

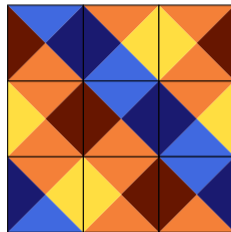
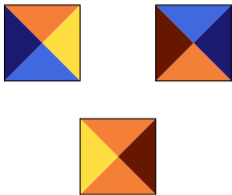
The structure of quasi-transitive graphs avoiding a minor

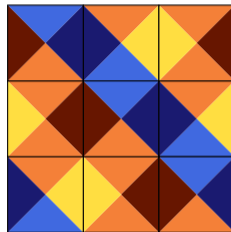
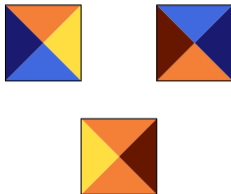
Clément Legrand-Duchesne

LaBRI, Bordeaux

February 6, 2024

Joint work with Louis Esperet and Ugo Giocanti.

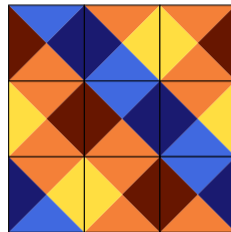
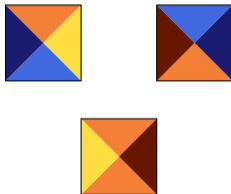




Wang's tiling problem

Entry: A finite family of tiles \mathcal{T}

Question: Does there exist a tiling of \mathbb{Z}^2 using tiles of \mathcal{T} ?



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Theorem

- Wang's tiling problem is undecidable.
- There exist aperiodic sets of tiles

Generalization of Wang's dominos

What about tiling other spaces ?

On \mathbb{Z} , \exists a tiling $\Leftrightarrow \exists$ a periodic tiling, so the problem is decidable.

Generalization of Wang's dominos

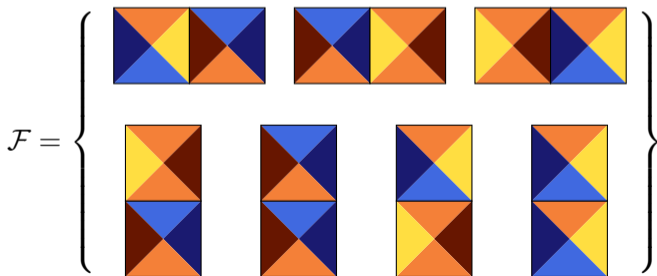
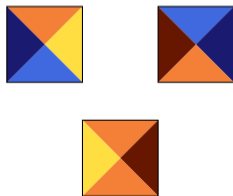
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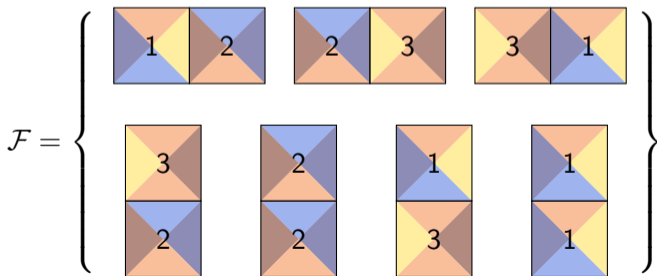
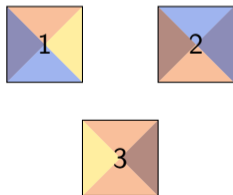
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Overview and intuition on different objects

- The domino problem
- Infinite graphs with lots of symmetries
- Tree-decompositions, treewidth and minors
- A small bit of group theory

Natural generalization of \mathbb{Z}^d : Cayley graphs

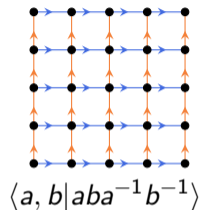
Group presentation of $\Gamma = \langle \Sigma | R \rangle$

- A finite set of generators and their inverses:
 $\Sigma = \{a, a^{-1}, b, b^{-1} \dots\}$
- A set of relations $R = \{aba^{-1}b^{-1}\}$
finitely presented if R is finite
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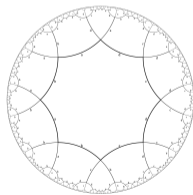
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Cayley graphs

Vertices are the elements of the group

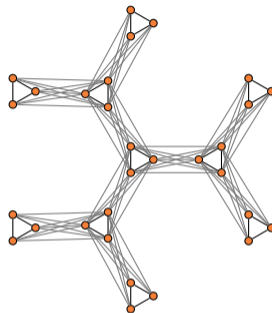
Edges are labelled by Σ



Why Cayley graphs ?

