_n) \in \mathbb{Q}[u][[t]]^n$, and all its components are algebraic over $\mathbb{Q}(t, u)$.

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[Proof sketch]

- There exists a **polynomial system** $\mathcal{S}$ defined over $\mathbb{Q}(t)$ in $\boldsymbol{nk(n+2)}$ equations and unknowns, that admits a solution $\mathcal{P}$ with $F_1(t, 1)$ as one of its coordinates,
- **The Jacobian of $\mathcal{S}$ is invertible at $\mathcal{P}$** $\implies$ $F_1(t, 1)$ is **algebraic** over $\mathbb{Q}(t)$.

Identifying more polynomial equations

Consider $\rightsquigarrow F_1, F_2 \equiv F_1(t,u), F_2(t,u) \in \mathbb{Q}[u][[t]]$

$$\begin{cases} 0 = (1-F_1)\cdot(\mathbf{u}-1) + t\mathbf{u}\cdot(2\mathbf{u}F_1^2 - \mathbf{u}F_1(t,1) + 2\mathbf{u}F_2(t,1) - 2F_1^2 + \mathbf{u} + F_1 - 2F_2(t,1) - 1), \\ 0 = F_2\cdot(1-\mathbf{u}) + t\mathbf{u}\cdot(2\mathbf{u}F_1F_2 + \mathbf{u}F_1 - 2F_1F_2 - F_1 + F_2 - F_2(t,1)). \end{cases}$$

Denote by $E_1, E_2 \in \mathbb{Q}(t)[x_1, x_2, z_0, z_1, \mathbf{u}]$ polynomials such that

$$\text{for } i \in \{1,2\}, \quad E_i(F_1(t,\mathbf{u}), F_2(t,\mathbf{u}), F_1(t,1), F_2(t,1), \mathbf{u}) = 0. \qquad (\equiv E_i(\mathbf{u}))$$

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Differentiating with respect to $\mathbf{u}$ yields

$$\begin{pmatrix} (\partial_{x_1}E_1)(\mathbf{u}) & (\partial_{x_2}E_1)(\mathbf{u}) \\ (\partial_{x_1}E_2)(\mathbf{u}) & (\partial_{x_2}E_2)(\mathbf{u}) \end{pmatrix} \cdot \begin{pmatrix} \partial_{\mathbf{u}}F_1 \\ \partial_{\mathbf{u}}F_2 \end{pmatrix} + \begin{pmatrix} (\partial_{\mathbf{u}}E_1)(\mathbf{u}) \\ (\partial_{\mathbf{u}}E_2)(\mathbf{u}) \end{pmatrix} = 0.$$

$$\textbf{For } \mathbf{U}(\mathbf{t}) \in \bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]], \begin{cases} \textbf{if} & (\partial_{x_1}E_1 \cdot \partial_{x_2}E_2 - \partial_{x_1}E_2 \cdot \partial_{x_2}E_1)(\mathbf{U}(\mathbf{t})) = 0, \\ \textbf{then} & (\partial_{x_1}E_1 \cdot \partial_{\mathbf{u}}E_2 - \partial_{x_1}E_2 \cdot \partial_{\mathbf{u}}E_1)(\mathbf{U}(\mathbf{t})) = 0. \end{cases}$$

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$$\begin{pmatrix} (\partial_{x_1}E_1)(\mathbf{u}) & (\partial_{x_2}E_1)(\mathbf{u}) \\ (\partial_{x_1}E_2)(\mathbf{u}) & (\partial_{x_2}E_2)(\mathbf{u}) \end{pmatrix} \cdot \begin{pmatrix} \partial_{\mathbf{u}}F_1 \\ \partial_{\mathbf{u}}F_2 \end{pmatrix} + \begin{pmatrix} (\partial_{\mathbf{u}}E_1)(\mathbf{u}) \\ (\partial_{\mathbf{u}}E_2)(\mathbf{u}) \end{pmatrix} = 0.$$

$$\textbf{For } \mathbf{U(t)} \in \bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]], \begin{cases} \textbf{if} & (\partial_{x_1}E_1 \cdot \partial_{x_2}E_2 - \partial_{x_1}E_2 \cdot \partial_{x_2}E_1)(\mathbf{U(t)}) = 0, \\ \textbf{then} & (\partial_{x_1}E_1 \cdot \partial_{\mathbf{u}}E_2 - \partial_{x_1}E_2 \cdot \partial_{\mathbf{u}}E_1)(\mathbf{U(t)}) = 0. \end{cases}$$

Does this yield an **elimination procedure**?

