

Dumont–Thomas numeration systems for \mathbb{Z}

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Outline

- 1 Introduction to numeration systems
- 2 Motivation: description of substitutive Wang tilings
- 3 Numeration systems for \mathbb{N} based on fixed points
- 4 Numeration systems for \mathbb{Z} based on periodic points
- 5 Properties of Dumont–Thomas numeration systems for \mathbb{Z}

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Binary numeration system for \mathbb{N}

n	$\text{rep}_2(n)$
7	111
6	110
5	101
4	100
3	11
2	10
1	1
0	ε

Binary numeration system for \mathbb{N}

- Let ψ be the substitution $\psi: a \mapsto ab, b \mapsto ba$ and denote its fixed point $\mathbf{t} = \psi(\mathbf{t}) = abbabaabbaab \dots$

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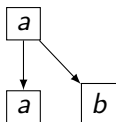
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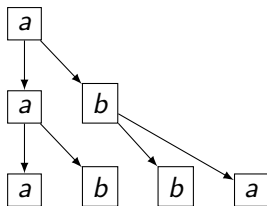
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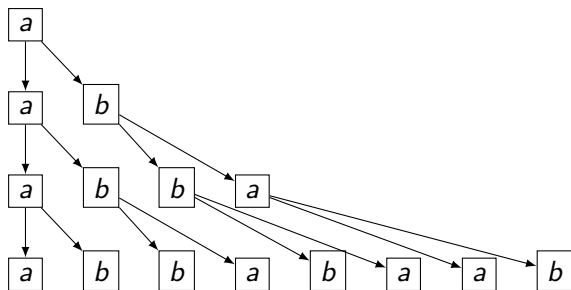
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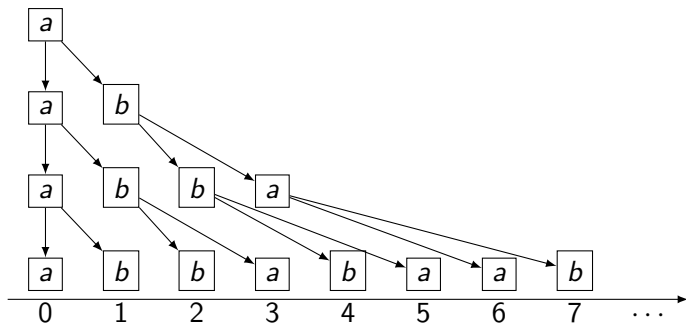
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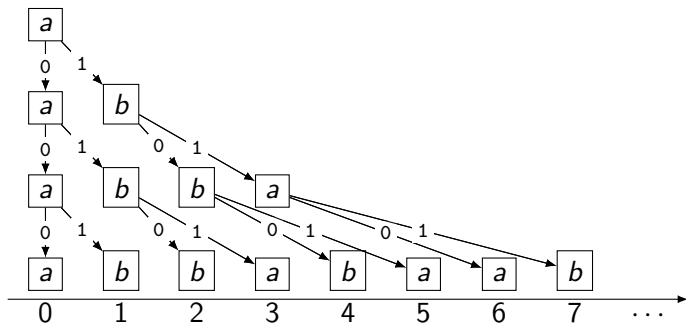
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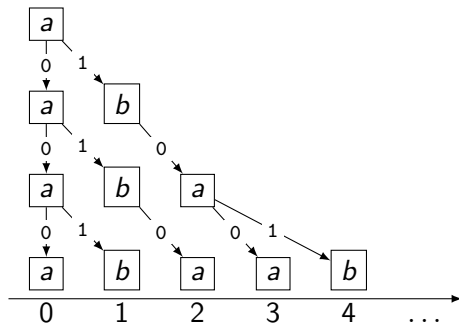


Labelling $\alpha \xrightarrow{j} \beta$ if and only if β is at position j in $\psi(\alpha)$.

Fibonacci numeration system for \mathbb{N}

- Let φ be the substitution $\varphi: a \mapsto ab, b \mapsto a$ and denote its fixed point $\mathbf{f} = \varphi(\mathbf{f}) = abaababaabaabab \dots$

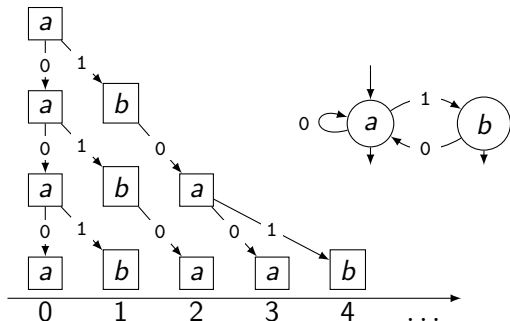
n	$\text{rep}_{\mathcal{F}}(n)$
7	1010
6	1001
5	1000
4	101
3	100
2	10
1	1
0	ε



