

Binomial³

Coefficients, equivalence, and complexities

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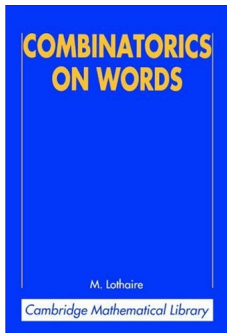
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LaBRI (Bordeaux, France)

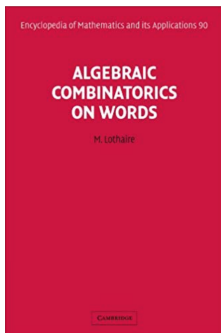
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Combinatorics on words

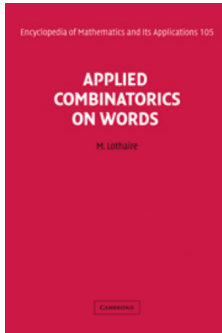
- Defined as the study of sequences of symbols
words letters
- Lothaire's books
(collective work, different sets of authors)



1983
overview



2002
overview



2005
applications

- Relatively new area of discrete mathematics
- Initiated by the Norwegian mathematician Axel Thue in 1906



(1863–1922)

- Some topics of interest
 - **combinatorial** structure of words
 - regularities and patterns in words
 - important classes/families of words (e.g., balanced, (a)periodic, automatic, regular, Lyndon, Sturmian, Arnoux-Rauzy, episturmian, de Bruijn)
 - equations on words

- Infinite words are written in **bold**
- $|w| = \#$ of letters in w
- $|w|_a = \#$ of occurrences of the letter a in w
- In a word
factor: subsequence of consecutive letters
(scattered) subword: subsequence of letters

Example: $|\mathbf{reappear}| = 8$ $|\mathbf{reappear}|_a = 2 = |\mathbf{reappear}|_e$

factor	subword
reappear	reappear
re appear	re appear
reappear 	reappear

- Length- n factors of \mathbf{x} : $\text{Fac}_n(\mathbf{x})$

How to study the combinatorial structure in words?

Factor complexity $p_{\mathbf{x}}: \mathbb{N} \rightarrow \mathbb{N}, n \mapsto \#\text{Fac}_n(\mathbf{x})$

Example: **Fibonacci** word

- Start: $f_0 = 0$ and $f_1 = 01$
- Induction: $f_{n+1} = f_n \cdot f_{n-1}$ (Fibonacci-number-like rule)

n	f_{n-1}	f_n	f_{n+1}
1	0	01	01 · 0
2	01	010	010 · 01
3	010	01001	01001 · 010
4	01001	01001010	01001010 · 01001

- Limit: $\mathbf{f} = 0100101001001 \dots$
- $p_{\mathbf{f}}$

n	$\text{Fac}_n(\mathbf{f})$	$p_{\mathbf{f}}(n)$
0	ε	1
1	0, 1	2
2	00, 01, 10	3
3	001, 010, 100, 101	4
4	0010, 0100, 0101, 1001, 1010	5

Theorem (Morse–Hedlund, 1938)

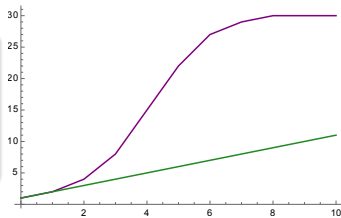
\mathbf{x} is ultimately periodic

iff $p_{\mathbf{x}}$ is bounded

iff $\exists n \in \mathbb{N}$ s.t. $p_{\mathbf{x}}(x) \leq n$

\mathbf{x} is Sturmian iff $p_{\mathbf{x}}(n) = n + 1$

(binary, aperiodic, minimal factor complexity)



$\mathbf{x} = y010101\dots$

$\mathbf{f} = 0100101001001\dots$

Variation: Count “different enough” factors: equivalence relations

First variation

- Abelian equivalence relation: $u \sim_{ab} v$ if $|u|_a = |v|_a \forall a \in A$

