

# Baxter $d$ -permutations and other pattern avoiding classes

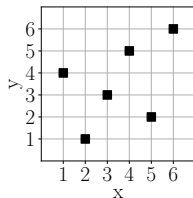
*Nicolas Bonichon* and Pierre-Jean Morel

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# Permutations and diagrams

A **permutation**  $\sigma = \sigma(1), \dots, \sigma(n)$  is a bijection from  $[n] := \{1, 2, \dots, n\}$  to itself.  $P_\sigma := \{(i, \sigma(i)), i \in [n]\}$  the **diagram of  $\sigma$** .

$$\sigma = 413526$$

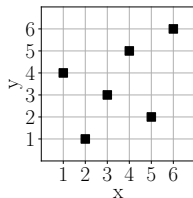


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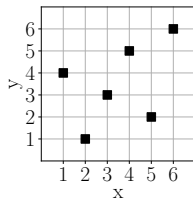
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A **diagram** (of size  $n$ ) is a point set on  $[n] \times [n]$  with exactly **1 point per row** and **1 point per column**.

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A **diagram** (of size  $n$ ) is a point set on  $[n] \times [n]$  with exactly **1 point per row and 1 point per column**.

$S_n$  the set of permutations of size  $n$ .

$$|S_n| = n!$$

# The Starting Question

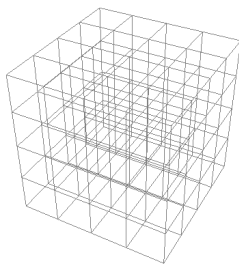
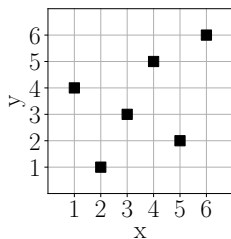
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- Permutations  $\Leftrightarrow$  Diagrams
- Diagrams are 2D objects
- What could be "3D" diagrams?
- $\Rightarrow$  "3D" Permutations?



## $d$ -Diagrams and $d$ -Permutations

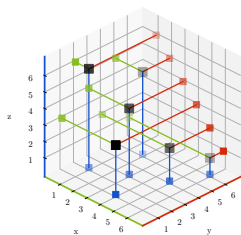
A **3-diagram** of size  $n$  is a point set on  $[n]^3$  such that

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## $d$ -Diagrams and $d$ -Permutations

A **3-diagram** of size  $n$  is a point set on  $[n]^3$  such that each plane of the grid (orthogonal to  $x$ ,  $y$  or  $z$ ) contains exactly 1 point.



Points:

(1,2,6)

(2,5,5)

(3,3,4)

(4,1,3)

(5,4,2)

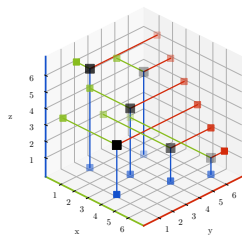
(6,6,1)

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A  **$d$ -diagram** is a point set of size  $n$  on  $[n]^d$  such that each **hyperplane** orthogonal  $x_i = j$  with  $i \in [d]$  and  $j \in [n]$  contains exactly 1 point.

# $d$ -Diagrams and $d$ -Permutations

A **3-permutation**  $\sigma := (\sigma_y, \sigma_z)$  is a couple of permutations.  
 $S_n^2 := \{3\text{-permutations of size } n\}$ .  $|S_n^2| = n!^2$ .



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(3,3,4)

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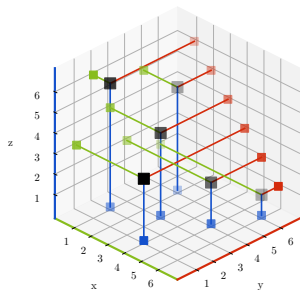
(6,6,1)

(253146, 654321)

---

A  $d$ -permutation of size  $n$ ,  $\sigma := (\sigma_1, \dots, \sigma_{d-1})$  is a sequence of  $d - 1$  permutations of size  $n$ .  $S_n^{d-1} := \{d\text{-permutations of size } n\}$ .  $|S_n^{d-1}| = n!^{d-1}$ .

# Projections



$$\text{proj}_{xy}(\sigma) = \sigma_y$$

$$\text{proj}_{xz}(\sigma) = \sigma_z$$

$$\text{proj}_{yz}(\sigma) = \sigma_z \sigma_y^{-1}$$

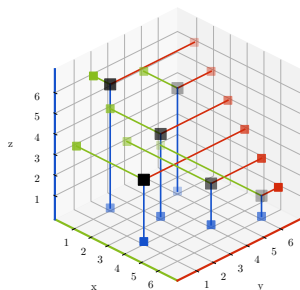
$$\text{proj}_{yz}((253146, 654321)) = 364251$$

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Let  $\bar{\sigma} := (\text{Id}_n, \sigma_1, \dots, \sigma_{d-1})$ . , the **projection** on  $i$  of  $d$ -permutation  $\sigma$  is the  $d'$ -permutation  $\text{proj}_i(\sigma) := \bar{\sigma}_{i_2} \bar{\sigma}_{i_1}^{-1}, \bar{\sigma}_{i_3} \bar{\sigma}_{i_1}^{-1}, \dots, \bar{\sigma}_{i_{d'}} \bar{\sigma}_{i_1}^{-1}$ .  $d'$  is the **dimension** of the projection.

