

Transposition factorisations, high genus maps, and the Plancherel–Hurwitz measure on partitions

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Permutations and partitions

Consider the set

$$\mathfrak{S}_n = \{\text{permutations of } (1, 2, \dots, n)\},$$

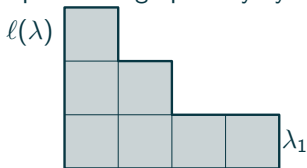
the **symmetric group of order n** .

Its natural counterpart is the set of positive integer sequences

$$\{\lambda \vdash n\} = \{(\lambda_1 \geq \lambda_2 \geq \dots \lambda_{\ell(\lambda)}) \mid \lambda_1 + \lambda_2 + \dots + \lambda_{\ell(\lambda)} = n\},$$

the **partitions of n** . They are represented graphically by Young diagrams,

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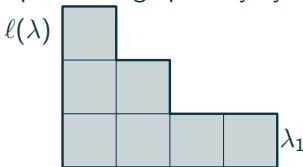
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We will treat a combinatorial problem on \mathfrak{S}_n in the limit as $n \rightarrow \infty$ related to high genus map enumeration, by studying the limit behaviour of a partition of n under a corresponding law.

Introduction: Two classical questions about random permutations answered using random partitions

Main object: From transposition factorisations and Hurwitz maps to the Plancherel–Hurwitz measure

Main result: Limit phenomenon for a large Plancherel–Hurwitz random partition in the high genus regime

Perspectives

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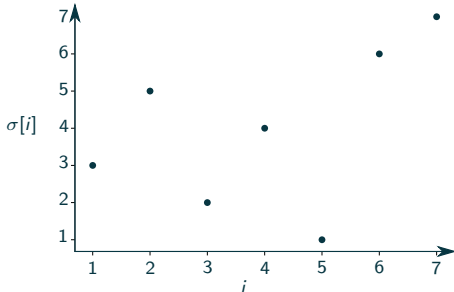
Perspectives

1. Length of the longest increasing subsequence of a uniform random permutation

Let σ be a uniform random element of \mathfrak{S}_n . What is the distribution of the length of the longest increasing subsequence of σ as $n \rightarrow \infty$?

e.g. $\sigma = (3, 5, 2, 4, 1, 6, 7) \in \mathfrak{S}_7$

Plotting σ ,

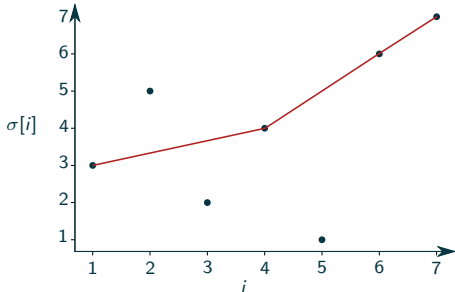


1. Length of the longest increasing subsequence of a uniform random permutation

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$$\begin{aligned} \text{e.g. } \sigma &= (3, 5, 2, 4, 1, 6, 7) \in \mathfrak{S}_7 \\ &\Rightarrow \text{LLIS}(\sigma) = 4 \end{aligned}$$

Plotting σ , LLIS(σ) is the length of the longest up/right path:



A measure on partitions

The **Robinson–Schensted** algorithm extracts the length of the longest increasing subsequence of a given permutation.

$$\sigma = (3, 5, 2, 4, 1, 6, 7)$$



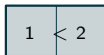
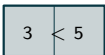
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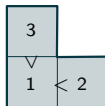
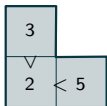
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3	<	5
∇		∇
2	<	4