Fibonacci numeration system for \mathbb{N}

- Let φ be the substitution $\varphi: a \mapsto ab, b \mapsto a$ and denote its fixed point $\mathbf{f} = \varphi(\mathbf{f}) = abaababaabaabab \dots$

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2	10
1	1
0	ε



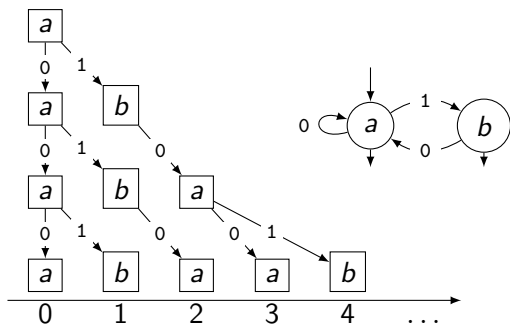
- It holds that $f_n = \mathcal{A}_{\mathbf{f}}(\text{rep}_{\mathcal{F}}(n))$, for all $n \in \mathbb{N}$.

\mathcal{F} may be viewed as a NS for \mathbb{N} described by Dumont and Thomas, 1989.

Fibonacci numeration system for \mathbb{N}

- Let φ be the substitution $\varphi: a \mapsto ab, b \mapsto a$ and denote its fixed point $\mathbf{f} = \varphi(\mathbf{f}) = abaababaabaabab \dots$

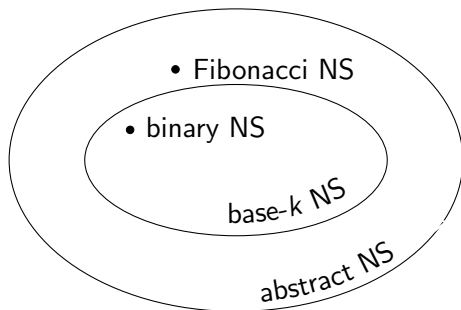
n	$\text{rep}_{\mathcal{F}}(n)$
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5	1000
4	101
3	100
2	10
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0	ϵ



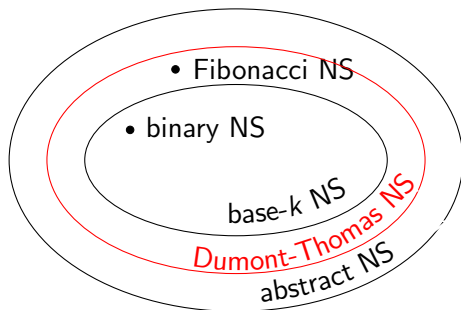
Theorem (P. Lecomte, M. Rigo, 2001)

An infinite word $\mathbf{x} = x_0x_1x_2\dots$ is an image under a coding of a fixed point of a substitution if and only if there exists deterministic finite automaton with output \mathcal{A} and an abstract numeration system \mathcal{S} s.t. $x_n = \mathcal{A}(\text{rep}_{\mathcal{S}}(n))$, for all $n \in \mathbb{N}$.

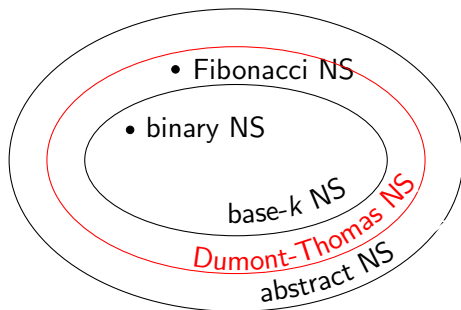
Numeration systems for \mathbb{N}



Numeration systems for \mathbb{N}



Numeration systems for \mathbb{N}



Goal: numeration systems based on ideas of Dumont and Thomas for \mathbb{Z} .

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Wang tiles

A **Wang tile** is a square tile with 4 edge labels and an index.

O J 0 D O	O H 1 D L	M D 2 J P	M D 3 D K	P J 4 H P	P H 5 H N	K D 6 H P	O I 7 B O
L E 8 I O	L C 9 I L	L I 10 A O	P I 11 E P	P I 12 I K	K B 13 I M	K A 14 I K	N I 15 C P

A particular set of 16 Wang tiles \mathcal{Z} .

Adjacent tiles with distinct label on common edge are **forbidden**:

O H 1 D L	M D 3 D K
-----------------	-----------------

Allowed.

M D 2 J P	M D 3 D K
-----------------	-----------------

Forbidden.

A **configuration** $x : \mathbb{Z}^2 \rightarrow \mathcal{Z}$ without forbidden pattern is said **valid**.

The set $\Omega_{\mathcal{Z}}$ of all valid configurations $x : \mathbb{Z}^2 \rightarrow \{0, \dots, 15\}$ is called the **Wang shift** $\Omega_{\mathcal{Z}}$ associated to the set \mathcal{Z} of Wang tiles.

