The algebra of cubic constraints for b-deformed constellations

joint work with V. Bonzom

Victor Nador

LIPN - LaBRI - Univ. de Bordeaux

30 Janvier 2023

Intro & Outline

A tour around the enumeration of maps and its generalizations:

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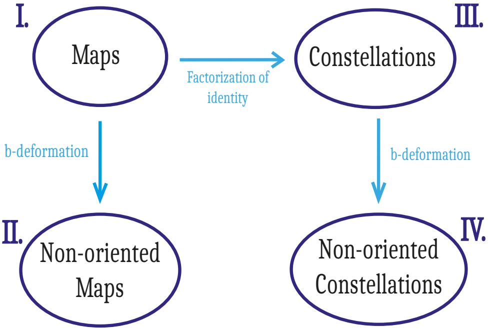

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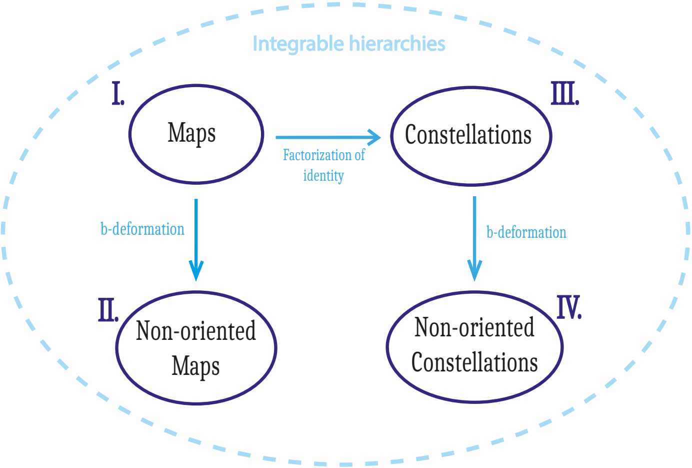

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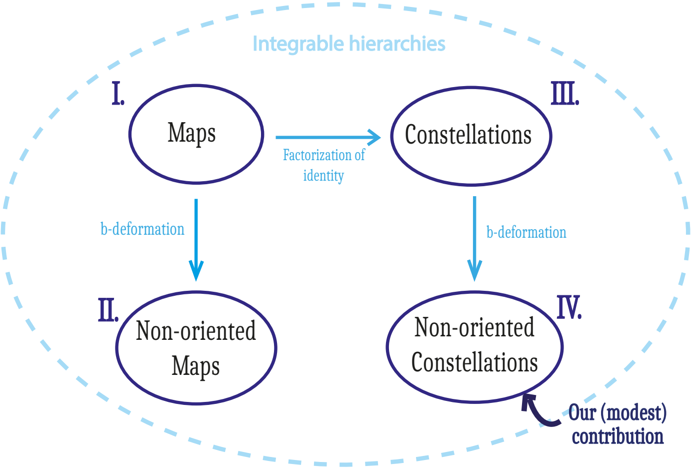

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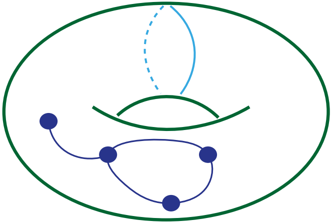

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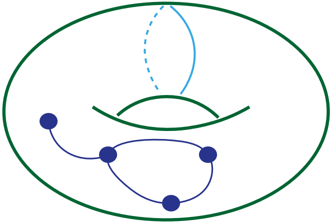

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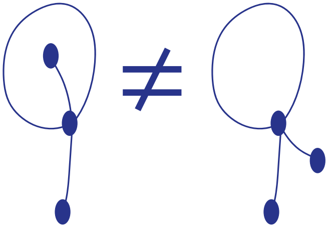

Same graphs but different maps.

Maps: an algebraic definition

Definition

A map is a triple of permutation (σ, α, ϕ) *such that*

- α *is an involution*
- $\sigma\alpha\phi^{-1} = id_{\mathfrak{S}_{2n}}$

where n is the number of edges of the map.