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∇		∇
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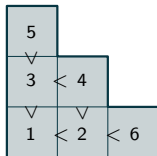
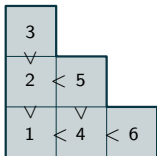
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∨	
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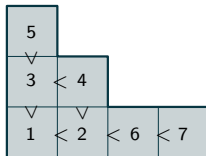
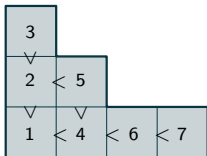
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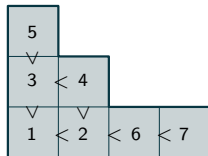
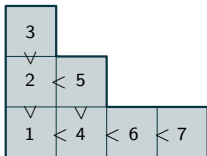
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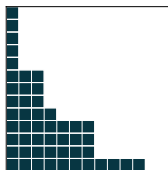
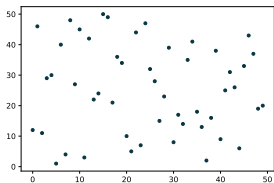
It associates each permutation $\sigma \in \mathfrak{S}_n$ bijectively with a pair of standard Young tableaux (SYT) of shape $\lambda \vdash n$ such that $\text{LLIS}(\sigma) = \lambda_1$.

It proves that $\sum_{\lambda \vdash n} f_\lambda^2 = n!$ where $f_\lambda = \#\{\text{SYT of shape } \lambda\}$, and produces the **Plancherel measure** on partitions of n

$$\mathbb{P}_n(\lambda) = \frac{1}{n!} f_\lambda^2$$

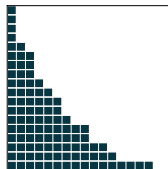
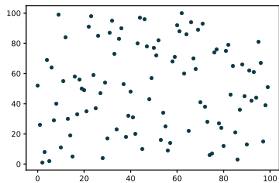
A limit shape

Looking at a random partition of n under \mathbb{P}_n as n grows, a shape emerges.



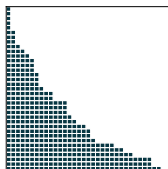
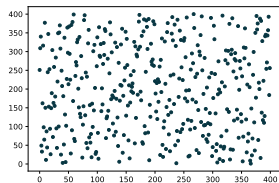
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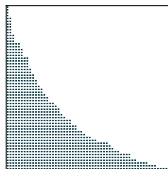
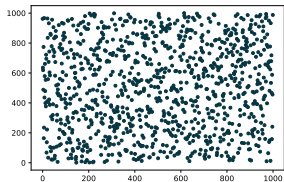
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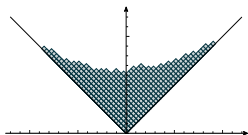
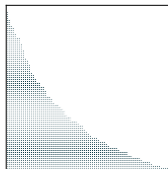
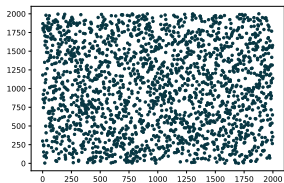
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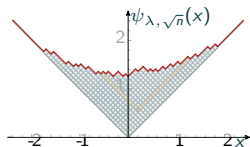
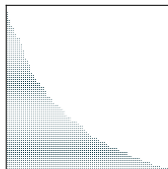
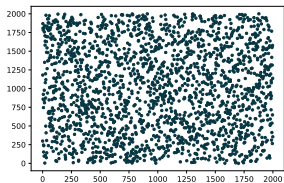
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Tilting the Young diagram by 45° , the shape is encoded by the curve $\psi_{\lambda, \sqrt{n}}(x)$ of the upper edge in coordinates scaling with $1/\sqrt{n}$.

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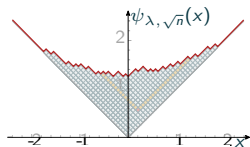
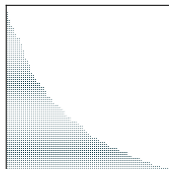
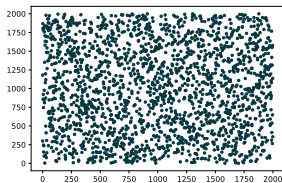
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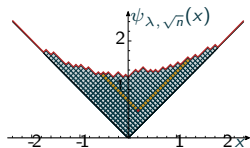
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The number of SYT f_λ can be expressed in terms of the **hook lengths** from each box to the upper edge, so that $\mathbb{P}_n(\lambda)$ is a functional of $\psi_{\lambda, \sqrt{n}}$.

