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From a lattice  
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# Interprétations combinatoires des identités de Macdonald pour les systèmes de racines affines

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Université Lyon 1 – Institut Camille Jordan

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# Ferrers diagram and hooks of partitions

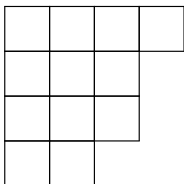
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$$(4, 3, 3, 2) \in \mathcal{P}$$

$$|\lambda| = 4 + 3 + 3 + 2 = 12$$

# Ferrers diagram and hooks of partitions

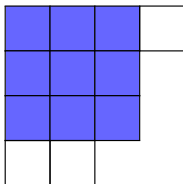
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$$(4, 3, 3, 2) \in \mathcal{P}$$

Durfee square

# Ferrers diagram and hooks of partitions

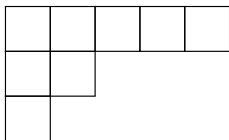
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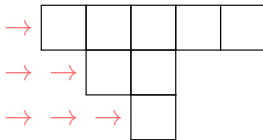
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$$(5, 2, 1) \in \mathcal{D}$$



$$(5, 2, 1) \in \mathcal{D}$$

# Ferrers diagram and hooks of partitions

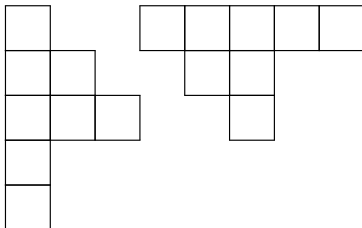
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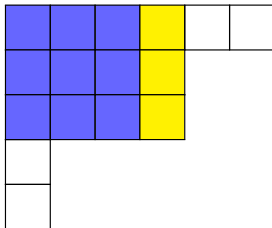
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Twice  $(5, 2, 1) \in \mathcal{D}$



$(6, 4, 4, 1, 1) \in \mathcal{DD}$

# Ferrers diagram and hooks of partitions

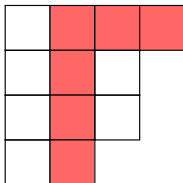
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$$(4, 3, 3, 2) \in \mathcal{P}$$

# Ferrers diagram and hooks of partitions

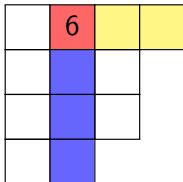
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$(4, 3, 3, 2) \in \mathcal{P}$

$$6 = 3 + 2 + 1$$

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7	6	4	1
5	4	2	
4	3	1	
2	1		

$$(4, 3, 3, 2) \in \mathcal{P}$$

- $\mathcal{H}(\lambda) := \{\text{hook-lengths}\}$

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7	6	4	1
5	4	2	
4	3	1	
2	1	$\mathcal{H}_3$	

$$(4, 3, 3, 2) \in \mathcal{P}$$

- $\mathcal{H}(\lambda) := \{\text{hook-lengths}\}$
- for  $t \in \mathbb{N}^*$ ,  $\mathcal{H}_t(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod{t}\}$

# Ferrers diagram and hooks of partitions

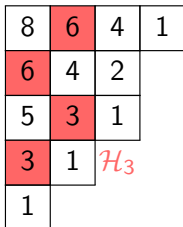
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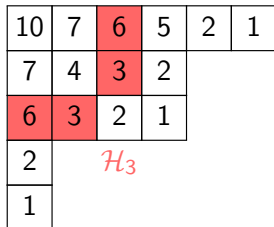
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$(4, 3, 3, 2, 1) \in \mathcal{P}$



$(6, 4, 4, 1, 1) \in \mathcal{DD}$

- $\mathcal{H}(\lambda) := \{\text{hook-lengths}\}$
- for  $t \in \mathbb{N}^*$ ,  $\mathcal{H}_t(\lambda) := \{h \in \mathcal{H}(\lambda) \mid h \equiv 0 \pmod{t}\}$