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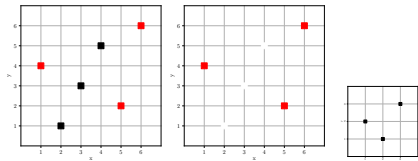
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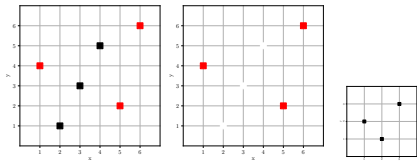


$\sigma = 413526$  contains the pattern  $\pi = 213$ .

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A permutation  $\sigma$  **contains** a permutation (or a **pattern**)  $\pi = \pi(1), \dots, \pi(k) \in S_k$  if there exist indices  $c_1 < \dots < c_k$  such that  $\sigma(c_1) \dots \sigma(c_k)$  is order-isomorphic to  $\pi$ . The set of points of indices  $c_1, \dots, c_k$  is an **occurrence** of the  $\pi$ .

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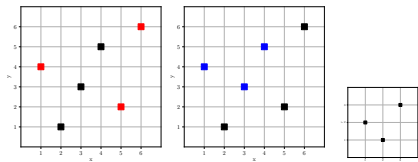


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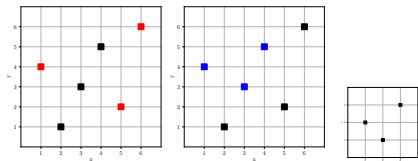


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$S_n(\pi)$  := the set of permutations that avoids  $\pi$ .

$S_n(\pi_1, \dots, \pi_k)$  := ... avoids **all**  $\pi_1, \dots, \pi_k$ .

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## Pattern avoidance classes

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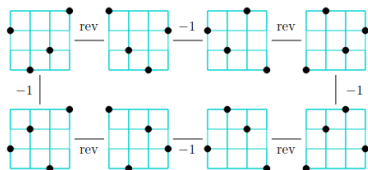
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$\pi$  and  $\tau$  are **trivially Wilf-equivalent** if there is a symmetry  $s$  of the square such that  $\forall \sigma, \sigma \in S(\pi)$  iff  $s(\sigma) \in S(s(\tau))$ .



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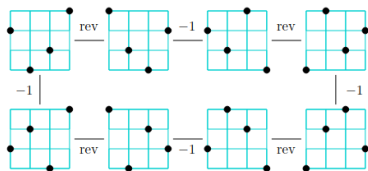
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312, 231, 132 and 213 are trivially Wilf-equivalent.

123 and 321 are trivially Wilf-equivalent.

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TWE: Number of trivially Wilf-Equivalent patterns.

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123	2	$\frac{1}{n+1} \binom{2n}{n} = 1, 2, 5, 14, 42, 132, \dots$	[Knuth 73]
123, 321	1	1, 2, 4, 4, 0, 0, 0, ...	[Simion 85]
213, 321	4	$1 + \frac{n(n-1)}{2} = 1, 2, 4, 7, 11, 16, 22, \dots$	[Simion 85]
312, 231	2	$2^{n-1} = 1, 2, 4, 8, 16, 32, 64, \dots$	[Simion 85]
231, 132	4	$2^{n-1} = 1, 2, 4, 8, 16, 32, 64, \dots$	[Simion 85]
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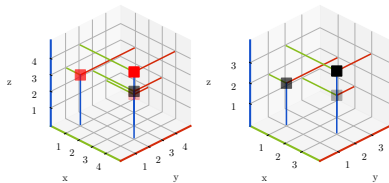
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312, 321	4	$2^{n-1} = 1, 2, 4, 8, 16, 32, 64, \dots$	[Simion 85]
213, 132, 123	2	Fibonacci: 1, 2, 3, 5, 8, 13, 21, ...	[Simion 85]
231, 213, 321	8	$n = 1, 2, 3, 4, 5, 6, 7, \dots$	[Simion 85]
312, 132, 213	4	$n = 1, 2, 3, 4, 5, 6, 7, \dots$	[Simion 85]
312, 321, 123	4	1, 2, 3, 1, 0, 0, 0, ...	
321, 213, 123	4	1, 2, 3, 1, 0, 0, 0, ...	
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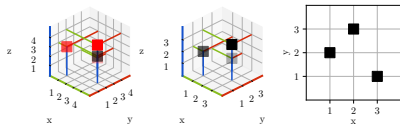
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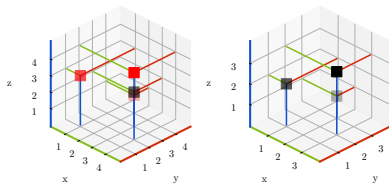


$(1432, 3124)$  contains the pattern  $(132, 213)$  and the pattern 231.

$\sigma$  **contains** a (2D) pattern  $\pi_1$  if  $\text{proj}_{xy}$ ,  $\text{proj}_{xz}$  or  $\text{proj}_{yz}$  contains  $\pi_1$ .

## patterns and $d$ -permutations

Let  $\sigma \in S_n^{d-1}$  and  $\pi \in S_k^{d'-1}$  with  $k \leq n$ . Then  $\sigma$  **contains the pattern**  $\pi$ , if there exist a **direct** projection  $\sigma' = \text{proj}_i(\sigma)$  of dimension  $d'$  that contains  $\pi$ .