Example:

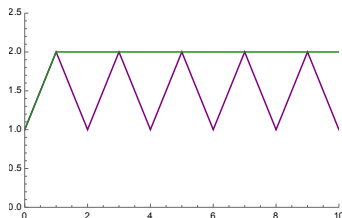
n	$\text{Fac}_n(\mathbf{f})$ up to \sim_{ab}	$a_{\mathbf{f}}(n)$
0	ε	1
1	$\{0\} \{1\}$	2
2	$\{00\} \{01, 10\}$	2
3	$\{001, 010, 100\} \{101\}$	2
4	$\{0010, 0100\} \{0101, 1001, 1010\}$	2

- Abelian complexity $a_{\mathbf{x}}: \mathbb{N} \rightarrow \mathbb{N}, n \mapsto \#(\text{Fac}_n(\mathbf{x})/\sim_{ab})$

Theorem (Coven–Hedlund, 1973)

\mathbf{x} is purely periodic iff $\exists n \in \mathbb{N} : a_{\mathbf{x}}(n) = 1$

\mathbf{x} is Sturmian iff \mathbf{x} is aperiodic and
 $a_{\mathbf{x}}(n) = 2 \forall n \geq 1$



$\mathbf{x}_2 = 010101\dots$

$\mathbf{f} = 0100101001001\dots$

Second variation

- Binomial coefficient

$$\binom{u}{v} = \# \text{ occurrences of } v \text{ as a subword of } u$$

Example: $\binom{101001}{101} = 6$ 101001 101001 101001
101001 101001 101001

Remark: $\binom{a^n}{a^k} = \binom{n}{k}$

- Binomial equivalence relation

$$k \geq 1 \quad u \sim_k v \text{ if } \binom{u}{x} = \binom{v}{x} \quad \forall x \text{ with } |x| \leq k$$

Example: $0110 \sim_2 1001$ but $0110 \not\sim_3 1001$

x	0	1	00	01	10	11	000	001
$\binom{0110}{x}$	2	2	1	2	2	1	0	0
$\binom{1001}{x}$	2	2	1	2	2	1	0	1

Remark: $\sim_1 = \sim_{ab}$; \sim_{k+1} refines \sim_k

Binomial coefficients (notably) appear in...

- Chapter 6 in Lothaire's book (1983) by Sakarovitch and Simon
- The **reconstruction problem**

Given an integer n , what is the smallest integer k such that each length- n word is uniquely determined by the set of all its length- k subwords (with multiplicities)?

Still open but bounds and several variations

(Kalashnik 1973 Krasikov–Roditty 1997 Levenshtein 2001 Dudik–Schulman 2003
Fleischmann–Lejeune–Manea–Nowotka–Rigo 2021 Richomme–Rosenfeld 2023 etc.)

- **Parikh matrices**
- **Piecewise testable languages**

regular languages defined by the presence and absence of given subwords

(Simon 1975)

- **p -group languages**

For a language $L \subseteq A^*$ and a prime p : \exists a p -group G and a monoid morphism $\alpha: A^* \rightarrow G$
s.t. $L = \alpha^{-1}(G)$ iff L is a finite Boolean combination of languages of the form
 $L_{v,r,p} = \{u \in A^* : \binom{u}{v} \equiv r \pmod{p}\}$.

(Eilenberg 1976)

- **Generalized Pascal's triangles** and related counting problems

(Leroy–Rigo–S. 2016-2017-2018 S. 2019)

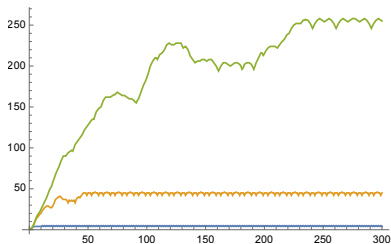
Definition (Rigo–Salimov 2015)

Let $k \geq 1$.