Sketch of strategy: duplicate variables

Notations: $E_i(\mathbf{u}) \equiv E_i(F_1(t,\mathbf{u}), F_2(t,\mathbf{u}), F_1(t,1), F_2(t,1), t, \mathbf{u})$.

- **If** $\mathrm{Det}(\mathbf{u}) = 0$ has **2 distinct solutions in $\mathbf{u}$** in $\bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]]$, define

$$\mathcal{S}_{\mathrm{dup}} := \begin{cases} E_1(u_i) = 0, E_2(u_i) = 0, \\ \mathrm{Det}(u_i) = 0, \mathrm{P}(u_i) = 0, \\ m \cdot (u_1 - u_2) - 1 = 0. \end{cases} \qquad \text{for } i = 1, 2.$$

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Then: 9 equations in **9** unknowns.

$$\underbrace{x_1, \dots, x_4}_{F_i(U_j)}, \underbrace{z_0, z_1}_{F_i(t,1)}, \underbrace{u_1, u_2}_{U_i} \Rightarrow 4 + 2 + 2 = \mathbf{8}.$$

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Hope: $\mathcal{S}_{\mathrm{dup}}$ admits **finitely many solutions** in $\overline{\mathbb{Q}(t)}^9$.

A degenerate toy (SDDEs)

Modelling m-row restricted slicings: [Beaton, Bouvel, Guerrini, Rinaldi '19]

$$\begin{cases} F_1 = t^2\mathbf{u} + t^2\mathbf{u}\,(F_1 + F_2), \\ F_2 = t^2\mathbf{u}\,\frac{F_2 - F_2(t,1)}{\mathbf{u}-1} + t^2\mathbf{u}\,\frac{F_1 - F_1(t,1)}{\mathbf{u}-1} + t^2\mathbf{u}\,F_2, \end{cases}$$

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Multiplying by $(u-1)$ gives

$$\begin{cases} E_1(\mathbf{u}) := -F_1 + t^2\mathbf{u} + t^2\mathbf{u}(F_1 + F_2) = 0, \\ E_2(\mathbf{u}) := -F_2(\mathbf{u}-1) + t^2\mathbf{u}F_1 + t^2\mathbf{u}F_2 - t^2\mathbf{u}F_1(t,1) - t^2\mathbf{u}F_2(t,1) = 0. \end{cases}$$

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$$\text{Det} := \begin{pmatrix} \partial_{x_1} E_1(\mathbf{u}) & \partial_{x_2} E_1(\mathbf{u}) \\ \partial_{x_1} E_2(\mathbf{u}) & \partial_{x_2} E_2(\mathbf{u}) \end{pmatrix} = t^2\mathbf{u}^2 + \mathbf{u} - 1.$$

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There exists **only one** solution $\mathbf{U}(t) \in \bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]]\ldots$

How to **create** more solutions in $\mathbf{u}$ to $\mathbf{Det}(\mathbf{u}) = \mathbf{0}$?

Symbolic deformation argument in the general case

$$\begin{cases} F_1 = f_1(\mathbf{u}) + t \cdot Q_1(F_1, F_2, \frac{F_1 - F_1(t,1)}{u-1}, \frac{F_2 - F_2(t,1)}{u-1}, t, \mathbf{u}), \\ F_2 = f_2(\mathbf{u}) + t \cdot Q_2(F_1, F_2, \frac{F_1 - F_1(t,1)}{u-1}, \frac{F_2 - F_2(t,1)}{u-1}, t, \mathbf{u}). \end{cases} \qquad \textbf{(SDDEs)}$$

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↓ Symbolic deformation ↓

For integers $\alpha >> \beta >> 0$ and a deformation parameter ϵ, consider

$$\begin{cases} G_1 = f_1(\mathbf{u}) + t^{\alpha} \cdot Q_1\left(G_1, G_2, \frac{G_1 - G_1(t,1,\epsilon)}{u-1}, \frac{G_2 - G_2(t,1,\epsilon)}{u-1}, t^{\alpha}, \mathbf{u}\right) + \epsilon t \frac{G_1 - G_1(t,1,\epsilon)}{u-1} + \epsilon t^{\beta} \frac{G_2 - G_2(t,1,\epsilon)}{u-1}, \\ G_2 = f_2(\mathbf{u}) + t^{\alpha} \cdot Q_2\left(G_1, G_2, \frac{G_1 - G_1(t,1,\epsilon)}{u-1}, \frac{G_2 - G_2(t,1,\epsilon)}{u-1}, t^{\alpha}, \mathbf{u}\right) + \epsilon t^{\beta} \frac{G_1 - G_1(t,1,\epsilon)}{u-1} + 4\epsilon t \frac{G_2 - G_2(t,1,\epsilon)}{u-1}. \end{cases}$$

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$G_i(t, \mathbf{u}, \epsilon)$ algebraic over $\mathbb{Q}(t, \mathbf{u}, \epsilon) \Rightarrow G_i(t, \mathbf{u}, 0) = F_i(t^{\alpha}, \mathbf{u})$ algebraic over $\mathbb{Q}(t, \mathbf{u})$.