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---

$i := i_1, \dots, i_{d'} \in [d]^{d'}$ , the projection  $\text{proj}_i$  is **direct** if  $i_1 < i_2 < \dots < i_{d'}$ .  
 $\text{proj}_{xy}$  is direct.  $\text{proj}_{yx}$  is **indirect**.



## 3-pattern avoidance classes

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$(12, 12), (12, 21),$ $(21, 12)$	4	$1, 1, 1, 1, 1, 1, \dots$	1 diagonal
$(12, 12), (12, 21),$ $(21, 12), (21, 21)$	1	$1, 0, 0, 0, 0, 0, \dots$	

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$(12, 12), (12, 21),$ $(21, 12)$	4	1, 1, 1, 1, 1, 1, $\dots$	1 diagonal
$(12, 12), (12, 21),$ $(21, 12), (21, 21)$	1	1, 0, 0, 0, 0, 0, $\dots$	
$(123, 123)$	4	1, 4, 35, 524, 11774, 366352, $\dots$	<i>new</i>
$(123, 132)$	24	1, 4, 35, 524, 11768, 365558, $\dots$	<i>new</i>
$(132, 213)$	8	1, 4, 35, 524, 11759, 364372, $\dots$	<i>new</i>

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$(132, 213)$	8	$1, 4, 35, 524, 11759, 364372, \dots$	<i>new</i>
$(12, 12), (132, 312)$	48	$(n+1)^{n-1} = 1, 3, 16, 125, 1296 \dots$	[Atkinson et al. 93,95]
$(12, 12), (123, 321)$	12	$1, 3, 16, 124, 1262, 15898, \dots$	distributive lattices inter.
$(12, 12), (231, 312)$	8	$1, 3, 16, 122, 1188, 13844, \dots$	A295928?

## 3-pattern avoidance classes

[A295928](#) Number of triangular matrices  $T(n,i,k)$ ,  $k \leq i \leq n$ , with entries "0" or "1" with the property that each triple  $\{T(n,i,k), T(n,i,k+1), T(n,i-1,k)\}$  containing a single "0" can be successively replaced by  $\{1, 1, 1\}$  until finally no "0" entry remains.

1, 3, 16, 122, 1188, 13844, 185448, 2781348, 45868268 ([list](#); [graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

OFFSET 1,2

COMMENTS A triple  $\{T(n,i,k), T(n,i,k+1), T(n,i-1,k)\}$  will be called a primitive triangle. It is easy to see that  $b(n) = n(n-1)/2$  is the number of such triangles. At each step, exactly one primitive triangle is completed (replaced by  $\{1, 1, 1\}$ ). So there are  $b(n)$  "0"- and  $n$  "1"-terms. Thus the starting matrix has no complete primitive triangle. Furthermore, any triangular submatrix  $T(m,i,k)$ ,  $k \leq i \leq m < n$  cannot have more than  $m$  "1"-terms because otherwise it would have less "0"-terms than primitive triangles. The replacement of at least one "0"-term would complete more than one primitive triangle. This has been excluded.

So  $T(n, i, k)$  is a special case of  $U(n, i, k)$ , described in [A101481](#):  $a(n) < A101481(n+1)$ .

A start matrix may serve as a pattern for a number wall used on worksheets for elementary mathematics, see link "Number walls". That is why I prefer the more descriptive name "fill matrix".

The algorithm for the sequence is rather slow because each start matrix is constructed separately. There exists a faster recursive algorithm which produces the same terms and therefore is likely to be correct, but it is based on a conjecture. For the theory of the recurrence, see "Recursive aspects of fill matrices". Probable extension  $a(10)$ - $a(14)$ : 821096828, 15804092592, 324709899276, 7081361097108, 163179784397820.

The number of fill matrices with  $n$  rows and all "1"- terms concentrated on the last two rows, is [A001960\( \$n\$ \).](#)

See link "Reconstruction of a sequence".

LINKS

[Table of  \$n, a\(n\)\$  for  \$n=1..9\$ .](#)

Gerhard Kirchner, [Recursive aspects of fill matrices](#)

Gerhard Kirchner, [Number walls](#)

Gerhard Kirchner, [VB-program](#)

Gerhard Kirchner, [Reconstruction of a sequence](#)

Ville Salo, [Cutting Corners](#), arXiv:2002.08730 [math.DS], 2020.

## 2-pattern avoidance classes

Patterns	TWE	Sequence	Comment
12	1	1, 0, 0, 0, 0, $\dots$	unavoidable pattern

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12	1	1, 0, 0, 0, 0, $\dots$	unavoidable pattern
21	1	1, 1, 1, 1, 1, $\dots$	1 diagonal



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12	1	1, 0, 0, 0, 0, ...	unavoidable pattern
21	1	1, 1, 1, 1, 1, ...	1 diagonal
123	1	1, 4, 20, 100, 410, 1224, 2232, ...	<i>new</i>
132	2	1, 4, 21, 116, 646, 3596, 19981, ...	<i>new</i>
231	2	1, 4, 21, 123, 767, 4994, 33584, ...	<i>new</i>
321	1	1, 4, 21, 128, 850, 5956, 43235, ...	<i>new</i>