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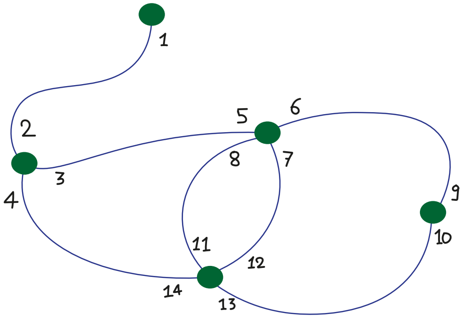

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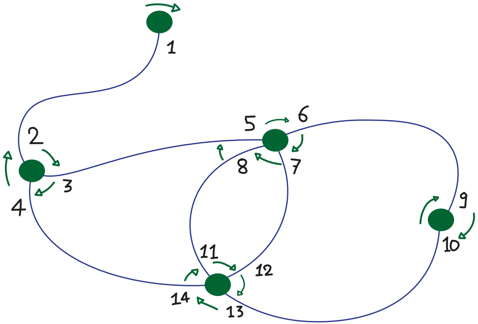

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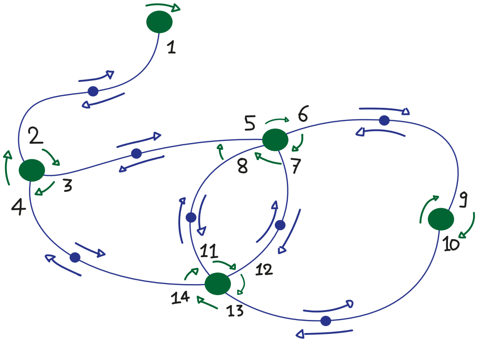

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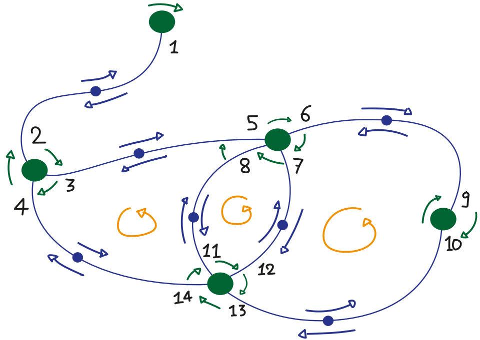

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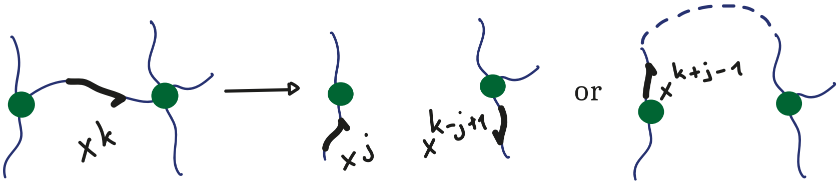
$$\phi = (1\,3\,6\,10\,14\,2)(8\,12)(5\,4\,11)(7\,13\,9e)$$

$$\alpha\sigma = \phi$$

Rooted (planar) map enumeration via Tutte equation

- We denote $F^{(0)}(t, p_k, x) = \sum_{k,n \geq 0} t^n x^k F_{n,k}(p_k)$ the generating function for rooted maps where
  - t counts the number of edges of the map
  - p_k counts the number of faces of length k
  - x tracks the size of the root face
- The Tutte equations connects the different term of the expansion of $F^{(0)}$.

$$F^{(0)}_{n,k+1} = \sum_{\substack{n_1+n_2=n-1 \\ 0 \leq j \leq k-1}} F^{(0)}_{n_1,j} F^{(0)}_{n_2,k-j-1} + \sum_{j \geq 1} t_j F^{(0)}_{n,k+j-1}$$