Configuration in a particular Wang shift Ω_Z

5	P I12I K	O I7B O	K B13I M	P I11E P	L E8I O	P I12I K	O I7B O	K B13I M	P I12I K	O I7B O	K B13I M	O I7B O	K B13I M				
4	K D6H P	O H1D L	M D2J P	P J4H P	O H1D L	K D6H P	O H1D L	M D3D K	K D6H P	O H1D L	M D2J P	O J0D O	M D3D K				
3	P I11E P	L E8I O	P I12I K	P I11E P	L E8I O	P I12I K	L I10A O	K A14I K	P I11E P	L E8I O	P I12I K	O I7B O	K B13I M				
2	P J4H P	O H1D L	K D6H P	P H5H N	O H1D L	K D6H P	O H1D L	K D6H P	P H5H N	O H1D L	K D6H P	O H1D L	M D3D K				
1	P I11E P	L E8I O	P I12I K	N I15C P	L C9I N	P I11E P	L E8I O	P I12I K	N I15C P	L C9I L	P I12I K	L I10A O	K A14I K				
0	P I12I K	O I7B O	K B13I M	P I11E P	L E8I O	P I12I K	O I7B O	K B13I M	P I12I K	L I10A O	K A14I K	O I7B O	K B13I M				
-1	K D6H P	O H1D L	M D2J P	P J4H P	O H1D L	K D6H P	O H1D L	M D3D K	K D6H P	O H1D L	M D2J P	O H1D L	M D3D K				
	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7				

A finite part of a particular configuration $y \in \Omega_Z$.

Configuration in a particular Wang shift $\Omega_{\mathbb{Z}}$

5	P I12I K	O I7B O	K B13I M	P I11E P	L E8I O	P I12I K	O I7B O	K B13I M	P I12I K	O I7B O	K B13I M	O I7B O	K B13I M
4	K D6H P	O H1D L	M D2J P	P J4H P	O H1D L	K D6H P	O H1D L	M D3D K	K D6H P	O H1D L	M D2J P	O J0D O	M D3D K
3	P I11E P	L E8I O	P I12I K	P I11E P	L E8I O	P I12I K	L I10A O	K A14I K	P I11E P	L E8I O	P I12I K	O I7B O	K B13I M
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-1	K D6H P	O H1D L	M D2J P	P J4H P	O H1D L	K D6H P	O H1D L	M D3D K	K D6H P	O H1D L	M D2J P	O D6H O	M D3D K
	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7

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Question

Does there exist a deterministic finite automaton with output \mathcal{A} and a numeration system rep for \mathbb{Z}^2 so that, for all $\mathbf{n} \in \mathbb{Z}^2$, $y_{\mathbf{n}} = \mathcal{A}(\text{rep}(\mathbf{n}))$?

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Does there exist a deterministic finite automaton with output \mathcal{A} and a numeration system rep for \mathbb{Z}^2 so that, for all $\mathbf{n} \in \mathbb{Z}^2$, $y_{\mathbf{n}} = \mathcal{A}(\text{rep}(\mathbf{n}))$?

Necessary step: extend Dumont–Thomas numeration systems to \mathbb{Z} and \mathbb{Z}^2 .

Configuration in a particular Wang shift $\Omega_{\mathbb{Z}}$

5	P I12I K	O I7B O	K B13I M	P I11E P	L E8I O	P I12I K	O I7B O	K B13I M	P I12I K	O I7B O	K B13I M	O I7B O	K B13I M
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	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7

A finite part of a particular configuration $y \in \Omega_{\mathbb{Z}}$.

Question

Does there exist a deterministic finite automaton with output \mathcal{A} and a numeration system rep for \mathbb{Z}^2 so that, for all $\mathbf{n} \in \mathbb{Z}^2$, $y_{\mathbf{n}} = \mathcal{A}(\text{rep}(\mathbf{n}))$?

Necessary step: extend Dumont–Thomas numeration systems to \mathbb{Z} and \mathbb{Z}^2 .

Moreover, we study two-sided periodic points instead of fixed points.

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Greedy algorithm

- Numeration system based on the sequence $(3^k)_{k=0}^{+\infty}$

k	0	1	2	3	...
3^k	1	3	9	27	...

$$11 = 9 + 2$$

$$= 1 \cdot 3^2 + 0 \cdot 3^1 + 2 \cdot 3^0$$

We obtain $\text{rep}_3(11) = 102$.

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- Numeration system based on the fixed point $\mathbf{r} = abcbaacbbbaaab \dots$ of the substitution $\rho: a \mapsto abc, b \mapsto baa, c \mapsto cbb$

k	0	1	2	3	...
$\rho^k(a)$	a	abc	$abcbaacbb$	$abcbaacbbbaabcabccbbbaabaa$...