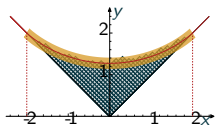
A limit shape theorem

Theorem (Vershik–Kerov–Logan–Shepp, 1977)

Let λ be a random partition of n under \mathbb{P}_n . Then, as $n \rightarrow \infty$,

$$\sup_x |\psi_{\lambda, \sqrt{n}}(x) - \Omega(x)| \xrightarrow{P} 0 \quad \text{and} \quad \lambda_1/\sqrt{n} \xrightarrow{P} 2, \quad \ell(\lambda)/\sqrt{n} \xrightarrow{P} 2.$$

$$\text{where } \Omega(x) = \begin{cases} \frac{2}{\pi} [x \arcsin(x) + \sqrt{4+x^2}], & |x| \leq 2 \\ |x|, & |x| \geq 2. \end{cases}$$



It follows that for uniform random $\sigma \in \mathfrak{S}_n$,
 $LLIS(\sigma)/\sqrt{n} \xrightarrow{P} 2$.

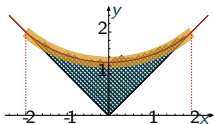
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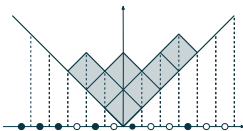


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Taking n to be a Poisson random variable, \mathbb{P}_n defines a **determinantal point process** on “fermion configurations” in bijection with partitions. We have

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left[\frac{\lambda_1 - 2n^{1/2}}{n^{1/6}} > s \right] = F_{\text{GUE}}(s)$$

where $F_{\text{GUE}}(s)$ is the GUE Tracy–Widom distribution (Baik, Deift & Johansson, 1998).



2. Mixing time for the transposition random walk

Let $\sigma^{*\ell} = \tau_\ell \cdots \tau_2 \cdot \tau_1$ be a random element of \mathfrak{S}_n generated by multiplying ℓ uniform random transpositions.

$$\sigma^{*0} = \text{id} : (1, 2, 3, 4, 5, 6, 7, 8, 9) \in \mathfrak{S}_9$$

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How fast does $\ell := \ell(n)$ need to grow with n such that $\sigma^{*\ell}$ is “almost uniform” as $n \rightarrow \infty$?

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How fast does $\ell := \ell(n)$ need to grow with n such that $\sigma^{*\ell}$ is “almost uniform” as $n \rightarrow \infty$?

Denoting the law of $\sigma^{*\ell} \in \mathfrak{S}_n$ by $\mathbb{P}^{*\ell}$, its proximity to the uniform measure is quantified by the total variation distance

$$\sup_{\sigma \in \mathfrak{S}_n} \left| \mathbb{P}^{*\ell}(\sigma) - \frac{1}{n!} \right|.$$

An answer with partitions

The transposition random walk is asymptotically well mixed if

$\ell > \frac{1}{2} n \log n$. Namely,

Theorem (Diaconis–Shahshahani, 1981)

For all $n \geq 10$, there is a constant b such that the total variation distance from \mathbb{P}^{ℓ} to the uniform measure is bounded as*

$$d_{TV} = \sup_{\sigma \in \mathfrak{S}_n} \left| \mathbb{P}^{*\ell}(\sigma) - \frac{1}{n!} \right| \leq b \exp \left[\frac{n \log n - 2\ell}{n} \right].$$

$\mathbb{P}^{*\ell}$ is approximated by “Fourier transforming” onto the irreducible representations of \mathfrak{S}_n . Starting from

$$\mathbb{P}(\sigma) = \binom{n}{2}^{-1} \mathbf{1}_{\sigma \text{ is a transposition}}$$

the Fourier transform of $\mathbb{P}(\sigma)$ onto an irreducible representation ρ

$$\rho(\mathbb{P}) = \sum_{\sigma \in \mathfrak{S}_n} \mathbb{P}(\sigma) \rho(\sigma), \quad \mathbb{P}(\sigma) = \sum_{\text{reps } \rho \text{ of } \mathfrak{S}_n} \dim(\rho) \operatorname{tr}_{\rho} \rho(\sigma) \cdot \rho(\mathbb{P})$$

satisfies $\rho(\mathbb{P}^{*\ell}) = \rho(\mathbb{P})^{\ell}$.