k -binomial complexity of \mathbf{x} : $\mathbf{b}_{\mathbf{x}}^{(k)} : \mathbb{N} \rightarrow \mathbb{N}$, $n \mapsto \#(\text{Fac}_n(\mathbf{x})/\sim_k)$

Observation: $\forall k \geq 1$, $u \sim_{k+1} v \implies u \sim_k v$, so

$$\mathbf{b}_{\mathbf{x}}^{(1)}(n) \leq \mathbf{b}_{\mathbf{x}}^{(2)}(n) \leq \dots \leq \mathbf{b}_{\mathbf{x}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{x}}^{(k+1)}(n) \leq \dots \leq \mathbf{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}$$



$b_x^{(k)}$ introduced in 2015 by Rigo and Salimov

Sturmian word \mathbf{s}	$b_s^{(k)} = p_s \forall k \geq 2$	Rigo–Salimov 2015
Tribonacci word \mathbf{z} (f.p. of $0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$)	$b_z^{(k)} = p_z \forall k \geq 2$	Lejeune–Rosenfeld–Rigo 2020 Lejeune’s PhD thesis 2021
Thue–Morse word \mathbf{t} (f.p. of $0 \mapsto 01, 1 \mapsto 10$) + wider class of words	$b_t^{(k)}$ bounded $\forall k \geq 1$	Rigo–Salimov 2015
Thue–Morse word \mathbf{t}	precise values of $b_t^{(k)}$ $\forall k \geq 1$	Lejeune–Leroy–Rigo 2020 Lejeune’s PhD thesis 2021
Generalized TM words	precise values of $b^{(2)}$	Lü–Chen–Wen–Wu 2021

	factor	abelian	k -binomial
theory (general behavior properties etc.)	rich	rich	not much is known

What is desirable

- Possible behavior of the binomial complexities
- Deduce structure on words from their binomial complexities
- Understand the binomial complexities of large classes of words

In this talk

- 2 characterizations of families of words
- 1 (or 2) question(s)

Thue–Morse word $\mathbf{t} = 01101001100101101001011001101001 \dots$
fixed point of $\varphi: 0 \mapsto 01, 1 \mapsto 10$

$b_{\mathbf{t}}^{(k)}$ is **bounded**

Theorem (Lejeune–Leroy–Rigo 2020)

$$\forall k \geq 1, b_{\mathbf{t}}^{(k)}(n) = \begin{cases} p_{\mathbf{t}}(n) & \text{if } n \leq 2^k - 1 \\ 3 \cdot 2^k - 3 & \text{if } n \equiv 0 \pmod{2^k} \text{ and } n \geq 2^k \\ 3 \cdot 2^k - 4 & \text{otherwise} \end{cases}$$

Observation:

$$\begin{pmatrix} |\varphi(0)|_0 \\ |\varphi(0)|_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} |\varphi(1)|_0 \\ |\varphi(1)|_1 \end{pmatrix}$$

Definition:

Parikh vector of w : $\Psi(w) = (|w|_a)_{a \in A}$

Parikh-constant if $\Psi(f(a)) = \Psi(f(b)) \forall a, b \in A$

Theorem (Rigo–Salimov 2015)

A fixed point of a Parikh-constant morphism has **bounded** binomial complexities.

Parikh-collinear if $\forall a, b \in A, \exists r_{a,b} \in \mathbb{Q}$ s.t. $\Psi(f(b)) = r_{a,b}\Psi(f(a))$

Example: $f: 0 \mapsto 000111, 1 \mapsto 0110$

$$\begin{aligned} \Psi(f(0)) &= \begin{pmatrix} |f(0)|_0 \\ |f(0)|_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} & \Psi(f(1)) &= \frac{2}{3}\Psi(f(0)) \\ \Psi(f(1)) &= \begin{pmatrix} |f(1)|_0 \\ |f(1)|_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{aligned}$$

Remark: Parikh-collinear iff the adjacency matrix of f has rank 1

Example: $\begin{pmatrix} |f(0)|_0 & |f(1)|_0 \\ |f(0)|_1 & |f(1)|_1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$

In terms of abelian complexity

Theorem (Cassaigne–Richomme–Saari–Zamboni 2011)

A morphism is Parikh-collinear iff it maps all infinite words to words with **bounded** $\mathbf{b}^{(1)}$.