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(slight abuse of notations) $\quad E_i(G_1, G_2, G_1(t,1,\epsilon), G_2(t,1,\epsilon), t, \mathbf{u}, \epsilon) = 0, \text{ for } i = 1, 2.$

The deformation of (SDDEs) ensures good properties

Lemma 1: There are 2 **distinct solutions** U_1, U_2 in $\overline{\mathbb{Q}}(\epsilon)[[\sqrt{t}]]$ to $\mathrm{Det}(\mathbf{u}) = 0$.

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Lemma 2: Let $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}$ be the Jacobian matrix of $\mathcal{S}_{\mathrm{dup}}$ w.r.t. $\underline{\mathbf{x}}, \underline{\mathbf{u}}, \underline{\mathbf{z}}$, then $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}(\mathcal{P})$ is invertible and the saturated ideal $\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^{\infty}$ is **0-dimensional** over $\mathbb{K}(t, \epsilon)$.

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Lemma 1: There are **2 distinct solutions** U_1, U_2 in $\overline{\mathbb{Q}}(\epsilon)[[\sqrt{t}]]$ to $\mathrm{Det}(\mathbf{u}) = 0$.

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All in all: Effective Algebraic Geometry

Main steps for proving the invertibility of the Jacobian matrix

- Up to elementary operations, $\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}}(\mathcal{P})$ is an **upper-block triangular matrix**:

$$\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}(\mathcal{P})} \sim \begin{pmatrix} \mathbf{A}(\mathbf{U_1}) & 0 & \star \\ 0 & \mathbf{A}(\mathbf{U_2}) & \star \\ 0 & 0 & \mathbf{\Lambda}(\mathbf{U_1}, \mathbf{U_2}) \end{pmatrix},$$

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- For $i = 1, 2$, we have $\mathbf{val_t}(\mathbf{det}(\mathbf{A}(\mathbf{U_i}))) < +\infty$,
- There exists $\gamma \in \mathbb{Z}_{>0}$ such that

$$\det(\mathbf{\Lambda}(\mathbf{U_1}, \mathbf{U_2})) = \boldsymbol{U_1}^{\gamma} \cdot \boldsymbol{U_2}^{\gamma} \cdot (\boldsymbol{U_1} - \boldsymbol{U_2}) \cdot \mathbf{H} \mod \boldsymbol{t}^{\boldsymbol{\alpha}},$$

for some $\mathbf{H} \in \mathbb{Q}[t, \epsilon] \setminus \{0\}$ whose degree is independent of $\boldsymbol{\alpha}$.

A first polynomial time algorithm with S. Yurkevich

$$F_1 = f_1(u) + t \cdot Q_1(F_1, \Delta F_1, F_2, \Delta F_2, t, u), F_2 = f_2(u) + t \cdot Q_2(F_1, \Delta F_1, F_2, \Delta F_2, t, u).$$

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Define the "numerators" E_1, E_2 and the polynomials

$$\mathrm{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \partial_{x_2} E_1 \\ \partial_{x_1} E_2 & \partial_{x_2} E_2 \end{pmatrix} \quad \text{and} \quad P := \det \begin{pmatrix} \partial_{x_1} E_1 & \partial_u E_1 \\ \partial_{x_1} E_2 & \partial_u E_2 \end{pmatrix}.$$

The **duplicated polynomial system** $\mathcal{S}$ vanishes at

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Compute a generator R of $\langle \mathcal{S}, m \cdot (U_1 - U_2) - 1 \rangle \cap \mathbb{Q}[t, z_0]$.

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[N., Yurkevich '23] $(\delta \geq \deg(\boldsymbol{\mathcal{S}}))$

- The polynomial R has its **total degree in $O(\delta^8)$**.
- Moreover, R can be computed in $O(\delta^{200})$ ops. in $\mathbb{Q}$.

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Proved in full generality

Conclusion and perspectives

- **Polynomial time algorithm** computing witnesses of algebraicity,
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Conclusion and perspectives

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Thank you for listening!

A polynomial system for systems of DDEs

Consider

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Compute a non-trivial element of $(\langle \mathcal{S}_{\mathrm{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\mathrm{dup}}})^\infty) \cap \mathbb{K}[t, z_0, \epsilon]$, then set ϵ to 0.