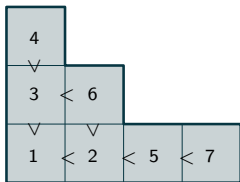
An answer with partitions

Both the conjugacy classes and the irreducible representations of \mathfrak{S}_n are indexed by partitions $\lambda \vdash n$.

- The dimension of the representation ρ_λ is

$$\dim(\rho_\lambda) = f_\lambda,$$

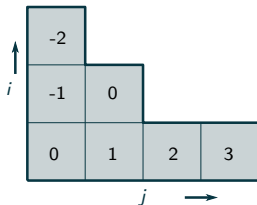
the number of standard Young tableaux of shape λ



- The normalised character of a transposition on ρ_λ is

$$\frac{\text{tr } \rho_\lambda(\tau)}{\dim \rho_\lambda} = C_\lambda = \sum_{\square \in \lambda} c(\square),$$

where $c(\square_{i,j}) = j - i$ is the **contents**



An answer with partitions

For the transposition random walk, we have

$$\sup_{\sigma \in \mathfrak{S}_n} \left| \mathbb{P}^{*\ell}(\sigma) - \frac{1}{n!} \right| \leq \frac{1}{n^{2\ell}} \sum_{\lambda \vdash n} f_\lambda^2 \left(n + C_\lambda \right)^{2\ell}$$

The bound is found only from the contributions from the “trivial partitions” on the right hand side

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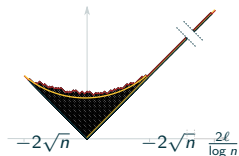
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From another perspective, λ and its conjugate are limiting partitions as $n \rightarrow \infty$ in an asymptotic regime $\ell > \frac{1}{2} n \log n$ for a partition under a measure

$$\mathbb{P}(\lambda) \propto f_\lambda^2 C_\lambda^\ell.$$

We consider a case where a measure of this form arises naturally, and study it in asymptotic regime where we find a new kind of limit phenomenon.



Introduction: Two classical questions about random permutations answered using random partitions

Main object: From transposition factorisations and Hurwitz maps to the Plancherel–Hurwitz measure

Main result: Limit phenomenon for a large Plancherel–Hurwitz random partition in the high genus regime

Perspectives

Transposition factorisations and Hurwitz maps

Consider a factorisation of the identity on \mathfrak{S}_n by ℓ transpositions,

$$(\tau_1, \tau_2, \dots, \tau_\ell) \in (\mathfrak{S}_n)^\ell, \tau_1 \cdot \tau_2 \cdots \tau_\ell = \text{id}.$$

It can be represented geometrically by a **pure Hurwitz map** (a graph with a system of orientation) with n vertices and ℓ edges:

- vertices are labelled $1, \dots, n$; edges are labelled $1, \dots, \ell$
- if $\tau_i = (a, b)$, the edge labelled i is incident on the vertices (a, b)
- edge labels increase cyclically around each vertex; each edge-label descent is incident on exactly one face.

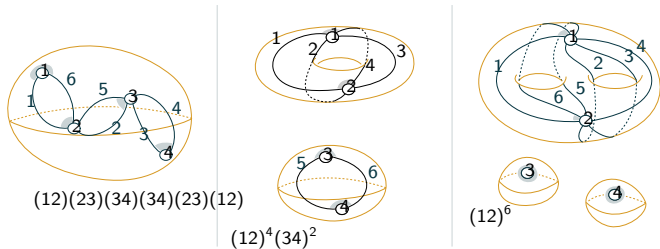
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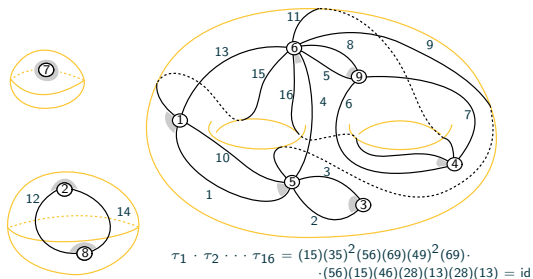
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For a non-transitive factorisation, we have an unconnected map.