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10	7	6	5	2	1
7	4	3	2		
6	3	2	1		
2		$\Delta$			
1					

(a)  $(6, 4, 4, 1, 1) \in \mathcal{DD}$

+	+	+	+	+	+
-	+	+	+		
-	-	+	+		
-		$\Delta$			
-					

(b)  $\varepsilon_s$

- $\mathcal{H}(\lambda) := \{\text{hook-lengths}\}$
- $\varepsilon_s = \begin{cases} -1 & \text{if } s \text{ is a box strictly below } \Delta \\ 1 & \text{otherwise} \end{cases}$

# Nekrasov–Okounkov formula

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Nekrasov–Okounkov (2006), Westbury (2006), Han (2008)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left( 1 - \frac{z^2}{h^2} \right) = (T; T)_{\infty}^{z^2-1}$$

where  $(a; q)_{\infty} := (1 - a)(1 - aq)(1 - aq^2) \dots$

# Nekrasov–Okounkov formula

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where  $(a; q)_{\infty} := (1 - a)(1 - aq)(1 - aq^2) \dots$

Note  $z = t \in \mathbb{N}^* \Rightarrow \lambda$  are  $t$ -core partitions :  $\mathcal{H}_t(\lambda) = \emptyset$ .

# A $q, t$ -Nekrasov–Okounkov formula

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Rains–Warnaar (2018), Carlsson–Rodriguez-Villegas (2018)

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{s \in \lambda} \frac{(1 - uq^{a(s)+1} t^{l(s)})(1 - u^{-1}q^{a(s)} t^{l(s)+1})}{(1 - q^{a(s)+1} t^{l(s)})(1 - q^{a(s)} t^{l(s)+1})} \\ = \prod_{i,j,k \geq 1} \frac{(1 - uq^i t^{j-1} T^k)(1 - u^{-1}q^{i-1} t^j T^k)}{(1 - q^{i-1} t^{j-1} T^k)(1 - q^i t^j T^k)} \end{aligned}$$

# A $q, t$ -Nekrasov–Okounkov formula

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Motivated by the study of an affine variety linked to mixed Hodge polynomials of character variety

Hausel–Rodriguez-Villegas (2008)

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$$= \prod_{i,j,k \geq 1} \frac{(1 - uq^i t^{j-1} T^k)(1 - u^{-1}q^{i-1} t^j T^k)}{(1 - q^{i-1} t^{j-1} T^k)(1 - q^i t^j T^k)}$$

- **R–W**: connection between Macdonald polynomials and mixed Hodge polynomials. Derivation of skew Cauchy like identity on Macdonald polynomials.
- **C–R–V**: proof using vertex operators and plethystic exponential.

# A $q, t$ -Nekrasov–Okounkov formula

Rains–Warnaar (2018), Carlsson–Rodriguez-Villegas (2018)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{s \in \lambda} \frac{(1 - uq^{a(s)+1} t^{l(s)})(1 - u^{-1}q^{a(s)} t^{l(s)+1})}{(1 - q^{a(s)+1} t^{l(s)})(1 - q^{a(s)} t^{l(s)+1})}$$

$$= \prod_{i,j,k \geq 1} \frac{(1 - uq^i t^{j-1} T^k)(1 - u^{-1}q^{i-1} t^j T^k)}{(1 - q^{i-1} t^{j-1} T^k)(1 - q^i t^j T^k)}$$

Special case  $q = t$

Dehaye–Han (2011), R–W (2018), C–R–V (2018)

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{(1 - uq^h)(1 - u^{-1}q^h)}{(1 - q^h)^2}$$

$$= \prod_{k,r \geq 1} \frac{(1 - uq^r T^k)^r (1 - u^{-1}q^r T^k)^r}{(1 - q^{r-1} T^k)^r (1 - q^{r+1} T^k)^r}$$

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# A $q, t$ -Nekrasov–Okounkov formula