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k	0	1	2	3	...
$\rho^k(a)$	a	abc	$abcbaacbb$	$abcbaacbbbaabcabccbbbaabaa$...

We aim to represent $n = 11$, thus we take the prefix of \mathbf{r} of length 11:

$$\begin{aligned}
 abcbaacbbba &= abcbaacbb \cdot \varepsilon \cdot ba \\
 &= \rho^2(a) \cdot \rho^1(\varepsilon) \cdot \rho^0(ba)
 \end{aligned}$$

We obtain $\text{rep}_{\mathbf{r}}(11) = |a| \cdot |\varepsilon| \cdot |ba| = 102$.

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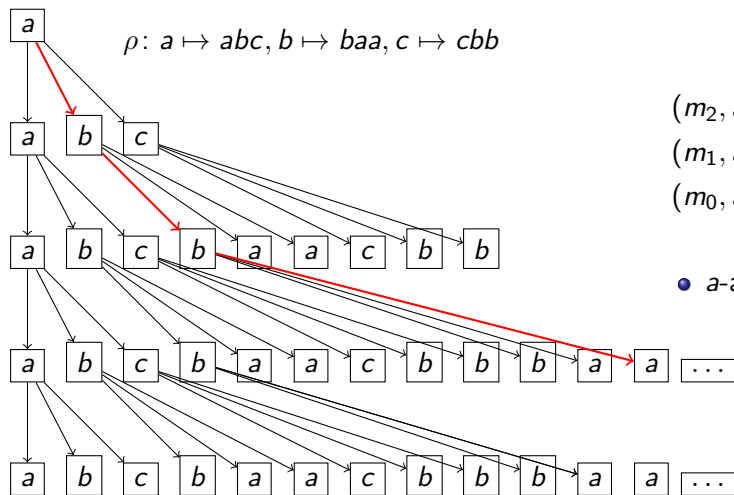
We aim to represent $n = 11$, thus we take the prefix of \mathbf{r} of length 11:

$$\begin{aligned}
 abcbaacbbba &= abcbaacbb \cdot \varepsilon \cdot ba \\
 &= \rho^2(a) \cdot \rho^1(\varepsilon) \cdot \rho^0(ba)
 \end{aligned}$$

We obtain $\text{rep}_{\mathbf{r}}(11) = |a| \cdot |\varepsilon| \cdot |ba| = 102$.

The sequence $(a, b), (\varepsilon, b), (ba, a)$ in $A^* \times A$ is called a -admissible.

Illustration of Dumont and Thomas theorem



$$(m_2, a_2) = (a, b),$$

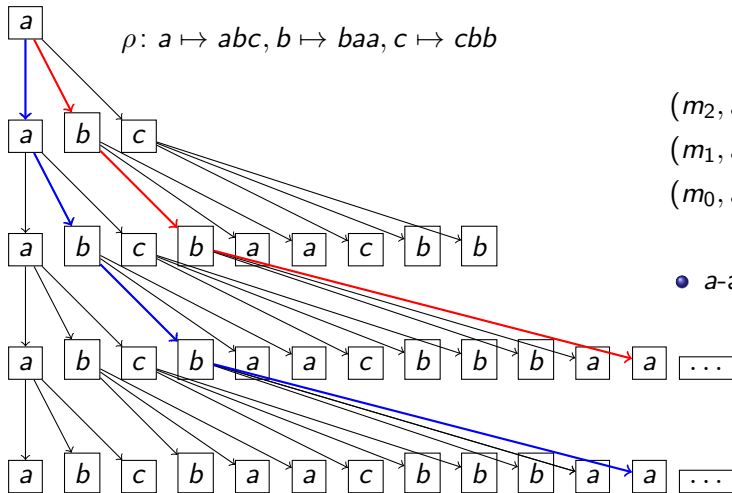
$$(m_1, a_1) = (\varepsilon, b),$$

$$(m_0, a_0) = (ba, a).$$

● *a*-admissible ✓

Recall: we obtained $\text{rep}_{\mathbf{r}}(11) = |a| \cdot |\varepsilon| \cdot |ba| = 102$, where $\mathbf{r} = \rho(\mathbf{r})$.

Illustration of Dumont and Thomas theorem



Recall: we obtained $\text{rep}_{\mathbf{r}}(11) = |a| \cdot |\varepsilon| \cdot |ba| = 102$, where $\mathbf{r} = \rho(\mathbf{r})$.

There exists a **unique** representation of every $n \geq 1$ with an a -admissible sequence $(m_i, a_i)_{i=0, \dots, k-1}$ such that $m_{k-1} \neq \varepsilon$.

Dumont–Thomas theorem

Theorem (Dumont, Thomas, 1989)

Let $a \in A$ and let $\eta : A^* \rightarrow A^*$ be a substitution. Let $\mathbf{u} = \eta(\mathbf{u})$ be a right-infinite fixed point of η with growing letter $u_0 = a$.