The Virasoro algebra

For the generating function $Z(N, p_k) = \sum_{g \geq 0} N^{2-2g} Z^{(g)}(N, p_k)$, we have the **Virasoro constraints**:

$$L_k = \sum_{j \geq 2} t_j (j+k) \frac{\partial}{\partial t_{j+k}} + \frac{1}{N^2} \sum_{j=1}^{k-1} j(k-j) \frac{\partial}{\partial t_j} \frac{\partial}{\partial t_k} \tag{1}$$

$$L_k \cdot Z(N, p_k) = 0 \tag{2}$$

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They form the **Virasoro algebra**:

$$[L_n, L_m] = (n-m) L_{n+m} \tag{3}$$

The algebra of symmetric function: power sums

For an infinite set of variables $\{x_i\}_{i\geq 1}$, a basis for the symmetric polynomials in x_i can be constructed using their **power-sum**

$$p_k(x_i) = \sum_{i\geq 1} x_i^k \tag{4}$$

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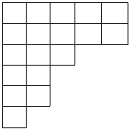

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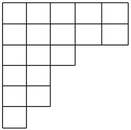

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This basis is orthogonal w.r.t the Hall scalar product

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda \qquad z_\lambda = \prod_{i\geq 1} m_i! i^{m_i} \tag{6}$$

The algebra of symmetric function: Schur functions

The unique orthogonal $\mathbb{Z}$-basis is given by the **Schur functions**. The one-part Schur [1] functions are obtained from the generating series

$$\exp(\sum_{j\geq 1}\frac{1}{j}x_j t^j) = \sum_{k\geq 1} s_k(x_i)t^k \tag{7}$$

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Change of basis from power-sums:

$$s_\lambda = \sum_\mu \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu \tag{9}$$

where $\chi_\lambda(\mu)$ is the character of the symmetric group in the irrep λ evaluated on conjugacy class μ.

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As a τ-function of the KP hierarchy

The generating function $Z(N, p_k)$ is a τ-function of the KP hierarchy

$$Z(N, p_k) = \sum_{\lambda} \underbrace{det\left(\int_{\mathbb{R}} dx e^{-\frac{x^2}{2}} x^{\lambda_i + N - i + j - 1}\right)}_{\text{satisfy Plucker relations}} s_\lambda(p_k) \tag{10}$$

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- New recurrence relations on coefficients of $F(N, p_k)$ e.g.

$$F_{2,2} - F_{3,1} + \frac{1}{2}F_{1,1}^2 + \frac{1}{12}F_{1,1,1,1} = 0 \quad \text{with} \quad F_i = \frac{\partial F}{\partial p_i} \tag{11}$$

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- Efficient for numerical computations [Goulden-Jackson 08]

$$(n+1)T_{g,n} = 4n(3n-2)(3n-4)T_{g-1,n-1} + 4\sum_{\substack{i+j=n-2\\ g_1+g_2=g}} (3i+2)(3j+2)T_{g_1,i}T_{g_2,i} \tag{12}$$

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- Other applications: local limit of maps of high genus [Budzinski-Louf 20]

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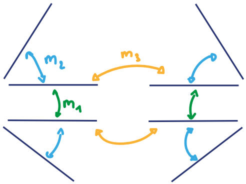
- Genus becomes a half-integer $g \in \mathbb{N}/2$

The b-deformation at $b = 1$

Most of the results obtained for orientable maps can be extended to the non-oriented case.

- Permutations are replaced by **matchings**



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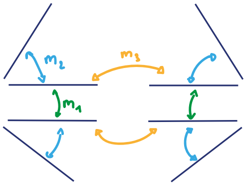

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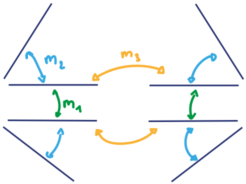

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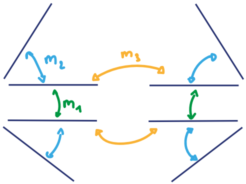

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- Non-oriented maps are associated to representation of the Gelfand pair $H_n \backslash \mathfrak{S}_{2n} / H_n$.
- From the point of view of integrable hierarchies, at $b = 1$ the hierarchy changes to the **BKP hierarchy**. [Bonzom-Chapuy-Dolega 21]

For general b

The introduction of b can be understood as a deformation of the algebra of symmetric function. It corresponds to **Jack polynomials** $J_\lambda^{(b)}$. In particular $J_\lambda^{(0)} = s_\lambda$ and $J_\lambda^{(1)} = Z_\lambda$.