In terms of binomial complexities

Theorem

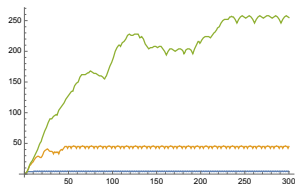
For a morphism $f: A^* \rightarrow B^*$

TFAE

- f is Parikh-collinear
- $\forall k \geq 1 \quad \forall u, v: u \sim_k v \Rightarrow f(u) \sim_{k+1} f(v)$
- $\exists k \geq 1: \forall u, v \quad u \sim_k v \Rightarrow f(u) \sim_{k+1} f(v)$
- $\forall k$ f words with **bounded** $\mathbf{b}^{(k)}$ \mapsto words with **bounded** $\mathbf{b}^{(k+1)}$
- $\exists k: f$ words with **bounded** $\mathbf{b}^{(k)}$ \mapsto words with **bounded** $\mathbf{b}^{(k+1)}$

Corollary

A fixed point of a Parikh-collinear morphism has **bounded** binomial complexities.



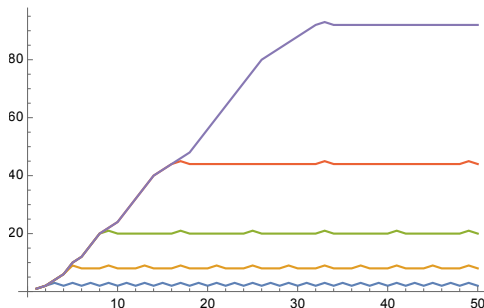
$$f: 0 \mapsto 000111, 1 \mapsto 0110$$

Words sharing binomial complexities with \mathbf{t}

Thue–Morse word $\mathbf{t} = 01101001100101101001011001101001 \dots$
fixed point of $\varphi: 0 \mapsto 01, 1 \mapsto 10$

Theorem (Lejeune–Leroy–Rigo 2020)

$$\forall k \geq 1, \mathbf{b}_{\mathbf{t}}^{(k)}(n) = \begin{cases} p_{\mathbf{t}}(n) & \text{if } n \leq 2^k - 1 \\ 3 \cdot 2^k - 3 & \text{if } n \equiv 0 \pmod{2^k} \text{ and } n \geq 2^k \\ 3 \cdot 2^k - 4 & \text{otherwise} \end{cases}$$



Definition: \mathbf{x} has \mathcal{P}_k if $\mathbf{b}_{\mathbf{x}}^{(j)} = \mathbf{b}_{\mathbf{t}}^{(j)} \forall j \in \{1, \dots, k\}$

Theorem (Richomme–Saari–Zamboni 2011)

An aperiodic word \mathbf{x} has \mathcal{P}_1 iff $\exists \mathbf{y}$ s.t. \mathbf{x} is a suffix of $\varphi(\mathbf{y})$.

Generalization of one direction

Theorem

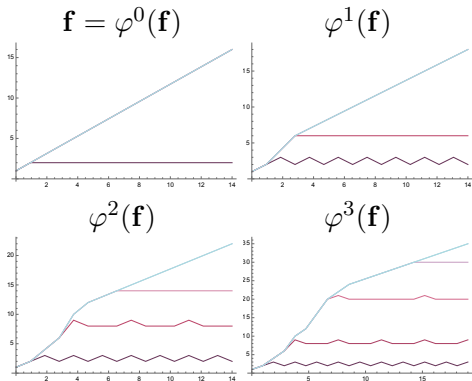
Let $k \geq 1$ and let \mathbf{y} be an **aperiodic** binary word. Then $\varphi^k(\mathbf{y})$ has \mathcal{P}_k .

