



Lattice properties of acyclic pipe dreams

Propriétés de treillis des arrangements de tuyaux acycliques

Noémie Cartier

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What is a lattice?



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Partially ordered set or **poset** : a set with a partial order relation

- reflexive

$$x \leq x$$

- transitive

$$x \leq y, y \leq z \Rightarrow x \leq z$$

- antisymmetric

$$x \leq y, y \leq x \Rightarrow x = y$$

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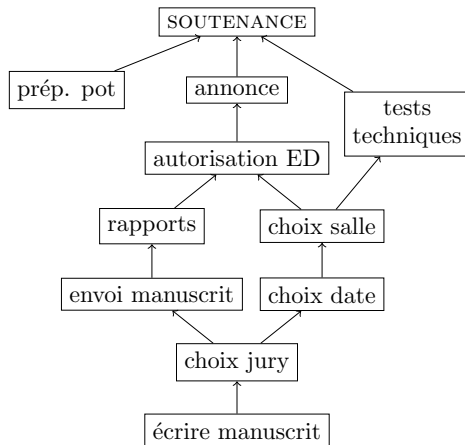
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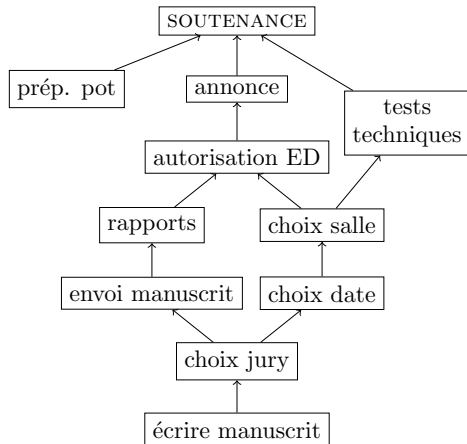
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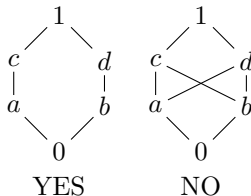
Linear extension: total order compatible with the partial order relations



What is a lattice?

A poset (X, \leq) is a **lattice** if any pair $a, b \in X$ has:

- a **join** or least upper bound $a \vee b$;
- a **meet** or greatest lower bound $a \wedge b$.



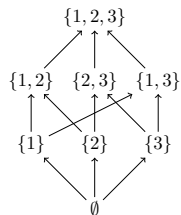
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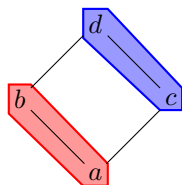
Classical examples:

- the **boolean lattice** $(\mathcal{P}(A), \subseteq)$: union and intersection;
- the **divisibility order** on positive integers: GCD and LCM.

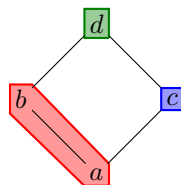


\equiv an equivalence relation on X a lattice is a **lattice congruence** if:

$$\begin{array}{l} x \equiv x' \\ y \equiv y' \end{array} \iff \begin{array}{l} x \vee y \equiv x' \vee y' \\ x \wedge y \equiv x' \wedge y' \end{array}$$



YES



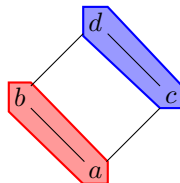
NO

$$a \vee c = c \neq d = b \vee c$$

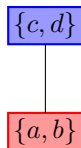


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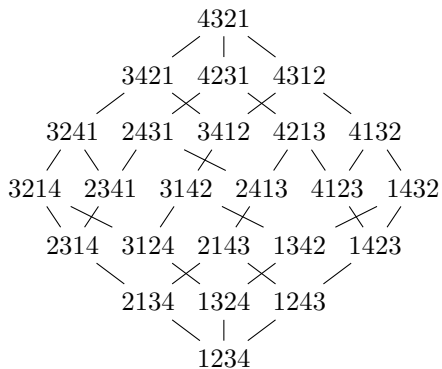
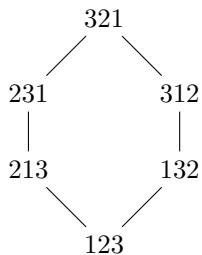
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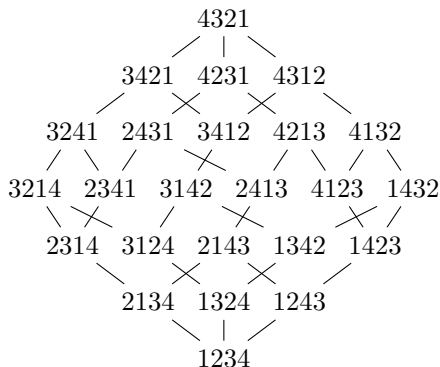
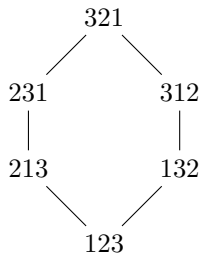
\Rightarrow **lattice quotient** X/\equiv : poset induced by \leq on the equivalence classes of \equiv