- The **Euler characteristic** χ of a pure Hurwitz map is

$$\chi = 2\kappa - 2 \sum_{c \text{ connected}} g_c = \#\text{vertices} - \#\text{edge} + \#\text{faces} = 2n - \ell,$$

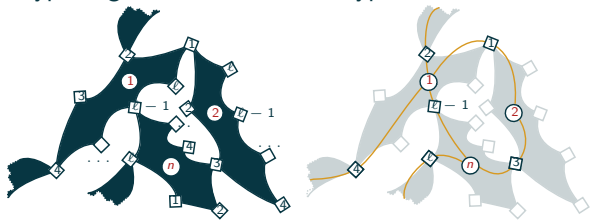
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- Hurwitz maps correspond to special cases of constellations, with only two hyperedges incident on each hypervertex.



They are counted by (unconnected) Hurwitz numbers

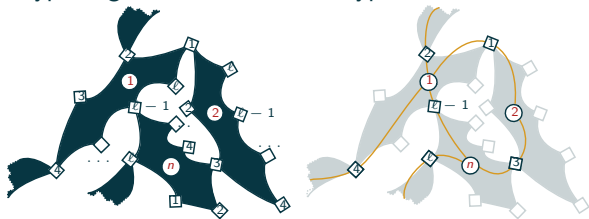
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- Uniform random connected Hurwitz maps are in the same universality class as e.g. uniform triangulations, quadrangulations, ... (Duchi, Poulalhan & Schaeffer, 2014).

A measure on partitions

We turn to an identity from the representation theory of \mathfrak{S}_n :

Frobenius' formula for unconnected Hurwitz numbers

$$\begin{aligned} H_{n,\ell} &= \#\{\tau_1 \cdot \tau_2 \cdots \tau_\ell = \text{id} \in \mathfrak{S}_n\} \\ &= \#\{\text{Hurwitz maps with } n \text{ vertices, } \ell \text{ edges}\} \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} f_\lambda^2 \left(\sum_{\square \in \lambda} c(\square) \right)^\ell \end{aligned}$$

This is a vast generalisation of the identity $n! = \sum_{\lambda \vdash n} f_\lambda^2$ from the Robinson–Schensted algorithm. It has no bijective proof.

For each even ℓ , we define the **Plancherel–Hurwitz measure** on partitions of n ,

$$\mathbb{P}_{n,\ell}(\lambda) = \frac{1}{H_{n,\ell}} \frac{1}{n!} f_\lambda^2 C_\lambda^\ell.$$

Aside: Plancherel–Hurwitz measure and the transposition walk

The classical Hurwitz number $H_{n,\ell}$ gives the **return probability** for the random walk by ℓ transpositions on \mathfrak{S}_n :

$$\mathbb{P}_{\text{return}}^{n,\ell} = \frac{H_{n,\ell}}{\binom{n}{2}^\ell} = \frac{1}{n! \binom{n}{2}^\ell} \sum_{\lambda \vdash n} f_\lambda^2 C_\lambda^\ell.$$

For $\ell > n \log n$, we have $\mathbb{P}_{\text{return}}^{n,\ell} \sim \frac{1}{n!}$ as $n \rightarrow \infty$.

The contribution from just

$$\lambda = (n) \quad \text{or} \quad \lambda' = (1, 1, \dots, 1)$$

to $H_{n,\ell}$ gives a good approximation, with

$$f_\lambda = 1, \quad C_\lambda = \binom{n}{2}, \quad \mathbb{P}_{\text{return}}^{n,\ell} \approx \frac{1}{n!}.$$

What happens below the mixing speed threshold?