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321	1	1, 4, 21, 128, 850, 5956, 43235, ...	<i>new</i>
123, 132	2	1, 4, 8, 8, 0, 0, 0, ...	
123, 231	2	1, 4, 9, 6, 0, 0, 0, ...	
123, 321	1	1, 4, 8, 0, 0, 0, 0, ...	
132, 213	1	1, 4, 12, 28, 58, 114, 220, ...	[Sun22+]
132, 231	4	1, 4, 12, 32, 80, 192, 448, ...	A001787[Sun22+]
132, 321	2	1, 4, 12, 27, 51, 86, 134, ...	A047732[Sun22+]
231, 312	1	1, 4, 10, 28, 76, 208, 568, ...	A026150[Sun22+]
231, 321	2	1, 4, 12, 36, 108, 324, 972, ...	A003946[Sun22+]

## 2-pattern avoidance classes

A001787	$a(n) = n \cdot 2^{n-1}$ . (Formerly M3444 N1398)	407
	0, 1, 4, 12, 32, 80, 192, 448, 1024, 2304, 5120, 11264, 24576, 53248, 114688, 245760, 524288, 1114112, 2359296, 4980736, 10485760, 22020096, 46137344, 96468992, 201326592, 419430400, 872415232, 1811939328, 3758096384, 7784628224, 16106127360, 33285996544 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> ; <a href="#">listen</a> ;	
A047732	First differences are <a href="#">A005563</a> .	1
	1, 4, 12, 27, 51, 86, 134, 197, 277, 376, 496, 639, 807, 1002, 1226, 1481, 1769, 2092, 2452, 2851, 3291, 3774, 4302, 4877, 5501, 6176, 6904, 7687, 8527, 9426, 10386, 11409, 12497, 13652, 14876, 16171, 17539, 18982, 20502, 22101, 23781, 25544, 27392, 29327 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> ; <a href="#">listen</a> ; <a href="#">history</a> ; <a href="#">text</a> ; <a href="#">internal format</a> )	
OFFSET	0, 2	
COMMENTS	Number of 3-permutations of $n$ elements avoiding the patterns 132, 321. See Bonichon and Sun. - <a href="#">Michel Marcus</a> , Aug 19 2022	
LINKS	Vincenzo Librandi, <a href="#">Table of <math>n</math>, <math>a(n)</math> for <math>n = 0..1000</math></a> Nicolas Bonichon and Pierre-Jean Morel, <a href="#">Baxter <math>d</math>-permutations and other pattern avoiding classes</a> , arXiv:2202.12677 [math.CO], 2022. Nathan Sun, <a href="#">On <math>d</math>-permutations and Pattern Avoidance Classes</a> , arXiv:2208.08506 [math.CO], 2022. <a href="#">Index entries for linear recurrences with constant coefficients</a> , signature (4, -6, 4, -1).	
FORMULA	$a(n) = A051925(n+1) + 1$ . - <a href="#">Alex Ratushnyak</a> , Jun 27 2012 From <a href="#">Vincenzo Librandi</a> , Jun 28 2012: (Start) G.f.: $(1 + 2x^2 - x^3)/(1-x)^4$ . $a(n) = 4a(n-1) - 6a(n-2) + 4a(n-3) - a(n-4)$ .	
A026150	$a(0) = a(1) = 1$ ; $a(n+2) = 2a(n+1) + 2a(n)$ .	53
	1, 1, 4, 10, 28, 76, 208, 568, 1552, 4240, 11584, 31648, 86464, 236224, 645376, 1763200, 4817152, 13160704, 35955712, 98232832, 268377088, 733219840, 2003193856, 5472827392, 14952042496, 40849739776 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> ; <a href="#">listen</a> ; <a href="#">history</a> ; <a href="#">text</a> ; <a href="#">internal format</a> )	
A003946	Expansion of $(1+x)/(1-3x)$ .	127
	1, 4, 12, 36, 108, 324, 972, 2916, 8748, 26244, 78732, 236196, 708588, 2125764, 6377292, 19131876, 57395628, 172186884, 516560652, 1549681956, 4649045868, 13947137604, 41841412812, 125524238436, 376572715308, 1129718145924 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> ; <a href="#">listen</a> ; <a href="#">history</a> ; <a href="#">text</a> ; <a href="#">internal format</a> )	

## 2- and 3-patterns avoidance classes

Patterns	TWE	Sequence	Comment
12, (12, 12)	1	1, 0, 0, 0, 0, ...	12
12, (21, 12)	3	1, 0, 0, 0, 0, ...	12
21, (12, 12)	1	1, 0, 0, 0, 0, ...	
21, (21, 12)	3	1, 1, 1, 1, 1, ...	21
123, (12, 12)	1	1, 3, 14, 70, 288, 822, 1260, ...	<i>new</i>
123, (12, 21)	3	1, 3, 6, 6, 0, 0, 0, ...	
132, (12, 12)	2	1, 3, 11, 41, 153, 573, 2157, ...	A281593?
132, (12, 21)	6	1, 3, 11, 43, 173, 707, 2917, ...	A026671?
231, (12, 12)	2	1, 3, 9, 26, 72, 192, 496, ...	A072863?
231, (12, 21)	4	1, 3, 11, 44, 186, 818, 3706, ...	<i>new</i>
231, (21, 12)	2	1, 3, 12, 55, 273, 1428, 7752, ...	A001764?
321, (12, 12)	1	1, 3, 2, 0, 0, 0, 0, ...	
321, (12, 21)	3	1, 3, 11, 47, 221, 1113, 5903, ...	A217216?