Weak order on permutations :



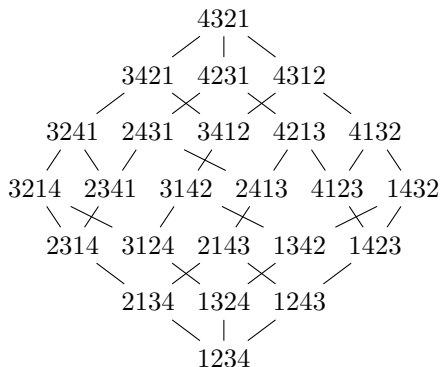
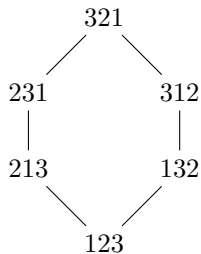
Weak order on permutations :



Defined by inclusion on **inversion sets**:

$$\text{inv}(\omega) := \{i < j \text{ and } \omega^{-1}(i) > \omega^{-1}(j)\} \quad \rightarrow (1, 2) \in \text{inv}(24135)$$

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The weak order on \mathfrak{S}_n is a **lattice** (Guilbaud–Rosenstiehl, '63).

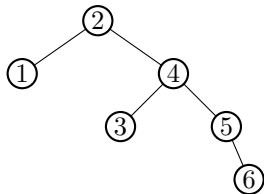


A lattice quotient of the weak order: the **Tamari lattice** (Tamari, '62)



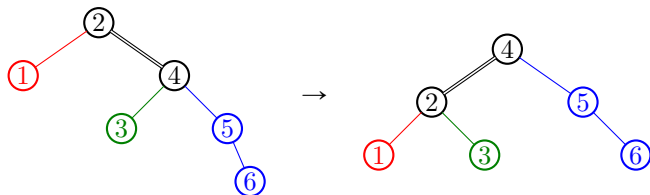
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Data structure: **binary search tree**



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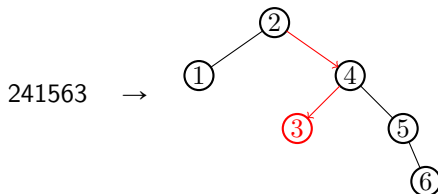
Data structure: **binary search tree**



Balancing operation: **tree rotation** (Adelson-Velsky–Landis, '62)

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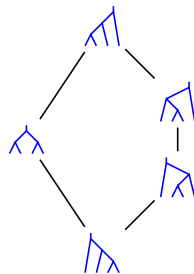
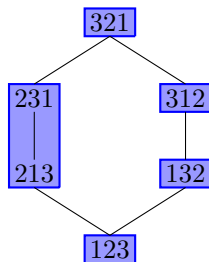
Data structure: **binary search tree**



From permutations to binary trees: **insertion into a BST**

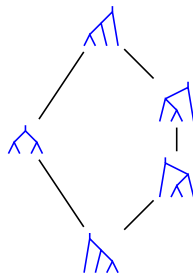
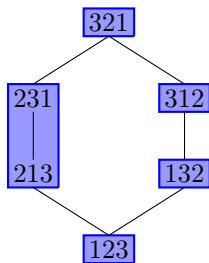


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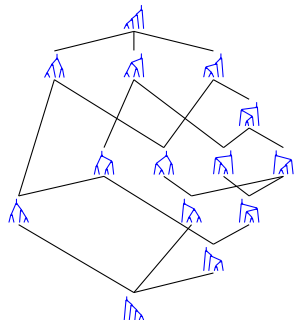
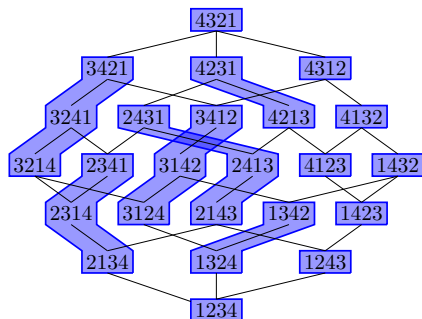
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The insertion into BST is a **lattice morphism** (Hivert–Novelli–Thibon, '05).