High genus map enumeration

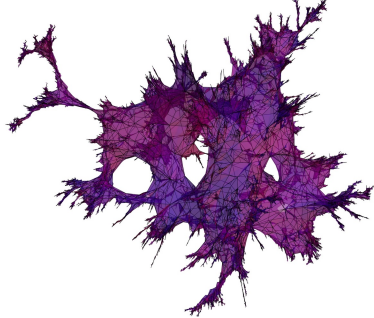
Planar and fixed genus maps have universal enumeration exponents.

In the **high genus** regime, where the genus grows linearly with map size, the first asymptotic estimates for connected map enumeration were obtained recently:

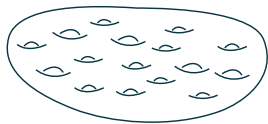
Theorem (Budzinski–Louf, 2021)

Let \mathcal{T}_{n,g_n} be the triangulations with n vertices and genus $g_n \sim qn$. Then, as $n \rightarrow \infty$,

$$|\mathcal{T}_{n,g_n}| = n^{2g_n} \exp[f(q)n + o(n)].$$



Simulation by J. Bettinelli.



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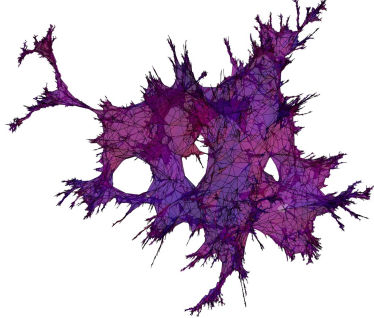
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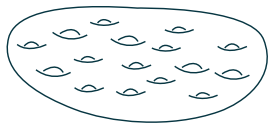
$$|\mathcal{T}_{n,g_n}| = n^{2g_n} \exp[f(q)n + o(n)].$$

To look at a similar enumeration problem, we study partitions under $\mathbb{P}_{n,\ell}$ in an asymptotic regime $\ell := \ell(n) \sim 2qn$.

Caveat: $H_{n,\ell}$ counts not necessarily connected, labelled maps.



Simulation by J. Bettinelli.



Introduction: Two classical questions about random permutations answered using random partitions

Main object: From transposition factorisations and Hurwitz maps to the Plancherel–Hurwitz measure

Main result: Limit phenomenon for a large Plancherel–Hurwitz random partition in the high genus regime

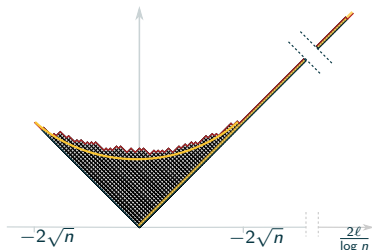
Perspectives

A limit phenomenon theorem

Theorem (Chapuy–Louf–W., 2022)

Let λ be a partition of n under the positive half of the Plancherel–Hurwitz measure $\mathbb{P}_{n,\ell}^+ = 2\mathbb{P}_{n,\ell}(\lambda)\mathbf{1}_{C_\lambda>0}$, where $\ell = \ell(n)$ is an even number and $\ell \sim 2qn$. Then, as $n \rightarrow \infty$:

- (i) $\lambda_1 \cdot \frac{\log n}{\ell} \xrightarrow{P} 2$;
- (ii) $\sup_x |\psi_{\tilde{\lambda}}(x) - \Omega(x)| \xrightarrow{P} 0$, $\tilde{\lambda} = (\lambda_2, \lambda_3, \dots)$;
- (iii) $\lambda_2/\sqrt{n} \xrightarrow{P} 2$ and $\ell(\lambda)/\sqrt{n} \xrightarrow{P} 2$.



- **New two-scale phenomenon**

Idea of the proof Like the limit behaviour itself, we work at two scales.