## 2- and 3-patterns avoidance classes

A281593	$a(n) = b(n) - \sum_{j=0..n-1} b(j)$ with $b(n) = \text{binomial}(2*n, n)$ .	1
	1, 1, 3, 11, 41, 153, 573, 2157, 8163, 31043, 118559, 454479, 1747771, 6740059, 26055459, 100939779, 391785129, 1523230569, 5931153429, 23126146629, 90282147849, 352846964649, 1380430179489, 5405662979649, 21186405207549, 83101804279101, 326199124351701 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> );	
A026671	Number of lattice paths from (0,0) to (n,n) with steps (0,1), (1,0) and, when on the diagonal, (1,1).	21
	1, 3, 11, 43, 173, 707, 2917, 12111, 50503, 211263, 885831, 3720995, 15652239, 65913927, 277822147, 1171853635, 4945846997, 20884526283, 88224662549, 372827899079, 1576001732485, 6663706588179, 28181895551161, 119208323665543, 504329070986033, 2133944799315027 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> );	
A072863	$a(n) = 2^{n-3} * (n^2 + 3*n + 8)$ .	7
	1, 3, 9, 26, 72, 192, 496, 1248, 3072, 7424, 17664, 41472, 96256, 221184, 503808, 1138688, 2555904, 5701632, 12648448, 27918336, 61341696, 134217728, 292552704, 635437056, 1375731712, 2969567232, 6392119296, 13723762688 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> ; <a href="#">listen</a> ; <a href="#">history</a> ; <a href="#">text</a> ; <a href="#">internal format</a> );	
A001764	$a(n) = \text{binomial}(3*n, n) / (2*n + 1)$ (enumerates ternary trees and also noncrossing trees). (Formerly M2926 N1174)	448
	1, 1, 3, 12, 55, 273, 1428, 7752, 43263, 246675, 1430715, 8414640, 50067108, 300830572, 1822766520, 11124755664, 68328754959, 422030545335, 2619631042665, 16332922290300, 102240109897695, 642312451217745, 4048514844039120, 25594403741131680, 162250238001816900 ( <a href="#">list</a> );	
A217216	Dimension of algebraic generators of the algebra "Baxter" of order n.	0
	0, 1, 1, 3, 11, 47, 221, 1113, 5903, 32607, 186143, 1092015, 6555515, 40137219, 249984481, 1580468321, 10125395007, 65639436955, 430048061915, 2844592155631, 18979693010495, 127641472658231, 864645413540671, 5896221199266519, 40455246946190079 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> ; <a href="#">listen</a> );	

# Separable permutations $Sep_n$

**direct sum**  $\sigma \oplus \pi$  : add  $\pi$  in the top right corner of  $\sigma$ .

$$132 \oplus 21 = 13254$$



**skew sum**  $\sigma \ominus \pi$  : add  $\pi$  in the bottom right corner of  $\sigma$ .

$$132 \ominus 21 = 35421$$



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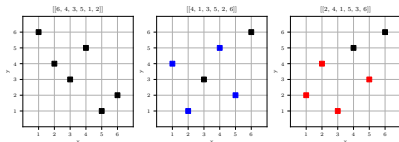


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**separable**: size 1 or a direct/skew sum separable permutations.



$$643512 = 1 \ominus ((1 \ominus 1) \oplus 1) \ominus (1 \oplus 1).$$

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## Separable permutations $Sep_n$

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[Brightwell 92]:

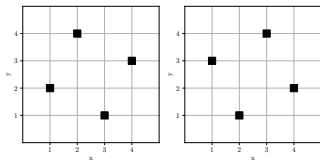
$$|Sep_n| = \frac{1}{n-1} \sum_{k=0}^{n-2} \binom{n-1}{k} \binom{n-1}{k+1} 2^{n-k-1}.$$

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$Sep_n = S_n(2413, 3142)$  [Bose Buss Lubiw 98]

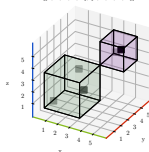


# Separable $d$ -permutations [Atkinson Mansour 10]

**$d$ -sum:**

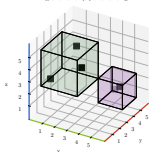
$$\pi \oplus^{+++} \sigma$$

[[1, 3, 2, 5, 4], [1, 3, 2, 4, 5]]



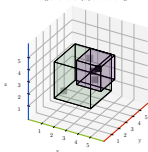
$$\pi \oplus^{++-} \sigma$$

[[1, 3, 2, 5, 4], [3, 5, 4, 1, 2]]



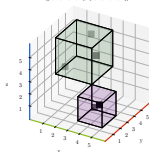
$$\pi \oplus^{+-+} \sigma$$

[[3, 5, 4, 2, 1], [1, 3, 2, 4, 5]]



$$\pi \oplus^{+--} \sigma$$

[[3, 5, 4, 2, 1], [3, 5, 4, 1, 2]]



Let  $\sigma$  and  $\pi$  two  $d$ -permutations and  $\text{dir} \in \{+, -\}^d$ . The  $d$ -sum with respect to direction  $\text{dir}$  is :

$$\sigma \oplus^{\text{dir}} \pi := \bar{\sigma}_2 \oplus_2^{\text{dir}} \bar{\pi}_2, \dots, \bar{\sigma}_d \oplus_d^{\text{dir}} \bar{\pi}_d,$$

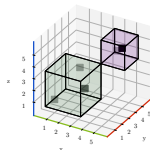
where  $\oplus_i^{\text{dir}}$  is  $\oplus$  if  $\text{dir}_i = +$  and  $\ominus$  if  $\text{dir}_i = -$ .