Example:

Fibonacci word

$\mathbf{f} = 010010100100\dots$

(fixed point of $0 \mapsto 01, 1 \mapsto 0$)



A converse?

Definition: \mathbf{x} is **recurrent** if each factor appears infinitely often

Example:

- Thue–Morse is recurrent
- Fibonacci is recurrent
- $101001000100001 \dots = 1010^210^310^41 \dots$ is not recurrent
e.g. 101 appears only once

Theorem

Let \mathbf{x} be a **recurrent** binary word having \mathcal{P}_k for some $k \geq 1$.
Then \mathbf{x} is a suffix of the word $\varphi^k(\mathbf{y})$ for some aperiodic binary word \mathbf{y} .



Goal: Get rid of the assumption on recurrence

Question B

Binomial complexities are increasingly nested

$$\mathbf{b}_x^{(1)}(n) \leq \mathbf{b}_x^{(2)}(n) \leq \dots \leq \mathbf{b}_x^{(k)}(n) \leq \mathbf{b}_x^{(k+1)}(n) \leq \dots \leq \mathbf{p}_x(n) \quad \forall n \in \mathbb{N}$$

Can the factor complexity coincide with any binomial complexity?

Notation: For two functions $f, g: \mathbb{N} \rightarrow \mathbb{N}$, we write $f \prec g$ when

- $f(n) \leq g(n) \quad \forall n \in \mathbb{N}$ and
- $f(n) < g(n)$ for infinitely many $n \in \mathbb{N}$

Question B (Stabilization)

For $k \geq 1$, does there exist \mathbf{w}_k s. t.

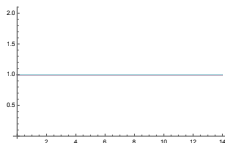
$$\mathbf{b}_{\mathbf{w}_k}^{(1)} \prec \mathbf{b}_{\mathbf{w}_k}^{(2)} \prec \dots \prec \mathbf{b}_{\mathbf{w}_k}^{(k-1)} \prec \mathbf{b}_{\mathbf{w}_k}^{(k)} = \mathbf{p}_{\mathbf{w}_k}?$$

Inspired by Lejeune's PhD thesis

First (naive) answer: periodic words

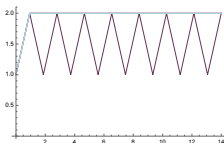
Examples:

$$\mathbf{w} = 000\dots$$



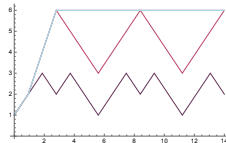
$$b_w^{(1)} = p_w$$

$$\mathbf{w} = 010101\dots$$



$$b_w^{(1)} \prec b_w^{(2)} = p_w$$

$$\mathbf{w} = 011001011001\dots$$



$$b_w^{(1)} \prec b_w^{(2)} \prec b_w^{(3)} = p_w$$



But these words have

- **bounded** complexities and
- rather **simple** structure

Let's propose more "interesting" words*

*No offense, I personally love periodic words

Recall

Theorem

Let $k \geq 1$ and let \mathbf{y} be an aperiodic binary word.
Then $\varphi^k(\mathbf{y})$ has \mathcal{P}_k .

Corollary

Let $k \geq 1$ and let \mathbf{y} be an aperiodic binary word.
For $\mathbf{x} = \varphi^k(\mathbf{y})$, we have $\mathbf{b}_x^{(1)} \prec \mathbf{b}_x^{(2)} \prec \dots \prec \mathbf{b}_x^{(k)} \prec \mathbf{b}_x^{(k+1)}$.