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Dehaye–Han (2011), R–W (2018), C–R–V (2018)

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} T^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \frac{(1 - uq^h)(1 - u^{-1}q^h)}{(1 - q^h)^2} \\ = \prod_{k,r \geq 1} \frac{(1 - uq^r T^k)^r (1 - u^{-1}q^r T^k)^r}{(1 - q^{r-1} T^k)^r (1 - q^{r+1} T^k)^r} \end{aligned}$$

Special case  $u = q^z$  and  $q \rightarrow 1$

# Macdonald identities

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Dehaye–Han’s starting point: a specialized Macdonald identity for affine type  $\tilde{A}_{t-1}$

- Macdonald identities are identities of the shape “sum = product” parametrised by sub-lattices of  $\mathbb{Z}$ , defined by an affine root systems.
- There is a bijective correspondence between the sub-lattices and some subsets of partitions.
- This allows us to give a combinatorial transcription of Macdonald identities and derive Nekrasov–Okounkov type formulas.

# Macdonald identities

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- **Macdonald** (1972): analogues of Weyl determinant formula for affine root systems  $R$  and the associated finite Weyl group,  $W$

$$\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho) - \rho} = \prod_{\substack{a > 0 \\ a \in R}} (1 - e^{-a})$$

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- **Macdonald** (1972): analogues of Weyl determinant formula for affine root systems  $R$  and the associated finite group of permutations, the Weyl group,  $W$

$$\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho) - \rho} = \prod_{\substack{a > 0 \\ a \in R}} (1 - e^{-a})$$

- Reformulation by **Stanton** (1989), **Rosengren–Schlosser** (2006) for the 7 infinite affine types  $A, B, B^\vee, C, C^\vee, D$  and  $BC$ .

# Macdonald identities

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- **Macdonald** (1972): analogues of Weyl determinant formula for affine root systems

$$\sum_{w \in W} \operatorname{sgn}(w) e^{w(\rho) - \rho} = \prod_{\substack{a > 0 \\ a \in R}} (1 - e^{-a})$$

reformulation by **Stanton** (1989), **Rosengren–Schlosser** (2006) for the 7 infinite affine types

- Setting  $x_j = e^{-\epsilon_j}$ ,  $T = e^{-1}$ , it can be seen as a multivariate series over  $\mathbb{Z}^t$  with a geometric condition.

# The example of affine type $A_{t-1}$

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$$W = S_t$$

Weyl determinant formula for affine type  $A_{t-1}$  (Vandermonde):

$$\det_{1 \leq i, j \leq t} (x_i^{j-1}) = \prod_{1 \leq i < j \leq t} (x_j - x_i)$$

# The example of affine type $A_{t-1}$

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Theorem [Macdonald (1972), Stanton (1989),  
Rosengren–Schlosser (2006)]

Set  $T \in \mathbb{C}$  such that  $|T| < 1$  and  $\mathbf{x} \in (\mathbb{C}^*)^t$ . Then

$$\begin{aligned} & (T; T)_{\infty}^{t-1} \prod_{1 \leq i < j \leq t} (x_j/x_i; T)_{\infty} (Tx_i/x_j; T)_{\infty} \\ &= \sum_{\substack{\mathbf{m} \in \mathbb{Z}^t \\ m_1 + \dots + m_t = 0}} \sum_{\sigma \in \mathcal{S}_t} \operatorname{sgn}(\sigma) \prod_{i=1}^t x_i^{tm_i + \sigma(i) - 1} T^{t \binom{m_i}{2} + m_i(\sigma(i) - 1)} \end{aligned}$$

# Schur polynomials

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The Schur polynomial associated to  $\lambda = (\lambda_1, \dots, \lambda_t)$  is:

$$s_\lambda(x_1, \dots, x_t) = \frac{\det_{1 \leq i, j \leq t} (x_i^{\lambda_j + t - j})}{\det_{1 \leq i, j \leq t} (x_i^{t - j})}.$$

# Schur polynomials

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- It is an orthonormal basis of symmetric polynomials for a well-chosen scalar product.
- It is a sum of monomials indexed by some combinatorial objects
- It is a type  $A$  character.