# Separable $d$ -permutations [Atkinson Mansour 10]

$d$ -sum:

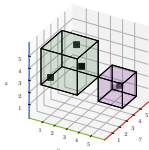
$$\pi \oplus^{(+++)} \sigma$$

[[1, 3, 2, 5, 4], [1, 3, 2, 4, 5]]



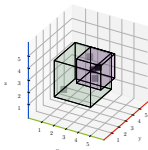
$$\pi \oplus^{(++-)} \sigma$$

[[1, 3, 2, 5, 4], [3, 5, 4, 1, 2]]



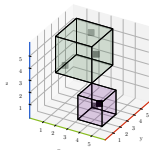
$$\pi \oplus^{(+--+)} \sigma$$

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$$\pi \oplus^{(+--)} \sigma$$

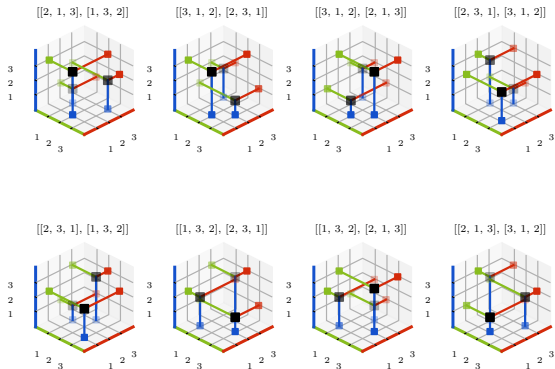
[[3, 5, 4, 2, 1], [3, 5, 4, 1, 2]]



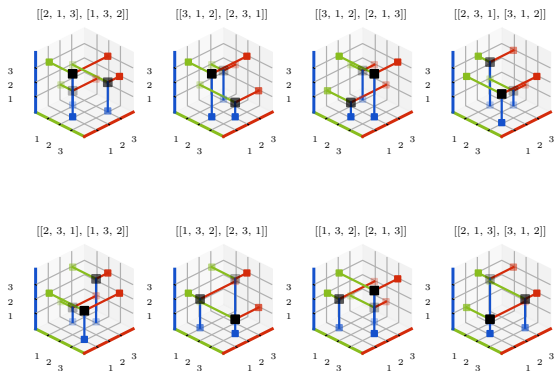
**separable  $d$ -permutation:** size 1 or a  $d$ -sum separable permutations.

$$p_1 = (132, 132) = (1, 1) \oplus^{(+++)} ((1, 1) \oplus^{(+--)} (1, 1))$$

# Separable $d$ -permutations [Atkinson Mansour 10]



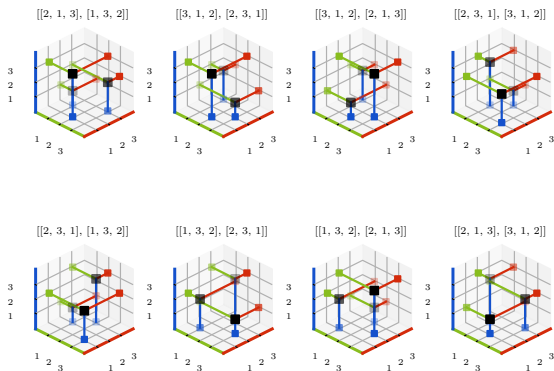
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$$|Sep_n^{d-1}| = \frac{1}{n-1} \sum_{k=0}^{n-2} \binom{n-1}{k} \binom{n-1}{k+1} (2^{d-1} - 1)^k (2^{d-1})^{n-k-1}.$$

# random generation and limit

## THE BROWNIAN LIMIT OF SEPARABLE PERMUTATIONS

FRÉDÉRIQUE BASSINO, MATHILDE BOUVEL, VALENTIN FÉRAY, LUCAS GERIN, AND ADELIN PIERROT

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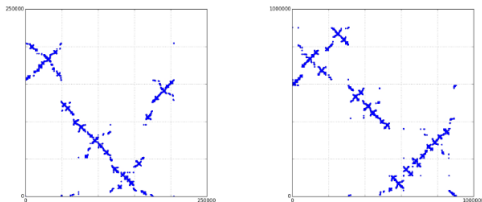


FIGURE 1. Two uniform random separable permutations of sizes respectively  $n = 204\,523$  and  $n = 903\,073$  (a permutation  $\sigma$  is represented here with its *diagram*: for every  $i \leq n$ , there is a dot at coordinates  $(i, \sigma_i)$ ).