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Cover relations of the weak order:

$$UabV \triangleleft UbaV$$

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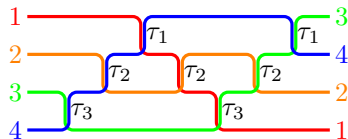
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Sorting network \leftrightarrow simple reflections product



Properties of words on S :

- minimal length for ω : $\ell(\omega) = |\text{inv}(\omega)|$ (**reduced** words)

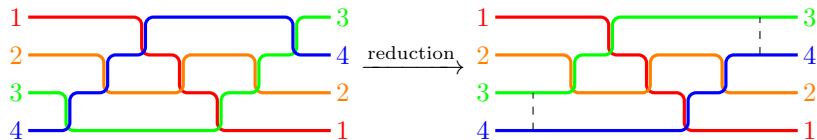
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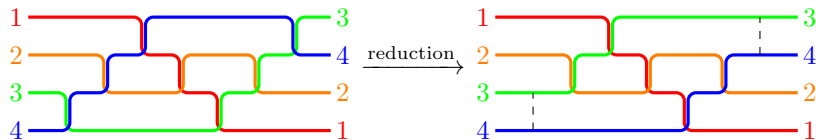
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- $\pi \leq \omega$ iff $\omega = \pi\sigma$ and $\ell(\omega) = \ell(\pi) + \ell(\sigma)$: π is a **prefix** of ω
- if $\pi \leq \omega$ then any reduced expression of ω has a reduced expression of π as a **subword**



$SC(Q, \omega)$ the **subword complex** on Q representing ω :

- ground set: indices of Q
- facets: complements of reduced subwords representing ω

(Knutson–Miller, '04)

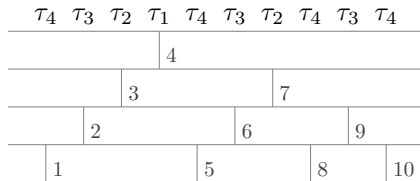


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An example:



Facet $\{1, 2, 3, 8, 9\}$ of $SC(\tau_4\tau_3\tau_2\tau_1\tau_4\tau_3\tau_2\tau_4\tau_3\tau_4, 25143)$

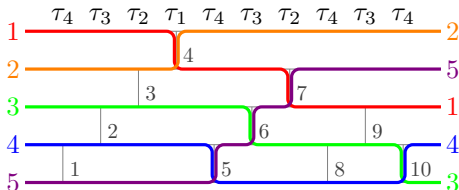


Fix Q word on S , $\omega \in \mathfrak{S}_n$

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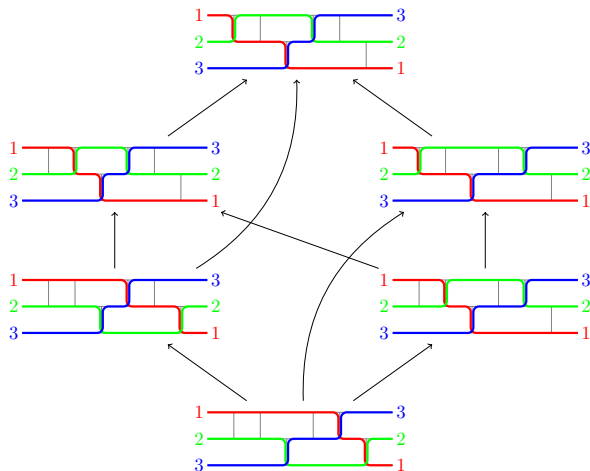
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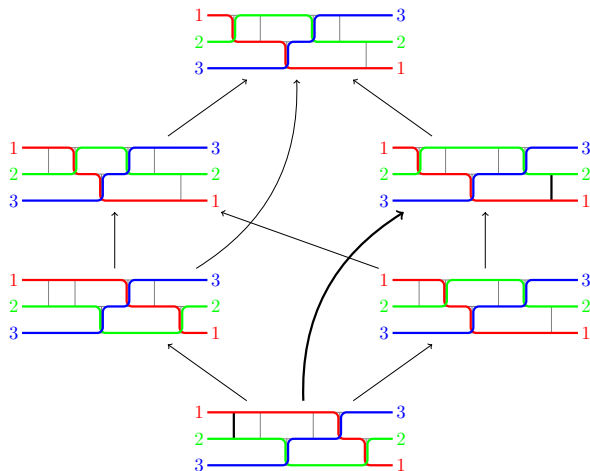