# Schur polynomials

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The Schur polynomial associated to  $\lambda = (\lambda_1, \dots, \lambda_t)$  is:

$$s_\lambda(x_1, \dots, x_t) = \frac{\det_{1 \leq i, j \leq t} (x_i^{\lambda_j + t - j})}{\det_{1 \leq i, j \leq t} (x_i^{t - j})}.$$

**Extension to other Weyl groups.** For instance, when  $W = \{\pm 1\}^t \times S_t$ , it becomes the symplectic Schur polynomial:

$$\text{sp}_\lambda(x_1, \dots, x_t) = \frac{\det_{1 \leq i, j \leq t} (x_i^{\lambda_j + t + 1 - j} - x_i^{-(\lambda_j + t + 1 - j)})}{\det_{1 \leq i, j \leq t} (x_i^{t + 1 - j} - x_i^{-(t + 1 - j)})}.$$

# Littlewood decomposition

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The key ingredient to reinterpret the geometric conditions and the quadratic forms into the theory of integer partitions.

# Littlewood decomposition

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Set  $\mathcal{A} \subseteq \mathcal{P}$ ,  $\mathcal{A}_{(t)} := \{\omega \in \mathcal{A} \mid \mathcal{H}_t(\omega) = \emptyset\}$ .

① **Partitions:**

$$\lambda \in \mathcal{P} \longleftrightarrow (\omega, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$$

# Littlewood decomposition

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From a lattice  
condition to  
subsets of  
partitions

Set  $\mathcal{A} \subseteq \mathcal{P}$ ,  $\mathcal{A}_{(t)} := \{\omega \in \mathcal{A} \mid \mathcal{H}_t(\omega) = \emptyset\}$ .

## 1 Partitions:

$$\lambda \in \mathcal{P} \longleftrightarrow (\omega, \underline{\nu}) \in \mathcal{P}_{(t)} \times \mathcal{P}^t$$

## 2 Double distinct partitions:

(a) for  $t$  odd:

$$\lambda \in \mathcal{DD} \longleftrightarrow (\omega, \mu, \underline{\nu}) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t-1)/2}$$

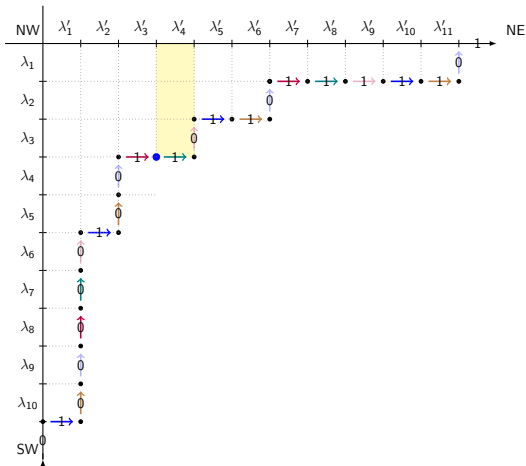
(b) for  $t$  even:

$$\lambda \in \mathcal{DD} \longleftrightarrow (\omega, \mu, \underline{\nu}, \kappa) \in \mathcal{DD}_{(t)} \times \mathcal{DD} \times \mathcal{P}^{(t/2-1)} \times \mathcal{SC}$$

Tools:  $\lambda \longleftrightarrow s(\lambda)$  bi-infinite word of 0's and 1's