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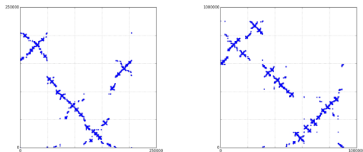
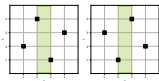


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Random separable 3-permutations of size  $\approx 100000$ .(with UsainBoltz)

# Vincular Patterns and Baxter permutations

**vincular pattern:** a pattern where some entries must be consecutive in the permutation (**adjacencies**). Ex:  $2413|_2$  and  $3142|_2$

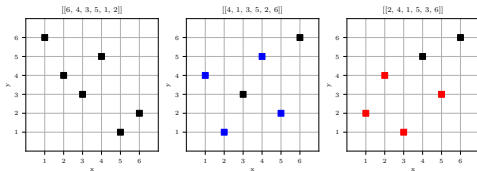
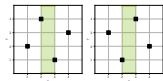


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A **vincular pattern**  $\pi|_X$ ,  $\pi \in S_k$ ,  $X \subseteq [k-1]$ .  $\sigma \in S_n$  **contains**  $\pi|_X$ ,  $\sigma_{i_1}, \sigma_{i_2} \dots \sigma_{i_k} \equiv \pi$  and that  $i_{j+1} = i_j + 1 \forall j \in X$ .

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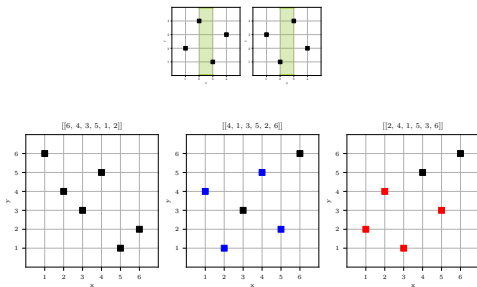



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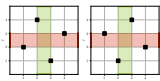
**Baxter permutations**:  $B_n := S_n(2413|_2, 3142|_2)$ .

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# Generalized vincular Patterns

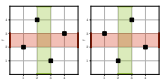
**generalized vincular pattern** : horizontal and/or vertical adjacencies. Ex:  $2413|_{2,2}$  and  $3142|_{2,2}$ .



$$B_n = S_n(2413|_{2,2}, 3142|_{2,2}).$$

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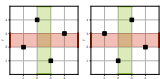


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Natural extension of generalized vincular patterns to  $d$ -permutations...

# Generalized vincular Patterns

**generalized vincular pattern** : horizontal and/or vertical adjacencies. Ex:  $2413|_{2,2}$  and  $3142|_{2,2}$ .



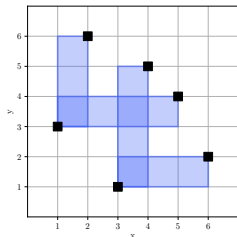
$$B_n = S_n(2413|_{2,2}, 3142|_{2,2}).$$

Natural extension of generalized vincular patterns to  
 $d$ -permutations...

But what could be a Baxter  $d$ -permutation...

# Well sliced permutations

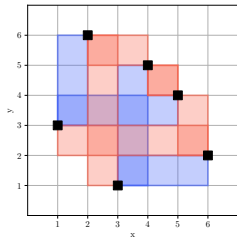
A **slice** is a rectangle defined by two adjacent points. **type**: horizontal or vertical. The **direction** of a slice is:  $+$  or  $-$ .





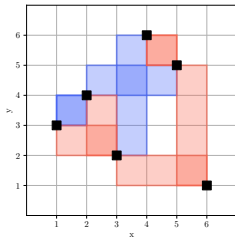
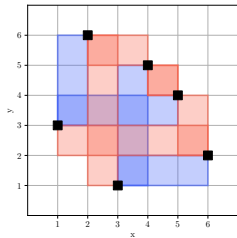
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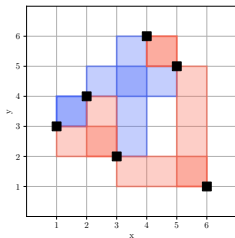
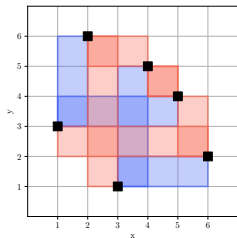
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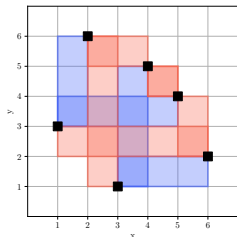
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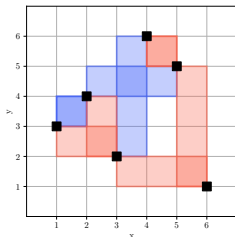
**well-sliced**: each vertical slice intersects exactly 1 horizontal slice and 2 intersecting slices share the same direction.

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not well sliced

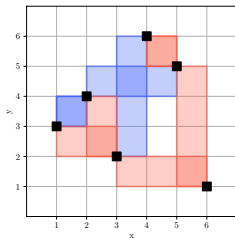
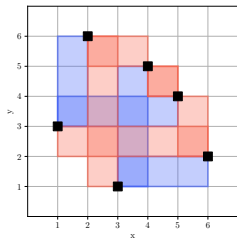


well sliced

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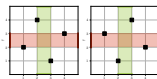
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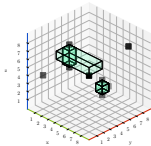
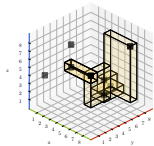
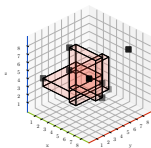
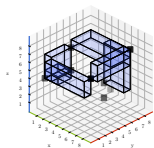
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**Proposition:** Baxter  $\Leftrightarrow$  well-sliced



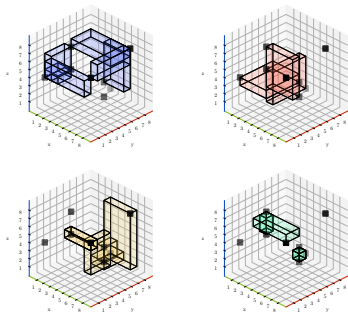
# Well-sliced $d$ -permutations

The **direction** of a slice is:  $++$ ,  $+ -$ ,  $- +$ ,  $-$ . The **type** of a slice is:  $x$ ,  $y$  or  $z$ .