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1. Macroscopic aspects: We compare entropies from

$$\begin{aligned} \frac{f_\lambda^2}{n!} &= \frac{n!}{\prod_{\square \in \lambda} \text{hook}_\lambda(\square)^2} \quad \text{and} \quad C_\lambda^\ell = \left[\sum_i \frac{1}{2} \lambda_i (\lambda_i - 2i + 1) \right]^\ell \\ &= e^{-n(1 + l_{\text{hook}}(\psi_\lambda)) + O(\sqrt{n} \log n)} \quad = e^{\ell [\log \sum_i \frac{1}{2} \lambda_i (\lambda_i - 2i + 1)]}. \end{aligned}$$

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1.1 Fix $\varepsilon = \frac{1}{100}$. Let $\lambda = \lambda^+ \sqcup \lambda^-$ where each $\lambda_i^+ > n^{1-\varepsilon}$, $\lambda_i^- < n^{1-\varepsilon}$. Then

$$\mathbb{P}_{n,\ell}(\lambda) \leq e^{2\ell - 2 \log \ell - (1-2\varepsilon)|\lambda^+| \log n + 2|\lambda^+| + 2 \log(|\lambda^+| \log n) + o(1)},$$

and we have $\mathbb{P}_{n,\ell}(|\lambda^+| \notin [0.4 \frac{\ell}{\log n}, 6 \frac{\ell}{\log n}]) \leq e^{-\ell/100 + o(n)}$.

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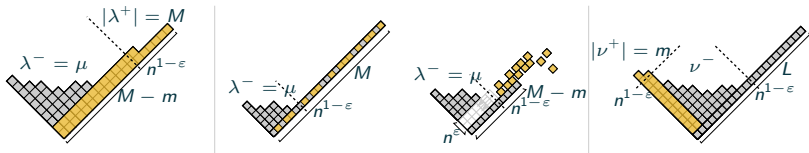
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1.2 Let $M \in [0.4 \frac{\ell}{\log n}, 6 \frac{\ell}{\log n}]$ and let

$$\Lambda_{M,m,\mu} = \{\lambda \vdash n \mid |\lambda^+| = M, \lambda_1 = M - m, \lambda^- = \mu\}.$$

For n large enough, $\mathbb{P}_{n,\ell}(\lambda \in \Lambda_{M,m,\mu}) \leq e^{-m/100}$. Hence, $\lambda_1 \in [0.4 \frac{\ell}{\log n}, 6 \frac{\ell}{\log n}]$ and $\lambda_2 < n^{1-\varepsilon}$ w.h.p. Following analogous steps we have $\ell(\lambda) < n^{1-\varepsilon}$ w.h.p.

Idea of the proof Like the limit behaviour itself, we work at two scales.

1. Macroscopic aspects:

1.3 For λ with $\lambda_1 \in [0.4 \frac{\ell}{\log n}, 6 \frac{\ell}{\log n}]$, $\lambda_2 < n^{1-\varepsilon}$ and $\ell(\lambda) < n^{1-\varepsilon}$,

$$\mathbb{P}_{n,\ell}(\lambda) \leq e^{\ell(2-2\log 2)+2\ell \log(\lambda_1 \log n)-2\ell \log \ell-\lambda_1 \log n-n(1+2I_{\text{hook}}(\psi_\lambda))+o(n)}$$

and looking at the maximiser of this bound, we find $\lambda_1 \cdot \frac{\log n}{\ell} \xrightarrow{P} 2$ as $n \rightarrow \infty$.

The remaining parts $\tilde{\lambda} = (\lambda_2, \lambda_3, \dots)$ optimise the Plancherel entropy and

$$\sup_x |\psi_{\tilde{\lambda}} - \Omega(x)| \xrightarrow{P} 0.$$

It follows that $\lambda_2, \ell(\lambda) \geq 2\sqrt{n}$ w.h.p.