Some examples of Cayley graphs

- \mathbb{Z}^d
- The infinite d -valent trees and their blow-ups



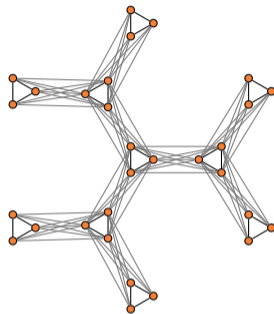
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Strong structural properties of Cayley graphs

- Regular
- **Transitive**: For all u, v , $\exists \phi \in \text{Aut}(G)$, $u = \phi(v)$
- Strong connections with expanders



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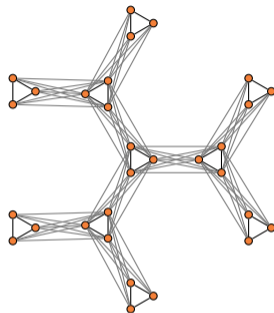
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Conjecture [Ballier and Stein 2018]

The domino problem is decidable in a group $\Gamma \Leftrightarrow \Gamma$ is virtually-free



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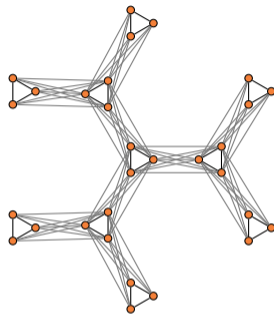
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Conjecture [Ballier and Stein 2018]

The domino problem is decidable in a group $\Gamma \Leftrightarrow \Gamma$ has a Cayley graph G of bounded treewidth



Definition

A **tree decomposition** of G is a tree T whose nodes are bags $X_i \subset V(G)$ s. t.

- $\bigcup_i X_i = V(G)$
- $\forall u \in V(G)$ the subgraph of nodes containing u is connected
- $\forall uv \in E(G), \exists X_i, \{u, v\} \subset X_i$

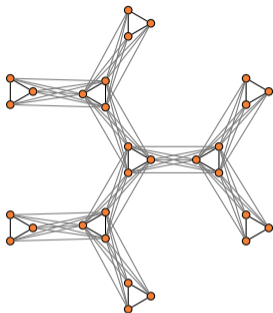
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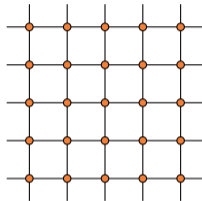
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A graph has **treewidth** at most k if it admits a tree decomposition with bags of size at most $k + 1$

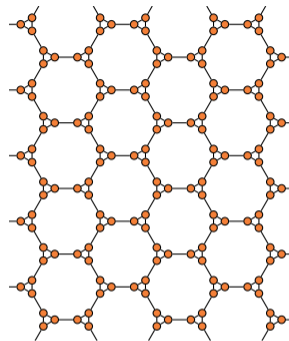
Treewidth of infinite graphs



Bounded treewidth

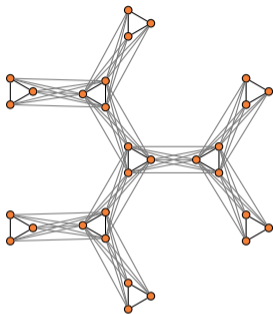


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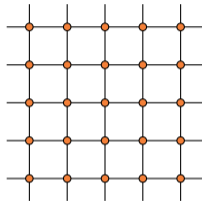


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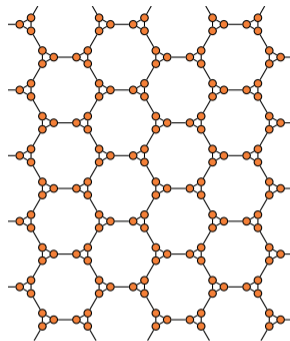
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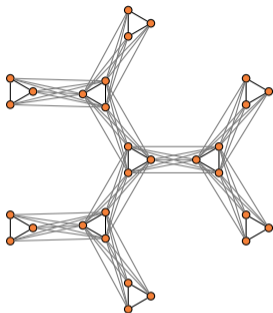