Structure given by **flips**: from one facet to another





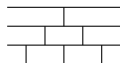
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A very special case

Q : triangular word

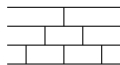


and $\omega = 1 n (n - 1) \dots 2$

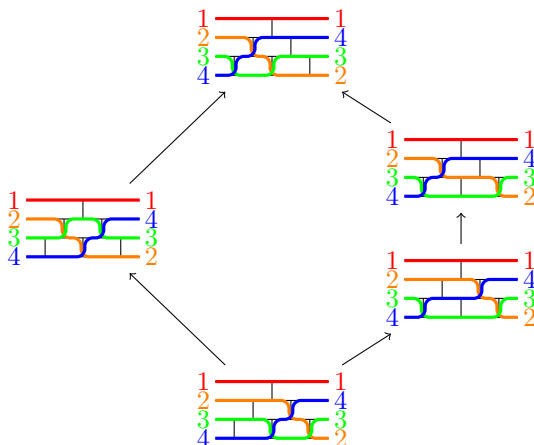


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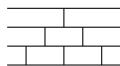
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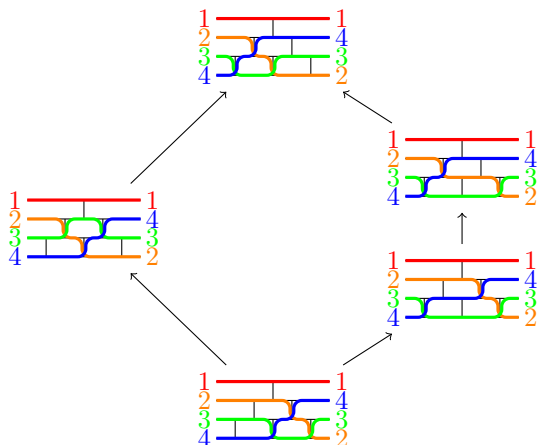


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⇒ this is the Tamari lattice!

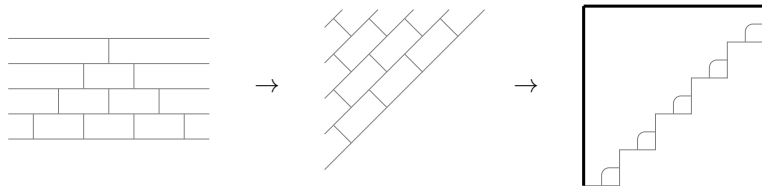


Why the Tamari lattice?



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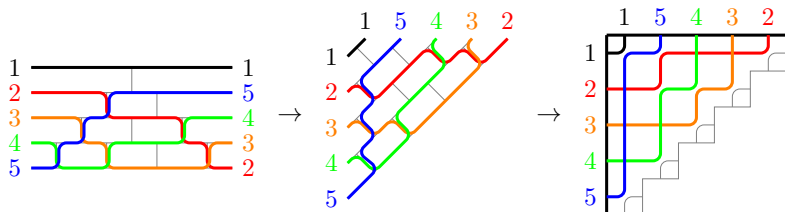
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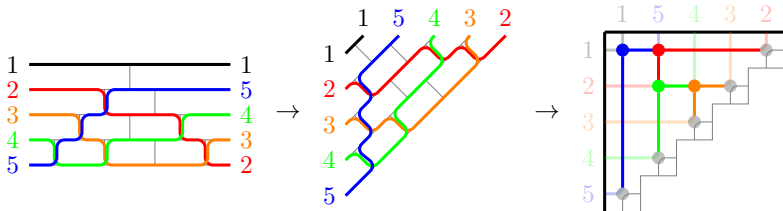
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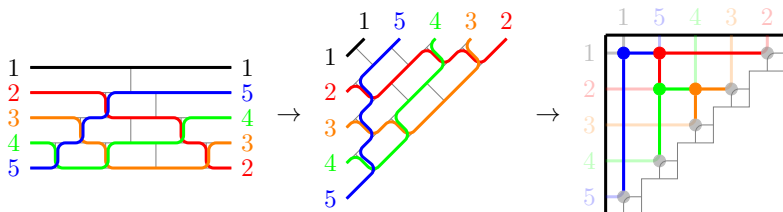
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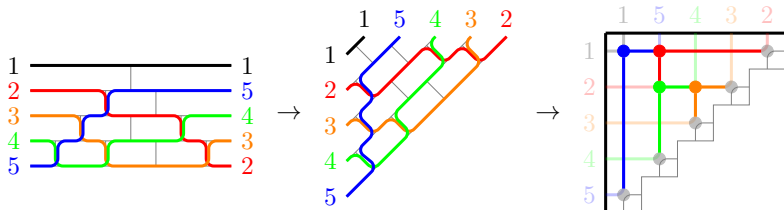
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A binary tree appears on the pipe dream \rightarrow bijection



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Tree rotations \equiv flips \rightarrow lattice isomorphism (Woo, 2004)

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Can we find other lattice quotients of parts of the weak order
 with pipe dreams?