For every $n \geq 1$, there exists a **unique** sequence $(m_i, a_i)_{i=0, \dots, k-1}$ s. t.

- this sequence is **a-admissible** and $m_{k-1} \neq \varepsilon$,
- $u_0 u_1 \cdots u_{n-1} = \eta^{k-1}(m_{k-1}) \eta^{k-2}(m_{k-2}) \cdots \eta^0(m_0)$.

Note: requiring $m_{k-1} \neq \varepsilon$ is analogical to forbidding **leading zeroes** in base- k numeration systems.

E.g. $\text{rep}_3(11) = 102$, $\text{rep}_3(11) \neq 0102$.

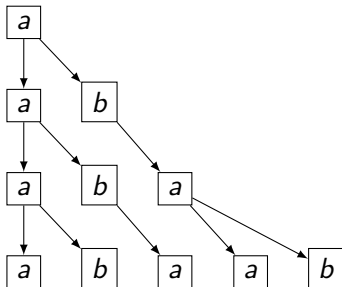
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Fibonacci substitution has no two-sided fixed point

Fibonacci substitution $\varphi: a \mapsto ab, b \mapsto a$ has

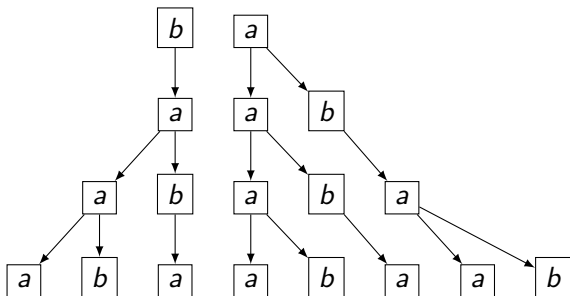
- a right-infinite **fixed point** with growing letter a ;



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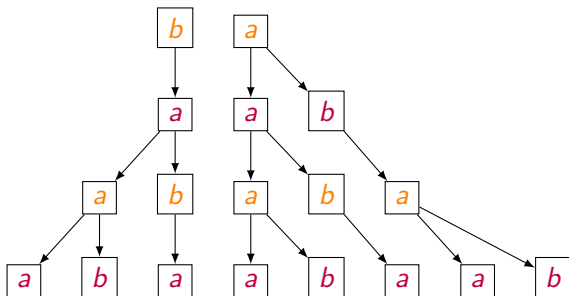
Fibonacci substitution $\varphi: a \mapsto ab, b \mapsto a$ has

- a right-infinite **fixed point** with growing letter a ;
- a two-sided **periodic point** with growing seed $b|a$ and period $p = 2$:

$$\mathbf{g} = \cdots abaab.abaababaab \cdots$$

- a two-sided **periodic point** with growing seed $a|a$ and period $p = 2$:

$$\mathbf{h} = \cdots ababa.abaababaab \cdots$$



Numeration systems for \mathbb{Z} based on periodic points

Theorem (Dumont, Thomas, 1989)

Let $a \in A$ and let $\eta : A^* \rightarrow A^*$ be a substitution. Let $\mathbf{u} = \eta(\mathbf{u})$ be a right-infinite fixed point of η with growing letter $u_0 = a$.

For every $n \geq 1$, there exists a **unique** sequence $(m_i, a_i)_{i=0, \dots, k-1}$ s. t.

- this sequence is **a-admissible** and $m_{k-1} \neq \varepsilon$,
- $u_0 u_1 \cdots u_{n-1} = \eta^{k-1}(m_{k-1}) \eta^{k-2}(m_{k-2}) \cdots \eta^0(m_0)$.

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- Our **Theorem A** is an analogy of Dumont–Thomas theorem for right-infinite periodic points with $u_0 = a$ and period $p \geq 1$, where
 - ▶ p divides k ,
 - ▶ $m_{k-1} m_{k-2} \cdots m_{k-p} \neq \varepsilon$.

Numeration systems for \mathbb{Z} based on periodic points

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Let $a \in A$ and let $\eta : A^* \rightarrow A^*$ be a substitution. Let $\mathbf{u} = \eta(\mathbf{u})$ be a right-infinite fixed point of η with growing letter $u_0 = a$.

For every $n \geq 1$, there exists a **unique** sequence $(m_i, a_i)_{i=0, \dots, k-1}$ s. t.

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- Our **Theorem A** is an analogy of Dumont–Thomas theorem for right-infinite periodic points with $u_0 = a$ and period $p \geq 1$, where
 - ▶ p divides k ,
 - ▶ $m_{k-1} m_{k-2} \cdots m_{k-p} \neq \varepsilon$.
- Our **Theorem B** is an analogy of Dumont–Thomas theorem for left-infinite periodic points with $u_{-1} = b$ and period $p \geq 1$, where
 - ▶ p divides k ,
 - ▶ $\eta^{p-1}(m_{k-1}) \eta^{p-2}(m_{k-2}) \cdots \eta^0(m_{k-p}) a_{k-p} \neq \eta^p(b)$.