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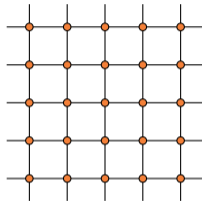
Intuition behind the conjecture

Bounded treewidth \Rightarrow tree-like structure with periodic colorings ✓

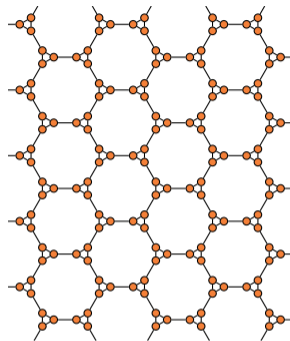
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Unbounded treewidth \Rightarrow infinite grid-like workspace

?

Definitions

Let T be a tree decomposition of G ,

Adhesion set: $X_i \cap X_j$ for some $i \neq j$

Adhesion: supremum size of an adhesion set

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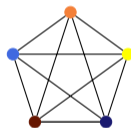
Adhesion: supremum size of an adhesion set

Torso of a bag X_i : graph $G[[X_i]]$ s. t.

- $G[X_i] \subset G[[X_i]]$
- add all edges uv s. t. u, v in an adhesion of X_i and connected by a path in $G \setminus E[X_i]$.

Definition

H **minor** of G : H can be obtained from G by contracting edges and by deleting vertices and edges.



Proposition

Having bounded treewidth is a minor closed property

Why is G H -minor free ?

Let G be a H -minor-free graph. Then G is piecewise

- too thin to contain H
- almost embeddable on surfaces too simple to contain H as a minor.

Robertson-Seymour's structure theorem on graph minors

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Robertson, Seymour 2003

Let H be a fixed graph, $\exists k$, s. t. any H -minor free graph G admits a tree-decomposition with :

- adhesion is at most k ,
- torsos are “almost” embeddable in a surface in which H does not embed (too low genus)

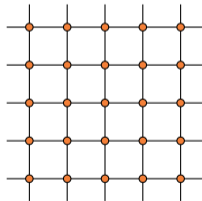
Diestel, Thomas 1999

The same holds for **locally-finite** graphs G that exclude some **finite** minor.

What about graphs with many symmetries ?

Definition

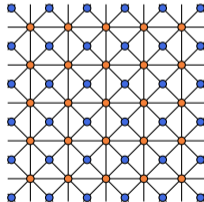
G **quasi-transitive**: \exists a t -coloring of G , s. t. $\forall u, v$ colored identically, $\exists \phi \in \text{Aut}(G)$ with $u = \phi(v)$
($V(G)$ has finitely many orbits under the action of $\text{Aut}(G)$)



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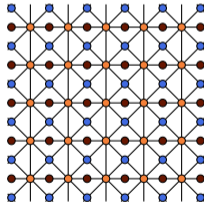
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Preserve the symmetric structure in \mathcal{T}

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Canonical tree decomposition: $\forall \phi \in \text{Aut}(G)$, ϕ maps bags on other bags
($\text{Aut}(G)$ induces an action on \mathcal{T} s. t. $\forall \phi, \forall i, \phi(X_i) = X_{i \cdot \phi}$)

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Esperet, Giocanti, L. 2023

Let G be a quasi-transitive locally finite graph G avoiding the countable clique as a minor.
Then G admits a **canonical** tree decomposition s. t.

Theorem 1 torsos are finite or planar

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Theorem 1 torsos are finite or planar

Theorem 2

- adhesion is at most 3
- torsos are minors of G
- torsos are planar or have bounded treewidth

Definition

Hadwiger number of G : supremum of the sizes of its complete minors.