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1.4 From a partition maximising the above entropy, we have an asymptotic estimate of the unconnected Hurwitz numbers:

Theorem

In the regime where ℓ is even and $\ell \sim 2qn$, as $n \rightarrow \infty$

$$H_{n,\ell} = \left(\frac{n}{\log n} \right)^{2\ell} \exp [- (2 - \log 2)\ell + o(n)].$$

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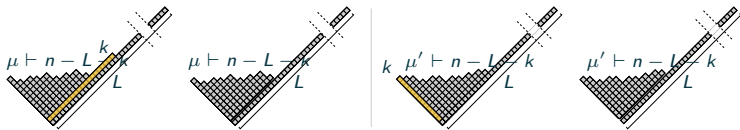
2. Microscopic aspects:

The upper bounds on $\lambda_2, \ell(\lambda)$ are improved in two rounds.

2.1 Fix $L := 2\ell / \log n$ and set $Z_n[L] = \sum_{\lambda \vdash n | \lambda_1 = L} \frac{1}{n!} f_\lambda^2 C_\lambda^\ell$. Comparing the partitions with their second part removed, we find

$$\mathbb{P}_{n,\ell}(\lambda_2 = k | \lambda_1 = L) \leq \binom{n}{k}^2 \frac{(n-k)!}{n!} \frac{Z_{n-k}[L]}{Z_n[L]} (1 + o(1)).$$

But $\frac{Z_{n-k}[L]}{Z_n[L](1+o(1))} = 1 + o(1)$, and we recover $\mathbb{P}(\lambda_2 = k | \lambda_1 = L) \leq \frac{n^k}{(k/e)^{2k}} e^{o(k)}$.



Considering a partition with each part decreased by 1 similarly bounds

$\mathbb{P}(\ell(\lambda) = k | \lambda_1 = L)$, and we have $\lambda_2, \ell(\lambda) \leq (e + o_p(1))\sqrt{n}$ as $n \rightarrow \infty$.

2.2 Once restricted to this bounding box we can use deviation bounds for the first part and content-sum under the pure Plancherel measure $\mathbb{P}_{n,0}$ to find similar deviation bounds for $\lambda_2, \ell(\lambda)$ above 2.

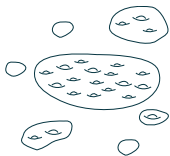
Relating back to high genus map enumeration

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cf. Budzinski–Louf result $|\mathcal{T}_{n,g_n}| = n^{2g_n} \exp[f(q)n + o(n)]$.



The maps counted by $H_{n,\ell}$ are not necessarily connected.

- Does a uniform random Hurwitz map have a high genus **giant component**?

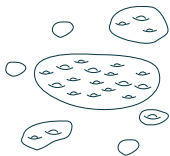
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- Does a uniform random Hurwitz map have a high genus **giant component**?

Theorem (Chapuy–Louf–W., 2022)

As $n \rightarrow \infty$, a uniformly random unconnected Hurwitz map with n vertices and $\ell \sim 2qn$ edges contains a connected component with at least $\gamma(q)\ell$ edges, for a $\gamma(q) > 0$. However, no connected component of such a map has more than $O_p(n/\log n)$ vertices.

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- an interpretations for the statistic λ_1 ? a connection between limiting partitions and maps?
- a bijective proof of Frobenius' formula?
- a determinantal point processes approach? integrability methods?
(cf. Okounkov, 2001; Dubrovin, Yang & Zagier, 2017)
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What about other asymptotic regimes for ℓ (in particular, $\ell \sim 2q\sqrt{n}$)?

Similar phenomenon via a bijection (Kammoun, 2023+): Consider uniform random $\sigma \in \mathfrak{S}_n$ conditioned to have $L(n) := 4qn/\log n$ fixed points. As $n \rightarrow \infty$, the Robinson Schensted algorithm recovers the same two-fold limit. Moreover, for $L(n) > 2n^{1/2+\varepsilon}$ the fluctuations in λ_1 are Gaussian, and for $L(n) < 2n^{1/2-\varepsilon}$ they are GUE Tracy–Widom.

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Thank you for your attention!