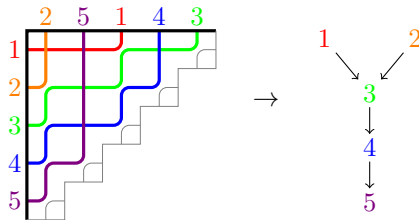


First extension: choose any exit permutation ω .

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Contact graph:

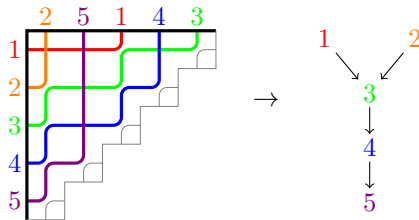
- vertices: pipes
- edges: from a to b if $a \curvearrowright b$ appears in the picture



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Acyclic contact graph \iff vertex of **brick polytope** (Pilaud–Santos, '12)

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What are the linear extensions of acyclic contact graphs?

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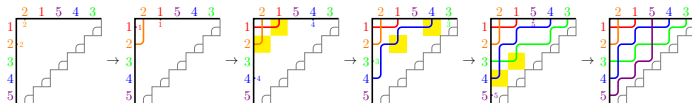
Theorem (Bergeron–C.–Ceballos–Pilaud)

For any $\omega \in \mathfrak{S}_n$, the ascending flip graph on $\Sigma(\omega)$ is a **lattice quotient** of the weak order interval $[\text{id}, \omega]$.

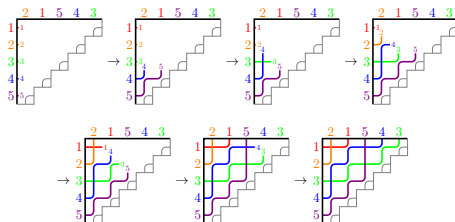
The map $\text{Ins}_\omega : [\text{id}, \omega] \mapsto \Sigma(\omega)$ is a **lattice morphism**.

Two algorithms to compute $\text{Ins}_\omega(\pi)$: (for $\omega = 21543$ and $\pi = 21435$)

■ insertion algorithm (pipe by pipe)



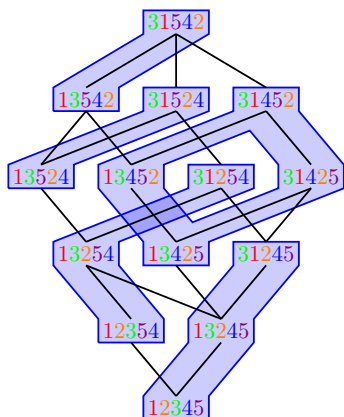
■ sweeping algorithm (cell by cell)



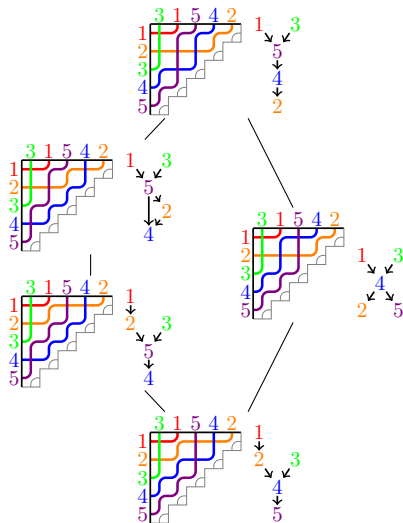


Triangular pipe dreams

An example: $\omega = 31542$



Ins_ω





Second extension: other sorting networks

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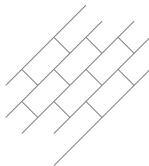
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Generalized pipe dreams