# An example for $\omega \in \mathcal{DD}_{(6)}$

$\omega = (11, 6, 4, 2, 2, 1, 1, 1, 1, 1)$  and its binary correspondence



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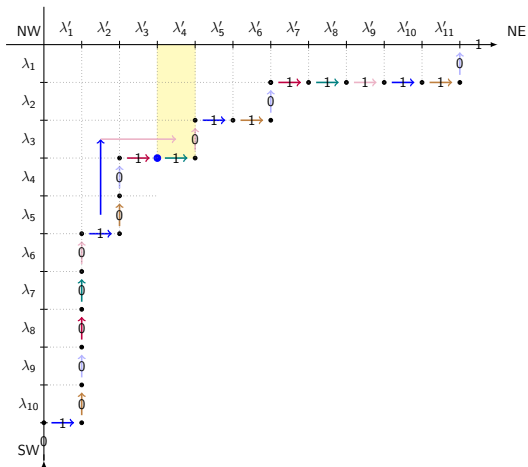
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$$\mathcal{DD}_{(t)} := \{\omega \in \mathcal{DD} \mid \mathcal{H}_t(\omega) = \emptyset\}.$$

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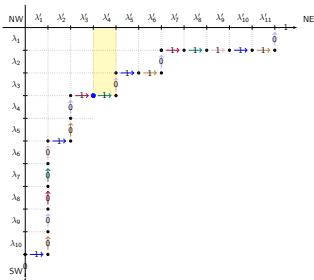
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$$s(w_0) = \cdots 000 | 111 \cdots$$

$$s(w_1) = \cdots 000 | 011 \cdots$$

$$s(w_2) = \cdots 011 | 111 \cdots$$

$$s(w_3) = \cdots 000 | 111 \cdots$$

$$s(w_4) = \cdots 000 | 001 \cdots$$

$$s(w_5) = \cdots 001 | 111 \cdots$$

$$s(\omega) = \cdots 0000001000001001 | 1011011111011111111 \cdots$$

# An example for $\omega \in \mathcal{DD}_{(6)}$

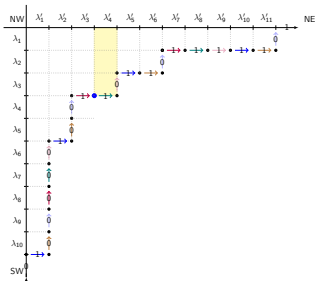
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Garvan–Kim–Stanton (1990):

$\omega \in \mathcal{P}_{(t)} \longleftrightarrow (n_0, \dots, n_{t-1}) \in \mathbb{Z}^t$  such that  $\sum_{i=0}^{t-1} n_i = 0$ . Here

$$(n_0, n_1, n_2, n_3, n_4, n_5) = (0, 1, -2, 0, 2, -1) \in \mathbb{Z}^6$$

# An example for $\omega \in \mathcal{DD}_{(6)}$

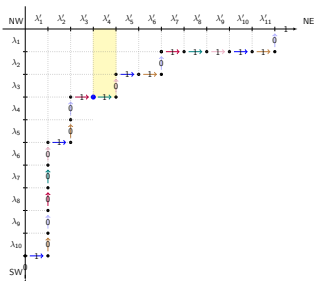
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# $t$ -cores and vectors of integers

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Garvan–Kim–Stanton (1990):

$$\omega \in \mathcal{DD}_{(2t+2)} \leftrightarrow (n_0, \dots, n_{t-1}) \in \mathbb{Z}^t .$$

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Garvan–Kim–Stanton (1990):

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A vector of integers  $(v_1, \dots, v_t) \in \mathbb{Z}^t$  is called a  $V_t$ -coding if:

- 1  $\#\{v_i - i \pmod{2t+2}, i = 1, \dots, t\} = t$
- 2  $v_i \not\equiv 0 \pmod{2t+2}, v_i \not\equiv t+1 \pmod{2t+2}$
- 3  $0 < v_t < v_{t-1} < \dots < v_1$

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These notions extend to the 7 infinite affine types.