Answer to Question B

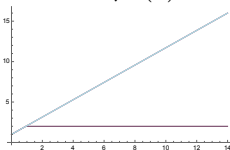
Theorem

Let $k \geq 1$ and let \mathbf{s} be a **Sturmian** word.
For $\mathbf{s}_k = \varphi^k(\mathbf{s})$, we have $\mathbf{b}_{\mathbf{s}_k}^{(1)} \prec \mathbf{b}_{\mathbf{s}_k}^{(2)} \prec \dots \prec \mathbf{b}_{\mathbf{s}_k}^{(k+1)} \prec \mathbf{b}_{\mathbf{s}_k}^{(k+2)} = \mathbf{p}_{\mathbf{s}_k}$.

Example: Fibonacci word $\mathbf{f} = 010010100100\dots$

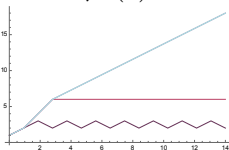
(fixed point of $0 \mapsto 01, 1 \mapsto 0$)

$$\mathbf{f} = \varphi^0(\mathbf{f})$$



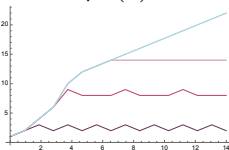
$$b^{(2)} = p$$

$$\varphi^1(\mathbf{f})$$



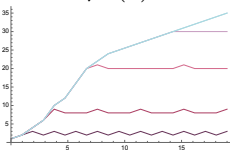
$$b^{(3)} = p$$

$$\varphi^2(\mathbf{f})$$



$$b^{(4)} = p$$

$$\varphi^3(\mathbf{f})$$



$$b^{(5)} = p$$

True for Sturmian words

(Rigo–Salimov 2015)



But... bounded $b_{s_k}^{(1)}, \dots, b_{s_k}^{(k+1)}$

Question C

For $k \geq 1$, does there exist \mathbf{w}_k s. t. $\mathbf{b}_{\mathbf{w}_k}^{(1)}$ is **unbounded** and

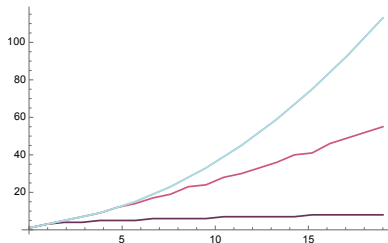
$$\mathbf{b}_{\mathbf{w}_k}^{(1)} \prec \mathbf{b}_{\mathbf{w}_k}^{(2)} \prec \dots \prec \mathbf{b}_{\mathbf{w}_k}^{(k-1)} \prec \mathbf{b}_{\mathbf{w}_k}^{(k)} = \mathbf{p}_{\mathbf{w}_k}?$$

Answer for $k = 3$:

$\mathbf{h} = 01121221222122221222 \dots$

(fixed point of $0 \mapsto 01, 1 \mapsto 12, 2 \mapsto 2$)

- $\mathbf{b}_{\mathbf{h}}^{(1)}$ **unbounded**
- $\mathbf{b}_{\mathbf{h}}^{(1)} \prec \mathbf{b}_{\mathbf{h}}^{(2)} \prec \mathbf{b}_{\mathbf{h}}^{(3)} = \mathbf{p}_{\mathbf{h}}$



Open question: What about larger values of k ?

If time permits: Question A

Binomial complexities are increasingly nested

$$\mathbf{b}_{\mathbf{x}}^{(1)}(n) \leq \mathbf{b}_{\mathbf{x}}^{(2)}(n) \leq \dots \leq \mathbf{b}_{\mathbf{x}}^{(k)}(n) \leq \mathbf{b}_{\mathbf{x}}^{(k+1)}(n) \leq \dots \leq \mathbf{p}_{\mathbf{x}}(n) \quad \forall n \in \mathbb{N}$$

Can the non-equality happen **infinitely** many times?

Question A

Does there exist \mathbf{w} s.t. $\mathbf{b}_{\mathbf{w}}^{(k)} \prec \mathbf{b}_{\mathbf{w}}^{(k+1)} \quad \forall k \geq 1$?