Second extension: other sorting networks

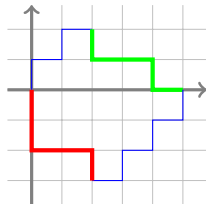
alternating sorting networks



↔

alternating shapes

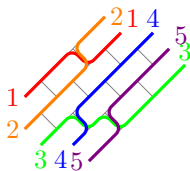
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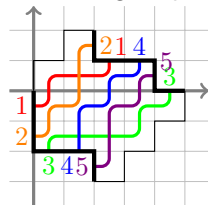


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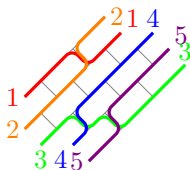


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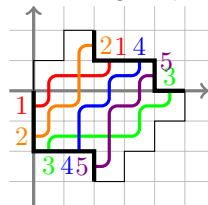
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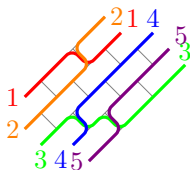


$\text{Ins}_{F,\omega}$ is still well defined on $[\text{id}, \omega]$, BUT...

- some linear extensions can be outside of $[\text{id}, \omega]$
- the flip graph is not always the image of the weak order

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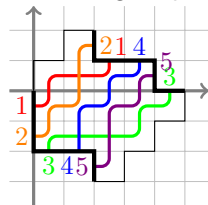
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alternating shapes

↔



$\text{Ins}_{F,\omega}$ is still well defined on $[\text{id}, \omega]$, BUT...

- some linear extensions can be outside of $[\text{id}, \omega]$
- the flip graph is not always the image of the weak order

Restrictions:

- $\Sigma_F(\omega)$ contains **strongly acyclic** pipe dreams
- order on $\Sigma_F(\omega)$: **acyclic order** (weaker than flip order)



Theorem (C.)

For any alternating shape F and $\omega \in \mathfrak{S}_n$ sortable on F , the acyclic order on $\Sigma_F(\omega)$ is a **lattice quotient** of the weak order interval $[\text{id}, \omega]$.

The map $\text{Ins}_{F,\omega} : [\text{id}, \omega] \mapsto \Sigma_F(\omega)$ is a **lattice morphism**.



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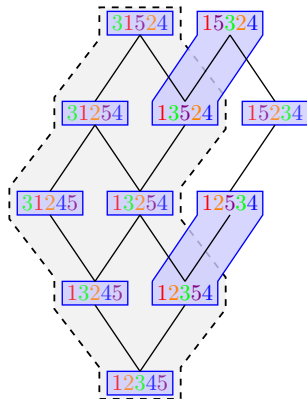
Theorem (C.)

If the maximal permutation $\omega_0 = n(n-1)\dots 21$ is sortable on F , then any linear extension of a pipe dream on F with exit permutation ω is in $[\text{id}, \omega]$, and **all acyclic pipe dreams are strongly acyclic**.

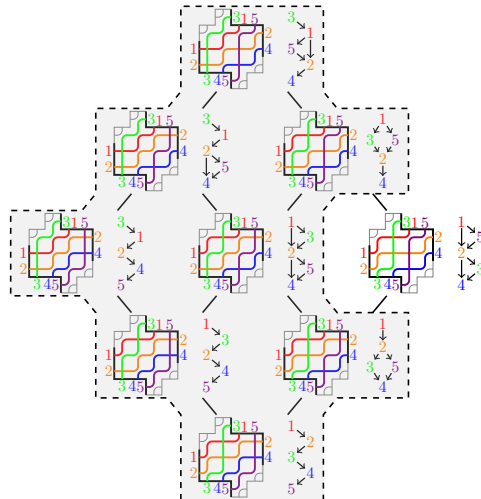


Generalized pipe dreams

An example: $\omega = 31524$



$\text{Ins}_{F, \omega}$





Further generalization: Coxeter groups

symmetric group \mathfrak{S}_n	Coxeter group W
simple transpositions	simple reflections
weak order	
subword complexes	
pair of pipes	root in Φ
$P^\#$	root cone
$\pi \in \text{lin}(P)$	root conf. $\subseteq \pi(\Phi^+)$



Theorem (BCCP)

*For any word Q on S and $w \in W$ sortable on Q , the map $\text{Ins}_{Q,w}$ is **well-defined** on the weak order interval $[e, w]$.*



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Conjecture

If Q is an alternating word on S and $w \in W$ is sortable on Q , then the application $\text{Ins}_{Q,w} : [e, w] \mapsto \text{SC}(Q, w)$ is a **lattice morphism** from the weak order on $[e, w]$ to the Brick polyhedron of $\text{SC}(Q, w)$.



Thank you for your attention!

Merci pour votre attention !