Numeration systems for \mathbb{Z} based on periodic points

Definition (Dumont–Thomas numeration systems for \mathbb{Z})

Let $\eta : A^* \rightarrow A^*$ be a substitution and \mathbf{u} be its two-sided periodic point with growing seed s and the period $p \geq 1$. We define

$$\text{rep}_{\mathbf{u}} : n \mapsto \begin{cases} 0 \cdot |m_{k-1}| \cdot |m_{k-2}| \cdot \dots \cdot |m_0|, & \text{if } n \geq 1; \\ 0, & \text{if } n = 0; \\ 1, & \text{if } n = -1; \\ 1 \cdot |m_{k-1}| \cdot |m_{k-2}| \cdot \dots \cdot |m_0|, & \text{if } n \leq -2, \end{cases}$$

where

- $(m_i, a_i)_{i=0, \dots, k-1}$ is the **unique** sequence from [Theorem A](#) if $n \geq 1$;
- $(m_i, a_i)_{i=0, \dots, k-1}$ is the **unique** sequence from [Theorem B](#) if $n \leq -2$.

Note that, for every $n \in \mathbb{Z}$, we have

$$|\text{rep}_{\mathbf{u}}(n)| = \ell p + 1, \text{ for some } \ell \in \mathbb{N}.$$

Two's complement numeration system for \mathbb{Z}

The Dumont–Thomas numeration system for \mathbb{Z} based on the fixed point of any 2-uniform substitution.

n	$\text{rep}_{2c}(n)$	n	$\text{rep}_{2c}(n)$	n	$\text{rep}_{2c}(n)$
-10	10110	0	0	10	01010
-9	10111	1	01	11	01011
-8	1000	2	010	12	01100
-7	1001	3	011	13	01101
-6	1010	4	0100	14	01110
-5	1011	5	0101	15	01111
-4	100	6	0110	16	010000
-3	101	7	0111	17	010001
-2	10	8	01000	18	010010
-1	1	9	01001	19	010011

This numeration system is positional with the value map

$$\text{val}_{2c} : w_{k-1} \cdots w_1 w_0 \mapsto \sum_{i=0}^{k-1} w_i 2^i - w_{k-1} 2^k.$$

Fibonacci complement numeration system \mathcal{F}_c for \mathbb{Z}

The Dumont–Thomas numeration system for \mathbb{Z} based on the periodic point of $\varphi: a \mapsto ab, b \mapsto a$ with growing seed $b|a$ and period $p = 2$.

n	$\text{rep}_{\mathcal{F}_c}(n)$	n	$\text{rep}_{\mathcal{F}_c}(n)$	n	$\text{rep}_{\mathcal{F}_c}(n)$
-10	1000100	0	0	10	0010010
-9	1000101	1	001	11	0010100
-8	1001000	2	010	12	0010101
-7	1001001	3	00100	13	0100000
-6	1001010	4	00101	14	0100001
-5	10000	5	01000	15	0100010
-4	10001	6	01001	16	0100100
-3	10010	7	01010	17	0100101
-2	100	8	0010000	18	0101000
-1	1	9	0010001	19	0101001

This numeration system is positional with the value map

$$\text{val}_{\mathcal{F}_c} : w_{k-1} \cdots w_1 w_0 \mapsto \sum_{i=0}^{k-1} w_i F_i - w_{k-1} F_k.$$

Outline

- 1 Introduction to numeration systems
- 2 Motivation: description of substitutive Wang tilings
- 3 Numeration systems for \mathbb{N} based on fixed points
- 4 Numeration systems for \mathbb{Z} based on periodic points
- 5 Properties of Dumont–Thomas numeration systems for \mathbb{Z}

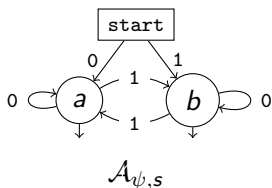
Periodic points as automatic sequences

Theorem (Labbé, L.)

Let $\eta : A^* \rightarrow A^*$ be a substitution and \mathbf{u} be its two-sided periodic point with growing seed $s = u_{-1}|u_0$. Then

$$u_n = \mathcal{A}_{\eta,s}(\text{rep}_{\mathbf{u}}(n)), \quad \text{for every } n \in \mathbb{Z}.$$

The Thue–Morse substitution $\psi : a \mapsto ab, b \mapsto ba$ has periodic point $\mathbf{t} = \cdots abbabaab.abbabaab \cdots$ with $s = b|a$ and period $p = 2$.



n	$\text{rep}_{\mathbf{t}}(n)$	n	$\text{rep}_{\mathbf{t}}(n)$
4	00100	-1	1
3	011	-2	110
2	010	-3	101
1	001	-4	100
0	0	-5	11011

Note that Thue–Morse substitution has **no two-sided fixed point**.