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However..

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$\rightarrow$ A combinatorial interpretation of the b-deformation was conjectured in [Goulden-Jackson 96] under the name **Matching-Jack** and **Hypermap-Jack** conjectures.

Constellations (or Hurwitz numbers) [Lando-Zvonkine 04]

Idea: Generalise the factorisation of identity to $d+1$ permutation

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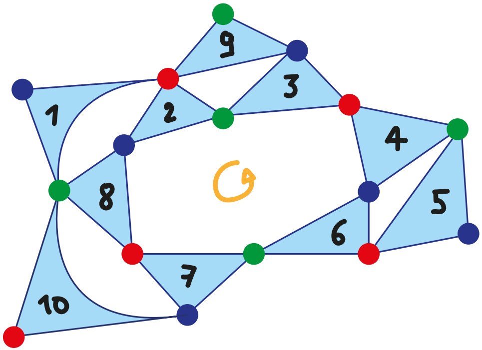

$\sigma_1 =$(2 1 9)(3 4)(5 6)
(7 8)(10)
$\sigma_2 =$(1 8 10)(6 7)(5 4)
(2 3)(9)
$\sigma_3 =$(1)(2 8)(3 9)
(7 10)(4 6)(5)

$\phi =$(2)(3 5 10 1)(7)
(9)(6)(8 4)

Weighted constellations (or Hurwitz number)

Adding weights

- t for the hyperedges
- $\{t^{(i)}\}_{k\geq 1}$ for vertices of color i
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The generating function of constellations is

$$Z^{(d+1)}(t_k^{(i)}, p_k, t) = \sum_n \frac{t^n}{n!} \sum_{\lambda^{(1)},..,\lambda^{(d)},\mu} h_{\lambda^{(1)},..,\lambda^{(d)},\mu} t^{(1)}_{\lambda_1} .. t^{(d)}_{\lambda_d} p_\mu \tag{13}$$

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- The coefficients $h_{\lambda^{(1)},..,\lambda^{(d)},\mu}$ are **Hurwitz numbers**
- They count ramified covering of the sphere with $d+1$ ramification points
- Maps are obtained for $d=2$ and $t_k^{(2)} = \delta_{2,k}$

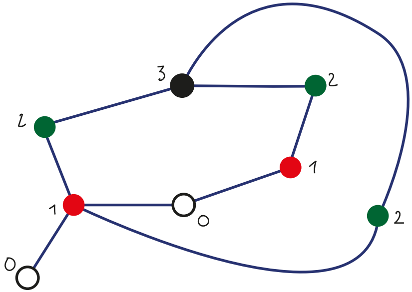
Weighted constellations (or Hurwitz number)

It is a τ-function of the KP hierarchy:

$$Z^{(d+1)}(t_k^{(i)}, p_k, t) = \sum_{\lambda} \frac{t^{|\lambda|}}{\mathrm{hook}(\lambda)^{1-d}} s_\lambda(t^{(1)})..s_\lambda(t^{(d)}) s_\lambda(p) \tag{14}$$

- Shown recently to satisfy **topological recursion** [Bonzom-Chapuy-Charbonnier-Garcia-Failde 22]
- Some open questions: what is the algebra of constraints satisfied by $Z^{(d+1)}(t_k^{(i)}, p_k, t)$?

Non-oriented constellations [Chapuy-Dolega 22]

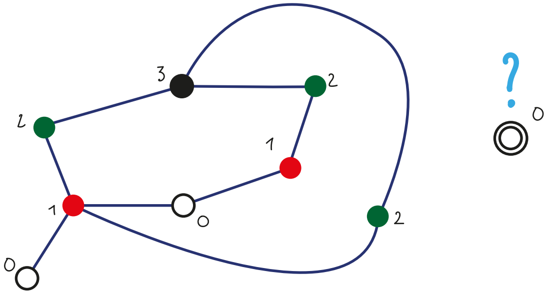


Two infinite families of weights:

- q_k for vertices of color 0 and valency k
- p_k for faces of length k

and weight u_i for each vertex of color i.

