The structure of quasi-transitive graphs avoiding a minor

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Joint work with Louis Esperet and Ugo Giocanti.
Wang’s dominos

Entry:
A finite family of tiles $T$

Question:
Does there exist a tiling of $\mathbb{Z}^2$ using tiles of $T$?

Theorem
• Wang’s tiling problem is undecidable.
• There exist aperiodic sets of tiles.
Wang’s dominoes

**Wang’s tiling problem**

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- There exist aperiodic sets of tiles
Generalization of Wang’s dominoes

What about tiling other spaces?

On $\mathbb{Z}$, $\exists$ a tiling $\iff \exists$ a periodic tiling, so the problem is decidable.
Generalization of Wang’s dominos

What about tiling other spaces?

On \( \mathbb{Z} \), \( \exists \) a tiling \( \Leftrightarrow \exists \) a periodic tiling, so the problem is decidable.

Alternative definition of the problem

\textbf{Entry:} \( k \) colors and a finite set of forbidden patterns \( \mathcal{F} \)

\textbf{Question:} Is there a coloring of \( \mathbb{Z}^2 \) that avoids \( \mathcal{F} \)?

\[ \mathcal{F} = \{ \text{patterns from images} \} \]
Generalization of Wang’s dominos

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Alternative definition of the problem

**Entry:** $k$ colors and a finite set of forbidden patterns $\mathcal{F}$

**Question:** Is there a coloring of $\mathbb{Z}^2$ that avoids $\mathcal{F}$?

\[
\mathcal{F} = \begin{Bmatrix}
\begin{array}{ccc}
1 & 2 \\
3 & 2 & 3
\end{array}, \\
\begin{array}{ccc}
3 & 2 \\
2 & 3 & 1
\end{array}, \\
\begin{array}{ccc}
2 & 3 \\
3 & 1 & 2
\end{array}
\end{Bmatrix}
\]
Goal of this talk

Overview and intuition on different objects

• The domino problem
• Infinite graphs with lots of symmetries
• Tree-decompositions, treewidth and minors
• A small bit of group theory
Natural generalization of $\mathbb{Z}^d$: Cayley graphs

Group presentation of $\Gamma = \langle \Sigma | R \rangle$

- A finite set of generators and their inverses:
  $\Sigma = \{a, a^{-1}, b, b^{-1}, \ldots\}$
- A set of relations $R = \{aba^{-1}b^{-1}\}$
  finitely presented if $R$ is finite
- The elements are the words on $\Sigma$, quotiented by patterns in $R$
Natural generalization of $\mathbb{Z}^d$: Cayley graphs

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Cayley graphs

Vertices are the elements of the group
Edges are labelled by $\Sigma$

Motivation: The domino problem
Why Cayley graphs?

Some examples of Cayley graphs

- $\mathbb{Z}^d$
- The infinite $d$-valent trees and their blow-ups
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Some examples of Cayley graphs
- \( \mathbb{Z}^d \)
- The infinite \( d \)-valent trees and their blow-ups

Strong structural properties of Cayley graphs
- Regular
- Transitive: For all \( u, v \), \( \exists \phi \in Aut(G), u = \phi(v) \)
- Strong connections with expanders
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Conjecture [Ballier and Stein 2018]
The domino problem is decidable in a group $\Gamma$ $\iff$ $\Gamma$ is virtually-free
Why Cayley graphs?

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Strong structural properties of Cayley graphs

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Conjecture [Ballier and Stein 2018]
The domino problem is decidable in a group $\Gamma \Leftrightarrow \Gamma$ has a Cayley graph $G$ of bounded treewidth
Crash course on treewidth

Definition

A tree decomposition of $G$ is a tree $T$ whose nodes are bags $X_i \subset V(G)$ s. t.

- $\bigcup_i X_i = V(G)$
- $\forall u \in V(G)$ the subgraph of nodes containing $u$ is connected
- $\forall uv \in E(G)$, $\exists X_i, \{u, v\} \subset X_i$
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A graph has treewidth at most $k$ if it admits a tree decomposition with bags of size at most $k + 1$
Treewidth of infinite graphs

Bounded treewidth

Unbounded treewidth

Unbounded treewidth

Intuition behind the conjecture

Bounded treewidth \implies tree-like structure with periodic colorings

Unbounded treewidth \implies infinite grid-like workspace
Treewidth of infinite graphs

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Bounded treewidth ⇒ tree-like structure with periodic colorings ✓
Treewidth of infinite graphs

Intuition behind the conjecture

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Unbounded treewidth ⇒ infinite grid-like workspace ?
Definitions
Let $T$ be a tree decomposition of $G$,
Adhesion set: $X_i \cap X_j$ for some $i \neq j$
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**Adhesion set:** $X_i \cap X_j$ for some $i \neq j$

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**Torso of a bag $X_i$:** graph $G[X_i]$ s. t.
- $G[X_i] \subset G[X_i]$
- add all edges $uv$ s. t. $u, v$ in an adhesion of $X_i$ and connected by a path in $G \setminus E[X_i]$. 
Definition

$H$ minor of $G$: $H$ can be obtained from $G$ by contracting edges and by deleting vertices and edges.