Characterization by a particular total order

n	-5	-4	-3	-2	-1	0	1	2	3	4
$\text{rep}_{2c}(n)$	1011	100	101	10	1	0	01	010	011	0100

Radix order: $u <_{rad} v$ if and only if $|u| < |v|$ or $|u| = |v|$ and $u <_{lex} v$

Reversed order: $u <_{rev} v$ if and only if $|u| > |v|$ or $|u| = |v|$ and $u <_{lex} v$

Characterization by a particular total order

n	-5	-4	-3	-2	-1	0	1	2	3	4
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Definition (total order \prec)

For every $u, v \in \{0, 1\}\mathcal{D}^*$, we define $u \prec v$ if and only if

- $u \in 1\mathcal{D}^*$ and $v \in 0\mathcal{D}^*$, or
- $u, v \in 0\mathcal{D}^*$ and $u <_{rad} v$, or
- $u, v \in 1\mathcal{D}^*$ and $u <_{rev} v$.

Characterization by a particular total order

n	-5	-4	-3	-2	-1	0	1	2	3	4
$\text{rep}_{2c}(n)$	1011	100	101	10	1	0	01	010	011	0100

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Theorem (Labbé, L.)

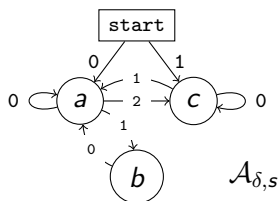
Let $\eta : A^* \rightarrow A^*$ be a substitution and \mathbf{u} be its two-sided periodic point with growing seed s and period $p \geq 1$. Let $f : \mathbb{Z} \rightarrow \{0, 1\}\mathcal{D}^*$ be a map.

Then $f = \text{rep}_{\mathbf{u}}$ if and only if f is **increasing** w.r.t. \prec , its image is

$$f(\mathbb{Z}) = \bigcup_{\ell \in \mathbb{N}} \mathcal{L}_{p+1}(\mathcal{A}_{\eta, s}) \setminus \{0W_{\min}, 1W_{\max}\}\mathcal{D}^* \text{ and } f(0) = 0.$$

Extension to \mathbb{Z}^2

A substitution $\delta: a \mapsto abc, b \mapsto a, c \mapsto ca$ has a periodic point $\mathbf{d} = \dots caabc.abcaacaabc \dots$ with growing seed $s = c|a$ and $p = 2$



n	$\text{rep}_{\mathbf{d}}(n)$	n	$\text{rep}_{\mathbf{d}}(n)$
6	00100	-1	1
5	021	-2	111
4	020	-3	110
3	010	-4	101
2	002	-5	100
1	001	-6	11102
0	0	-7	11101

We denote $W_{\min} = 00$ and $W_{\max} = 12$. Let $w \in \mathcal{L}(\mathcal{A}_{\delta,s})$. Then

$$\mathcal{A}_{\delta,s}(w) = \begin{cases} \mathcal{A}_{\delta,s}(0(W_{\min})^*v), & \text{if } w = 0v; \\ \mathcal{A}_{\delta,s}(1(W_{\max})^*v), & \text{if } w = 1v. \end{cases}$$

This is why W_{\min} and W_{\max} are called **neutral words**.

We represent $\mathbf{n} = (6, -5) \in \mathbb{Z}^2$ as

$$\begin{pmatrix} \text{rep}_{\mathbf{d}}(6) \\ \text{rep}_{\mathbf{d}}(-5) \end{pmatrix} = \begin{pmatrix} 00100 \\ 100 \end{pmatrix} = \begin{pmatrix} 00100 \\ 11200 \end{pmatrix}.$$

Extension to \mathbb{Z}^d

Definition (Numeration system for \mathbb{Z}^d)

Let $\eta : A^* \rightarrow A^*$ be a substitution and \mathbf{u} its periodic point with growing seed. For every $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$, we define

$$\text{rep}_{\mathbf{u}}(\mathbf{n}) = \begin{pmatrix} \text{pad}_t(\text{rep}_{\mathbf{u}}(n_1)) \\ \text{pad}_t(\text{rep}_{\mathbf{u}}(n_2)) \\ \dots \\ \text{pad}_t(\text{rep}_{\mathbf{u}}(n_d)) \end{pmatrix} \in \{0, 1\}^d (\mathcal{D}^d)^*,$$

where pad_t **inserts** corresponding neutral word W_{\min} or W_{\max} into representations **up to length** $t = \max\{|\text{rep}_{\mathbf{u}}(n_i)| : 1 \leq i \leq d\}$.