Question: How to add a vertex of color 0 with valency k in a constellation ?



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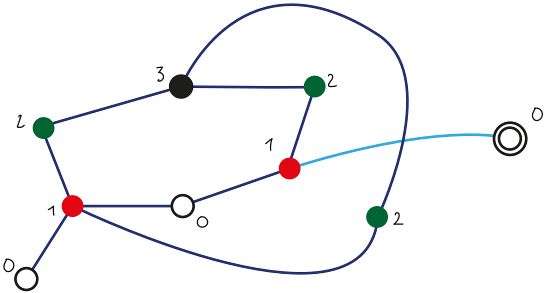

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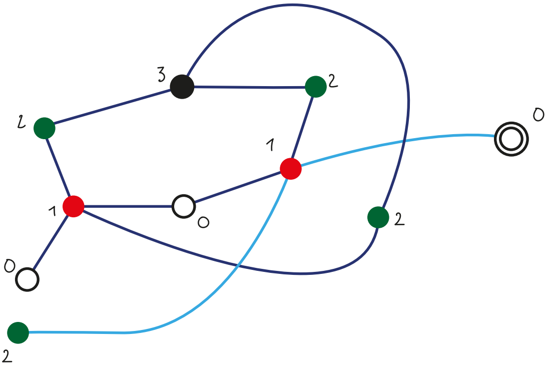

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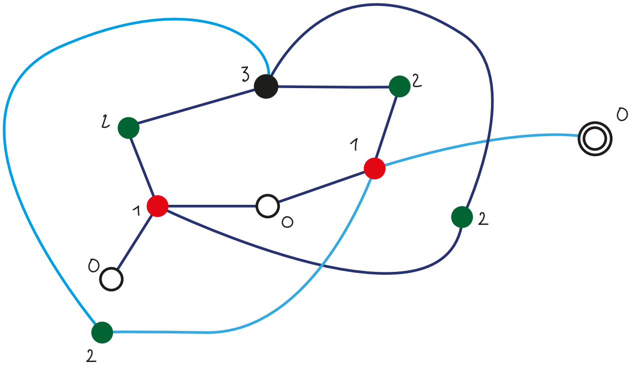

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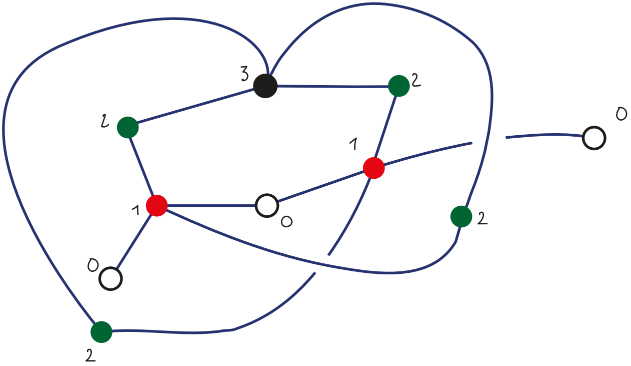

This process can be repeated k times, increasing the size of the constellations by k.

Evolution equation for constellations

Introduce operators

$$J_k = \begin{cases} k\frac{\partial}{\partial p_k} & \text{for } k > 0 \\ p_{-k} & \text{for } k < 0 \end{cases} \tag{15}$$

and the **cut-and-join operator**

$$\Lambda_X = \sum_{i \neq 0} X_i J_i \tag{16}$$

Then the operator creating the complete right-path is

$$M^{(d)}(u_1, ..., u_d) = X_+ \prod_{i=1}^{d} (\Lambda_X + u_i) \tag{17}$$

Evolution equation for constellations

Introduce operators

$$J_k = \begin{cases} k\frac{\partial}{\partial p_k} & \text{for } k > 0 \\ p_{-k} & \text{for } k < 0 \end{cases} \tag{15}$$