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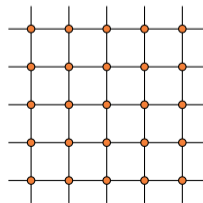
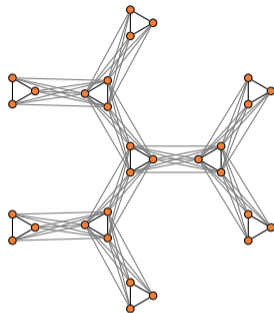
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“ K_∞ minor free $\Rightarrow K_t$ minor free for some t ”

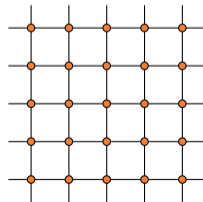
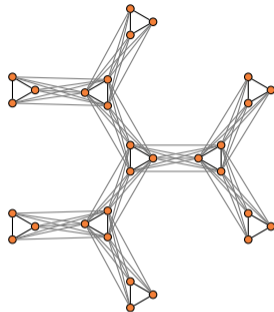
Definitions

- **Ray**: infinite one-way path in G
- Two rays r_1 and r_2 are **equivalent** if \forall finite subgraph C of G , \exists a connected component of $G \setminus C$ intersecting r_1 an infinite number of times, and r_2 too
- **End** of G : equivalence class of rays



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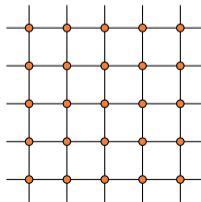
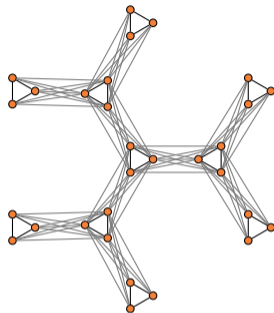


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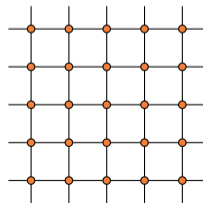
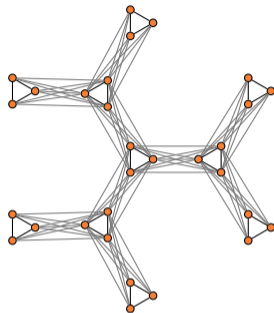
Hopf 1944 & Diestel, Jung, Möller 1993

A quasi-transitive graph has 0,1,2 or an infinite number of ends



Definitions

- A finite set C **separates** two ends if they have an infinite number of vertices in distinct components of $G \setminus C$
- A graph G is **vertex-accessible** if there is a $k < \infty$ s. t. any two ends can be separated by a set of size k .



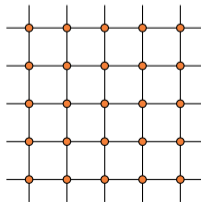
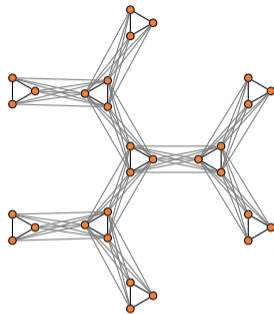
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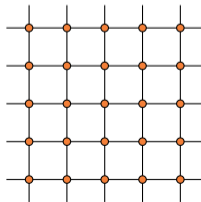
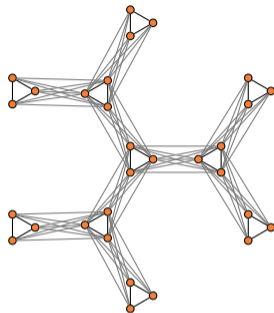
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Esperet, Giocanti, L. 2023

Quasi-transitive graphs K_∞ -minor free graphs are vertex-accessible.



Stallings 1972

Γ a finitely generated group. Γ has more than one end $\Leftrightarrow \Gamma$ can be decomposed as a product of two groups (amalgamated free-product or HNN-extension over a finite group)

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Esperet, Giocanti, L. 2023 Finitely generated K_∞ minor free groups are accessible and finitely presented

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Bungaard, Nielsen 46 & Fox 52

One-ended planar groups contain the fundamental group of a closed orientable surface as a subgroup of finite index

Domino conjecture on groups avoiding a minor

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Esperet, Giocanti, L. 2023

The domino conjecture holds in groups with no K_∞ -minor

Key ideas to take away

- Among quasi-transitive graphs, planar graphs and graphs excluding a minor are much alike
- For a quasi-transitive graph, K_∞ -minor free $\Rightarrow K_t$ minor free for some t

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Thanks!