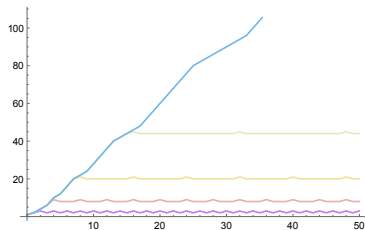
Inspired by Lejeune's PhD thesis

Answers to Question A

Thue–Morse word \mathbf{t}

- bounded $b_{\mathbf{t}}^{(k)}$

(Lejeune–Leroy–Rigo 2020)

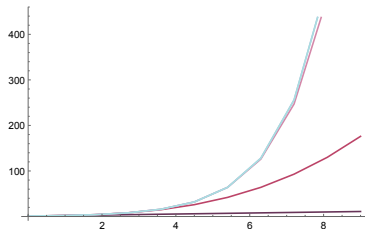


Binary Champernowne word

$$\mathbf{c} = 011011100101110111 \dots$$

(the concatenation of all binary representations)

- unbounded $b_{\mathbf{c}}^{(1)}$
- not morphic
- not uniformly recurrent



A festival of answers: more “structured” words

Answers to Question A

$$\mathbf{v} = \tau(g^\omega(a))$$

$$g: a \mapsto a0\alpha, 0 \mapsto 01, 1 \mapsto 10, \alpha \mapsto \alpha^2$$

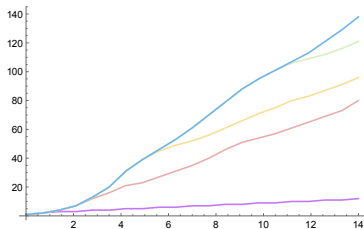
$$g^\omega(a) := \lim_{n \rightarrow \infty} g^n(a)$$

$$\tau: a \mapsto \varepsilon, 0 \mapsto 0, 1 \mapsto 1, \alpha \mapsto 1$$

- unbounded $\mathbf{b}_\mathbf{v}^{(1)}$
- binary
- morphic

(morphic: the image of a fixed point of a morphism

under a coding)



binomial · binomial · binomial

Grillenberger's word

$$\mathbf{w} = 0100010101100111 \dots$$

- unbounded $\mathbf{b}_\mathbf{w}^{(1)}$
- binary
- uniformly recurrent

(uniformly recurrent: every factor occurs infinitely many times within bounded gaps).

Construction (Grillenberger 1973)

Start: $D_0 = \{0, 1\}$ with $0 < 1$

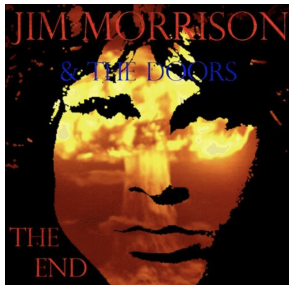
Induction:

w_n := concatenation of words in D_n in lexicographic order

Set $D_{n+1} := w_n D_n^2$

Output: uniformly recurrent

$$\mathbf{w} = \lim_{n \rightarrow \infty} w_n$$



- Characterization 1

A f. p. of a Parikh-collinear morphism has bounded binomial complexities.

- Characterization 2

An aperiodic and recurrent binary word has \mathcal{P}_k ($k \geq 1$) iff it is the suffix of $\varphi^k(\mathbf{y})$ for some aperiodic \mathbf{y} .

- Question A

$\exists \mathbf{w}$ s.t. $\mathbf{b}_{\mathbf{w}}^{(k)} \prec \mathbf{b}_{\mathbf{w}}^{(k+1)} \forall k \geq 1$?

- Question B

For $k \geq 1$, $\exists \mathbf{w}_k$ s. t. $\mathbf{b}_{\mathbf{w}_k}^{(1)} \prec \mathbf{b}_{\mathbf{w}_k}^{(2)} \prec \dots \prec \mathbf{b}_{\mathbf{w}_k}^{(k-1)} \prec \mathbf{b}_{\mathbf{w}_k}^{(k)} = \mathbf{p}_{\mathbf{w}_k}$?

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