and the **cut-and-join operator**

$$\Lambda_X = \sum_{i \neq 0} X_i J_i \tag{16}$$

Then the operator creating the complete right-path is

$$M^{(d)}(u_1, ..., u_d) = X_+ \prod_{i=1}^{d} (\Lambda_X + u_i) \tag{17}$$

Using these operator, we can write an **evolution equation** satisfied by the generating function

$$t\frac{\partial}{\partial t} F^{(d+1)} = \sum_{k \geq 1} \Theta_X t^k q_k {M^{(d)}}^k x_0 \cdot F^{(d+1)} \tag{18}$$

Exctracting constraints from the evolution equation

Adapting a Lemma from [Bonzom-Chapuy-Dolega 22] , we can extract constraints of the partition function from the evolution equation

Lemma

If the following are satisfied

- $\left(t\frac{\partial}{\partial t} + t^k A\right) \cdot F = 0$ *for* A *independent of* t *with* $A = \sum_{j \geq 1} p_j \left(\frac{j}{t^k}\frac{\partial}{\partial p_j} + L_k^{(d)}\right)$
- F *satisfies* $\frac{\partial}{\partial p_i} F_{|_{t=0}} = 0$
- $L_k^{(d)}$ *are close under bracketing i.e. there exists differential operator* D_{ij}^k *such that* $\left[L_i^{(d)}, L_j^{(d)}\right] = D_{ij}^k M_k^{(d)}$

Then we have $L_k^{(d)} \cdot F = 0$

$\rightarrow$ This requires to compute the commutator of operators $L_k^{(d)}$!

Commutation relations for $L_k^{(d)}$

- The difficulty to compute commutators of $L_k^{(d)}$ depends on the order of J in the evolution equation.

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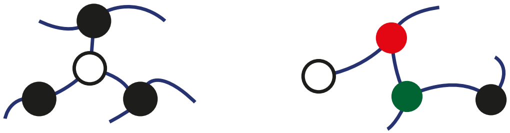
- If J appear quadratically (i.e. for maps and bipartite maps), then we get the **Viraroso algebra**.

- First unknown cases are cubic in J:
  1. 4-constellations, obtained with $q_k = \delta_{k,1}$ and $M^{(3)}$.
  2. Bipartite maps with 3-valent white vertices, obtained with $q_k = \delta_{k,3}$ and ${M^{(1)}}^3$.



Commutation and constraints for 3-constellations

For 3-constellations, we get the following commutator

$$\begin{aligned}\left[M_n^{(3)}, M_m^{(3)}\right] &= 2(1+b)(n-m)\sum_{k\geq M} J_{M+\mu-1-k}M_k^{(3)} \\ &+ (1+b)\,\mathrm{sgn}(n-m)\sum_{k=\mu}^{M-1}(2k-3\mu+1)J_{M+\mu-1-k}M_k^{(3)} \\ &+ (1+b)(n-m)\sum_{k=M}^{M+\mu-1} J_{M+\mu-1-k}M_k^{(3)} \\ &+ (1+b)b(n-m)(n+m)M_{n+m-1}^{(3)}\end{aligned} \tag{19}$$

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\end{aligned} \tag{19}
$$

- This allows us to prove that $\{M_n^{(3)}\}_{n\geq 1}$ are the constraints satisfied by the partition function for all b.
- Q: What is this algebra ? Can we find variables such that the commutator takes a "nicer" form ? Conjecture: It is a W_3 algebra in the orientable case ?

Some questions

- Can we generalise this result to arbitrary d ?
  - Study constraints in the orientable case for generic d via integrability (WIP)

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  - Study constraints in the orientable case for generic d via integrability (WIP)
- Is there combinatorial methods to interpret
  - The equations of the KP hierarchy ?
  - The b-weight for arbitrary values of b ?
- Does integrability properties hold for arbitrary values of b ?
  - Yes : Goulden and Jackson conjecture, constraints form an algebra for maps and in the case studied here
  - No : a priori no representation theory and no Lie group matrix model

Thank you for your attention !