Proposition

Having bounded treewidth is a minor closed property.
Robertson-Seymour’s structure theorem on graph minors

Why is $G$ $H$-minor free?

Let $G$ be a $H$-minor-free graph. Then $G$ is piecewise

- too thin to contain $H$
- almost embeddable on surfaces too simple to contain $H$ as a minor.

Robertson, Seymour 2003

Diestel, Thomas 1999

The same holds for locally-finite graphs $G$ that exclude some finite minor.
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Robertson, Seymour 2003
Let $H$ be a fixed graph, $\exists k$, s. t. any $H$-minor free graph $G$ admits a tree-decomposition with :
- adhesion is at most $k$,
- torsos are “almost” embeddable in a surface in which $H$ does not embed (too low genus)

Diestel, Thomas 1999
The same holds for locally-finite graphs $G$ that exclude some finite minor.
What about graphs with many symmetries?

**Definition**

*G* quasi-transitive: ∃ a *t*-coloring of *G*, s. t. ∀ *u*, *v* colored identically, ∃φ ∈ Aut(*G*) with *u* = φ(*v*)

(*V(G)* has finitely many orbits under the action of Aut(*G*))
What about graphs with many symmetries?

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\( G \) **quasi-transitive**: \( \exists \) a \( t \)-coloring of \( G \), s. t. \( \forall u, v \) colored identically, \( \exists \phi \in Aut(G) \) with \( u = \phi(v) \)

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Definition

Canonical tree decomposition: \( \forall \phi \in Aut(G), \phi \) maps bags on other bags \((Aut(G)\) induces an action on \( T \) s. t. \( \forall \phi, \forall i, \phi(X_i) = X_{i,\phi} \)\)
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Esperet, Giocanti, L. 2023

Let \( G \) be a quasi-transitive locally finite graph \( G \) avoiding the countable clique as a minor. Then \( G \) admits a canonical tree decomposition s. t.

Theorem 1 torsos are finite or planar
Definition

Canonical tree decomposition: $\forall \phi \in Aut(G)$, $\phi$ maps bags on other bags $(Aut(G)$ induces an action on $T$ s. t. $\forall \phi, \forall i, \phi(X_i) = X_{i,\phi}$)

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Let $G$ be a quasi-transitive locally finite graph $G$ avoiding the countable clique as a minor. Then $G$ admits a canonical tree decomposition s. t.

Theorem 1  torsos are finite or planar

Theorem 2  
- adhesion is at most 3
- torsos are minors of $G$
- torsos are planar or have bounded treewidth
**Definition**

Hadwiger number of $G$: supremum of the sizes of its complete minors.
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Every locally finite quasi-transitive graph attains its Hadwiger number.

"$K_\infty$ minor free $\Rightarrow K_t$ minor free for some $t$"
Definitions

- Ray: infinite one-way path in $G$
- Two rays $r_1$ and $r_2$ are equivalent if $\forall$ finite subgraph $C$ of $G$, $\exists$ a connected component of $G \setminus C$ intersecting $r_1$ an infinite number of times, and $r_2$ too
- End of $G$: equivalence class of rays
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Hopf 1944 & Diestel, Jung, Möller 1993

A quasi-transitive graph has 0,1,2 or an infinite number of ends
Separating the ends

Definitions

- A finite set $C$ separates two ends if they have an infinite number of vertices in distinct components of $G \setminus C$.
- A graph $G$ is vertex-accessible if there is a $k < \infty$ s. t. any two ends can be separated by a set of size $k$.  

Dunwoody 2007

Planar quasi-transitive graphs are vertex-accessible.

Esperet, Giocanti, L. 2023

Quasi-transitive graphs $K_\infty$-minor free graphs are vertex-accessible.
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Decomposing and presenting groups

Stallings 1972

Γ a finitely generated group. Γ has more than one end ⇔ Γ can be decomposed as a product of two groups (amalgamated free-product or HNN-extension over a finite group)

Definition

Γ accessible: Stallings’ inductive decomposition terminates
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Esperet, Giocanti, L. 2023 Finitely generated $K_\infty$ minor free groups are accessible and finitely presented
For any \( g \geq 1 \), the fundamental group of the closed orientable surface of genus \( g \) has undecidable domino problem.
Domino conjecture on groups avoiding a minor

Aubrun, Barbieri, Moutot 2019
For any $g \geq 1$, the fundamental group of the closed orientable surface of genus $g$ has undecidable domino problem

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Esperet, Giocanti, L. 2023
The domino conjecture holds in groups with no $K_\infty$-minor
Key ideas to take away

- Among quasi-transitive graphs, planar graphs and graphs excluding a minor are much alike
- For a quasi-transitive graph, $K_\infty$-minor free $\implies K_t$ minor free for some $t$
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- Among quasi-transitive graphs, planar graphs and graphs excluding a minor are much alike
- For a quasi-transitive graph, $K_\infty$-minor free $\Rightarrow$ $K_t$ minor free for some $t$

Thanks!