# Well-sliced $d$ -permutations

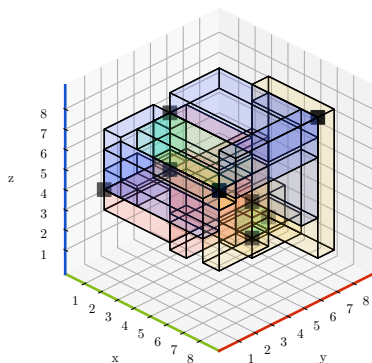
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**well-sliced**: each slice intersects exactly 1 slice of each type and 2 intersecting slices share the same direction.

## Well-sliced $d$ -permutations

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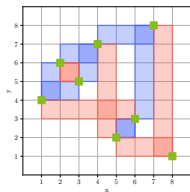
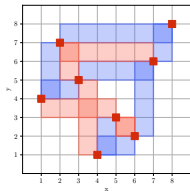
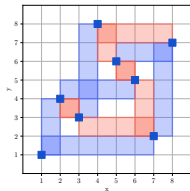
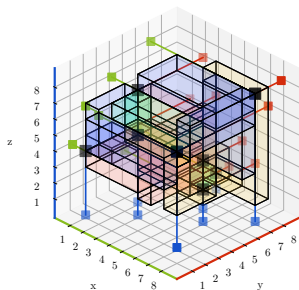


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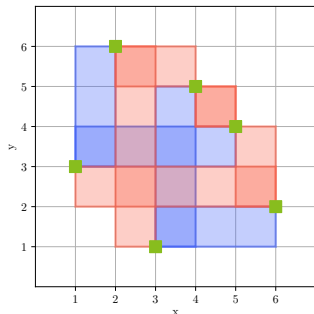


# Baxter $d$ -permutation

A **Baxter  $d$ -permutation** is a  $d$ -permutation such that each of its  $d' \leq d$  projection is well-sliced.



# Well-sliced $d$ -permutations vs Baxter $d$ -permutation



well-sliced but not Baxter.

# Baxter $d$ -permutation characterisation

## Theorem

$$B_n^{d-1} = S_n^{d-1}(\text{Sym}(2413|_{2,2}), \text{Sym}((312, 213)|_{1,2,.}), \\ \text{Sym}((3412, 1432)|_{2,2,.}), \text{Sym}((2143, 1423)|_{2,2.})).$$

## Baxter $d$ -permutation enumeration

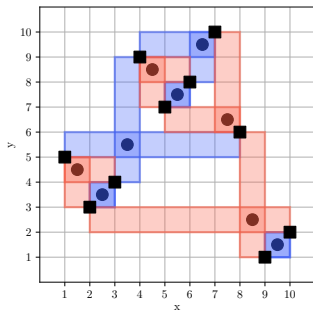
$$|B_n^{d-1}|$$

$n \setminus d$	2	3	4	5
1	1	1	1	1
2	2	4	8	16
3	6	28	120	496
4	22	260	2440	20816
5	92	2872	59312	1035616
6	422	35620		
7	2074	479508		

**open problem:** Enumeration formula?

# Baxter / Complete Baxter / Anti-Baxter permutations

dots: intersection of slices



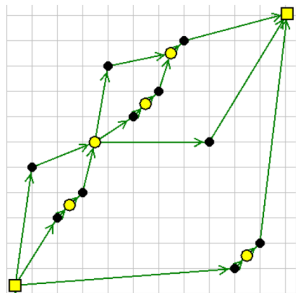
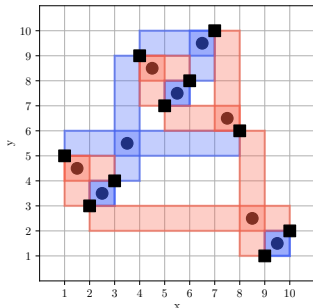
squares: Baxter permutations

squares + dots : Complete Baxter permutations

dots: Anti-Baxter permutation

# Baxter / Complete Baxter / Anti-Baxter permutations

dots: intersection of slices



[B. Bousquet Fusy 10] Baxter permutations  $\leftrightarrow$  plane bipolar orientations.

# Baxter / Complete Baxter / Anti-Baxter $d$ -permutations

# Anti-Baxter $d$ -permutation enumeration

$$|A_n^{d-1}|$$

$n \setminus d$	2	3	4	5
1	1	1	1	1
2	2	4	8	16
3	6	36	216	1296
4	22	444	7096	
5	88	5344		
6	374	64460		
7	1668			

**open problem:** Enumeration formula?



## conclusion/perspectives

- general framework for pattern avoidance in  $d$ -permutations  
+ easy results
- generalisation of Baxter permutation
- work in progress (with Thomas Muller and Adrian Tanasa): links between  $2^{d-1}$  floorplans and  $d$ -permutations.
- lot of open problems
- implementation available  
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Have fun !