# Hook-length formulas for $t$ -core partitions

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## Theorem [W., 2022+]

Set  $\omega \in \mathcal{DD}_{(2t+2)}$  and  $\mathbf{v} \in \mathbb{Z}^t$  its associated  $V_t$ -coding, and set  $w_i = v_i - t - 1$ . Then for any function  $\tau$  defined on  $\mathbb{Z}$  we have:

$$|\omega| = \frac{1}{2t+2} \sum_{i=1}^t w_i^2 - \frac{(2t+2)^2 + 2}{12}$$

and if we set  $\alpha_i(\omega) = \#\{s \in \omega, h_s = (2t+2) - i, \varepsilon_s = 1\}$ :

$$\begin{aligned} & \prod_{\substack{s \in \omega \\ h_s \in \mathcal{H}(\omega)}} \frac{\tau(h_s - \varepsilon_s(2t+2))}{\tau(h_s)} \\ &= \prod_{i=1}^{2t+1} \left( \frac{\tau(-i)}{\tau(i)} \right)^{\alpha_i(\omega)} \prod_{i=1}^t \frac{\tau(w_i)}{\tau(i)} \prod_{1 \leq i < j \leq t} \frac{\tau(w_i - w_j)}{\tau(j-i)} \frac{\tau(w_i + w_j)}{\tau(i+j)}. \end{aligned}$$

# Macdonald identity as sum of symplectic Schur functions

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Define  $K_T(t, \mathbf{x})$  as follows:

$$\prod_{i=1}^t \left( T x_i^2, T x_i^{-2}; T \right)_{\infty} \prod_{1 \leq i < j \leq t} \left( T x_i x_j, T x_i^{-1} x_j^{-1}, T x_i x_j^{-1}, T x_j x_i^{-1} \right)_{\infty}$$

## Theorem [W., 2022+]

The Macdonald identity for affine type  $C_t$  can be rewritten:

$$\sum_{\omega \in \mathcal{DD}_{(2t+2)}} (-1)^{d_{\omega} + |H_+|} T^{|\omega|/2} \text{sp}_{\mu}(\mathbf{x}) = (T; T)_{\infty}^t K_T(t, \mathbf{x})$$

where  $\mu$  is defined bijectively from the  $V_t$ -coding of  $\omega$ .

# A $q$ -Nekrasov–Okounkov analogue for affine type $C$

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$$K_m(T, u, q) := \prod_{r \geq 1} \frac{(1 - u^{-2} q^{2r} T^m) (1 - u q^r T^m)}{(1 - u^2 q^{2r} T^m) (1 - u^{-1} q^r T^m)} \\ \frac{(1 - u^2 q^{r+1} T^m)^{r - \lfloor r/2 \rfloor} (1 - u^{-2} q^r T^m)^{r - \lfloor r/2 \rfloor}}{(1 - q^r T^m)^{r - \lfloor r/2 \rfloor} (1 - q^{r+1} T^m)^{r - \lfloor r/2 \rfloor}}$$

## Theorem [W., 2022+]

We have:

$$\sum_{\lambda \in DD} (-uq)^{d_\lambda} T^{|\lambda|/2} \prod_{s \in \lambda} \frac{1 - u^{-2\varepsilon_s} q^{h_s - 2\varepsilon_s}}{1 - q^{h_s}} \\ \times \prod_{\substack{s \in \Delta \\ s \notin H_+}} \frac{1 + uq^{1+h_s/2}}{1 + u^{-1}q^{-1+h_s/2}} = \prod_{m \geq 1} \frac{(1 - u^{-2} T^m)}{(1 - u^{-1} T^m)} K_m(T, u, q)$$

# A $q$ -Nekrasov–Okounkov analogue for affine type $C$

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## Theorem [W., 2022+]

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$u = q^z, q \rightarrow 1 \Rightarrow C\text{-N-O type formula (Pétréolle, 2016)}$

# Perspectives

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- Links with vertex operators
- Links with Macdonald polynomials
- $q, t$  analogues ?

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Thank you for your attention.