Fibonacci complement numeration system $\mathcal{F}c$ for \mathbb{Z}^2

7	10000 01010	10001 01010	10010 01010	10100 01010	10101 01010	00000 01010	00001 01010	00010 01010	00100 01010	00101 01010	01000 01010	01001 01010	01010 01010
6	10000 01001	10001 01001	10010 01001	10100 01001	10101 01001	00000 01001	00001 01001	00010 01001	00100 01001	00101 01001	01000 01001	01001 01001	01010 01001
5	10000 01000	10001 01000	10010 01000	10100 01000	10101 01000	00000 01000	00001 01000	00010 01000	00100 01000	00101 01000	01000 01000	01001 01000	01010 01000
4	10000 00101	10001 00101	10010 00101	10100 00101	10101 00101	00000 00101	00001 00101	00010 00101	00100 00101	00101 00101	01000 00101	01001 00101	01010 00101
3	10000 00100	10001 00100	10010 00100	10100 00100	10101 00100	00000 00100	00001 00100	00010 00100	00100 00100	00101 00100	01000 00100	01001 00100	01010 00100
2	10000 00010	10001 00010	10010 00010	100 010	101 010	000 010	001 010	010 010	00100 00010	00101 00010	01000 00010	01001 00010	01010 00010
1	10000 00001	10001 00001	10010 00001	100 001	101 001	000 001	001 001	010 001	00100 00001	00101 00001	01000 00001	01001 00001	01010 00001
0	10000 00000	10001 00000	10010 00000	100 000	1 0	0 0	001 000	010 000	00100 00000	00101 00000	01000 00000	01001 00000	01010 00000
-1	10000 10101	10001 10101	10010 10101	100 101	1 1	0 1	001 101	010 101	00100 10101	00101 10101	01000 10101	01001 10101	01010 10101
-2	10000 10100	10001 10100	10010 10100	100 100	101 100	000 100	001 100	010 100	00100 10100	00101 10100	01000 10100	01001 10100	01010 10100
-3	10000 10010	10001 10010	10010 10010	10100 10010	10101 10010	00000 10010	00001 10010	00010 10010	00100 10010	00101 10010	01000 10010	01001 10010	01010 10010
-4	10000 10001	10001 10001	10010 10001	10100 10001	10101 10001	00000 10001	00001 10001	00010 10001	00100 10001	00101 10001	01000 10001	01001 10001	01010 10001
-5	10000 10000	10001 10000	10010 10000	10100 10000	10101 10000	00000 10000	00001 10000	00010 10000	00100 10000	00101 10000	01000 10000	01001 10000	01010 10000
	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7

Description of a particular Wang configuration

The numeration system $\mathcal{F}c$ extended to \mathbb{Z}^2 describes the Wang configuration $y \in \Omega_{\mathcal{Z}}$.

5	P I12I K	O I7B O	K B13I M	P I11E P	L E8I O	P I12I K	O I7B O	K B13I M	P I12I K	O I7B O	K B13I M	O I7B O	K B13I M				
4	K D6H P	O H1D L	M D2J P	P J4H P	O H1D L	K D6H P	O H1D L	M D3D K	K D6H P	O H1D L	M D2J P	O J0D O	M D3D K				
3	P I11E P	L E8I O	P I12I P	P I11E P	L E8I O	P I12I K	L I10A O	K A14I K	P I11E P	L E8I O	P I12I P	O I7B O	K B13I M				
2	P J4H P	O H1D L	K D6H P	P H5H P	O H1D L	K D6H P	O H1D L	K D6H P	P H5H P	O H1D L	K D6H P	O H1D L	M D3D K				
1	P I11E P	L E8I O	P I12I K	N I15C P	L C9I L	P I11E K	L E8I O	P I12I K	N I15C P	L C9I L	P I12I K	L I10A O	K A14I K				
0	P I12I K	O I7B O	K B13I M	P I11E P	L E8I O	P I12I K	O I7B O	K B13I M	P I12I K	L I10A O	K A14I K	O I7B O	K B13I M				
-1	K D6H P	O H1D L	M D2J P	P J4H P	O H1D L	K D6H P	O H1D L	M D3D K	K D6H P	O H1D L	K D6H P	O H1D L	M D3D K				
	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7				

Theorem (Labbé, L.)

There exists a deterministic finite automaton with output \mathcal{A} so that

$$y_n = \mathcal{A}(\text{rep}_{\mathcal{F}c}(n)), \quad \text{for every } n \in \mathbb{Z}^2.$$

Open problems

- Find necessary and sufficient conditions for the Dumont–Thomas numeration systems to be positional.
- Explore links between Dumont–Thomas numeration systems for \mathbb{Z}^d and periodic points of d -dimensional substitutions.
- Find other Wang shifts with an automatic characterization.

Thank you for your attention!