MINOR-UNIVERSAL GRAPH FOR BOUNDED GENUS GRAPHS

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February 1st, 2023
Graph Minors

Introduction

On planar graph

Surfaces of bounded genus

Proof

Conclusion
Graph Minors

$H$ is a minor of $G$ if $H$ can be obtained from $G$ by
- vertex deletion
- edge deletion
- edge contraction
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**Minor-universal graph for bounded genus graphs**
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Minor-universality

Let $\mathcal{F}$ be a family of finite graphs.
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Let $\mathcal{F}$ be a family of finite graphs. $U$ is minor-universal for $\mathcal{F}$ if any $G \in \mathcal{F}$, $G$ is a minor of $U$. → What is the order of a smallest $U$?
Minor-universality

Let $\mathcal{F}$ be a \textit{family of finite graphs}. $U$ is \textit{minor-universal} for $\mathcal{F}$ if any $G \in \mathcal{F}$, $G$ is a minor of $U$.

$\rightarrow$ What is the order of a smallest $U$ given a certain \textit{property}?
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Let $F$ be a family of finite graphs. $U$ is minor-universal for $F$ if any $G \in F$, $G$ is a minor of $U$.

→ What is the order of a smallest $U$ given a certain property?

Examples:

▶ Order of a tree minor-universal for the trees on $n$ vertices?
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→ [O. Bodini ’03] $\Omega(n \log n)$ and $O(n^{1.984\ldots})$. 
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- Order of a planar graph minor-universal for the planar $n$-graphs?
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- Order of a tree minor-universal for the trees on $n$ vertices?
  → [O. Bodini ’03] $\Omega(n \log n)$ and $O(n^{1.984...})$.

- Order of a planar graph minor-universal for the planar $n$-graphs?
  → [RST ’94] $O(n^2)$ with the $2n \times 2n$-grid.
Genus of graphs

Simple connected graphs on compact connected orientable surfaces.
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Simple connected graphs on compact connected orientable surfaces.
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\[ g = 0 \]

\[ g = 1 \]

Simple connected graphs on compact connected orientable surfaces.
Genus of graphs

$g = 0$

$g = 1$

$g = 2$

$g = 3$

$g = 4$

... 

Simple connected graphs on compact connected orientable surfaces.
Result
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_N. Robertson, P. Seymour, R. Thomas, 1994_

For all $n$, there is a **planar** graph on $O(n^2)$ vertices minor-universal for the **planar $n$-graphs**.
## Result

### N. Robertson, P. Seymour, R. Thomas, 1994

For all $n$, there is a planar graph on $O(n^2)$ vertices minor-universal for the planar $n$-graphs.

### C. Gavoille, C. H., 2023+

For all $n, g \geq 1$, there is a graph with genus $g$ and $O(g^2(n + g)^2)$ vertices minor-universal for the $n$-graphs with genus $g$. 
Minor-universal grid for planar graphs
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Step 1: Getting a Hamiltonian planar major

Every planar $n$-graph is a minor of a Hamiltonian planar $2n$-graph.
Minor-universal grid for planar graphs

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Step 2: Getting the grid

Every Hamiltonian planar $N$-graph is a minor of the $N \times N$-grid.
Construction of a Hamiltonian major

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![Graph Diagram]
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Polygonal schema for surfaces

\[ \sigma = a_1 \bar{a}_1 b_1 \bar{b}_1 \ldots a_g \bar{a}_g b_g \]
Polygonal schema for surfaces

Classification Theorem (subpart)

Every compact, connected, orientable surface of genus $g \geq 1$ is homeomorphic to a polygonal surface given by the following polygonal schemata, called canonical:

$$\sigma = a_1 \bar{a}_1 b_1 \bar{b}_1 ... a_g b_g \bar{a}_g \bar{b}_g$$

[Éric Colin de Verdière]
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Polygonal embedding for graphs

\[ G \text{ has a polygonal embedding characterized by:} \]

1. Borders of the \( k \)-gon respecting the schemata \( \sigma \).
2. At most \( N \) external vertices on each border (\( p_1, p_2, p_3, q_1, q_2, q_3 \)).
3. At most \( n \) internal vertices (1, 2, 3, 4, 5).
Polygonal embedding for graphs

\[ G \] has a polygonal embedding characterized by:

- \( \sigma \leq N \) external vertices on each border (\( p_1, p_2, p_3, q_1, q_2, q_3 \)).
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- $\rightarrow, \leftarrow$: borders of the $k$-gon respecting the schemata $\sigma$. 
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$G$ has a polygonal embedding of type $P_{\sigma,k}(N,n)$:
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Polygonal embedding for graphs

F. Lazarus, M. Pocchiola, G. Vegter, A. Verrous, 2001
(version E. Colin de Verdière)

Let $G$ be an $n$-graph embedded on an orientable surface $S$ of genus $g \geq 1$.
One can compute in $O(g(g + n))$ time a canonical polygonal schema of $S$ such that each border intersects at most 4 times each edge of $G$.

Corollary

Every $n$-vertex graph of genus $g \geq 1$ has a polygonal embedding $P_{\sigma,4g}(4(3n + 6g - 6), n)$ with $\sigma = a_1b_1\bar{a}_1\bar{b}_1 \cdots a_g b_g \bar{a}_g \bar{b}_g$. 
Sketch of the proof

C. Gavoille, C. H., 2023+

For all $n, g \geq 1$, there is a graph with genus $g$ and $O(g^2(n + g)^2)$ vertices minor-universal for the $n$-graphs with genus $g$. 
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For all $n, g \geq 1$, there is a graph with genus $g$ and $O(g^2(n + g)^2)$ vertices minor-universal for the $n$-graphs with genus $g$.

Technical theorem

$\forall \sigma, k, N, n$, there is a graph with a polygonal embedding $P_{\sigma,k}(N + 2n, k^2(N + 2n)^2)$, minor universal for the graphs with a polygonal embedding $P_{\sigma,k}(N, n)$. 

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Step 1/2: getting an outerplanar-like embedding

Let $G$ be a graph with a polygonal embedding $P_{\sigma,k}(N, n)$. $G$ is minor of a graph with polygonal embedding $P_{\sigma,k}(N + 2n, 0)$. 
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![Diagram of the graph $P_{\sigma,k}(N, n)$ and its modifications to create an outerplanar-like embedding.]
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Let $G$ be a graph with a **polygonal embedding** $P_{\sigma,k}(N, n)$. $G$ is minor of a graph with polygonal embedding $P_{\sigma,k}(N + 2n, 0)$. 
step 2/2: grid-like minor-universal graph

There is a graph with \textit{polygonal embedding} \( P_{\sigma,k}(N', \frac{(kN')^2}{2}) \) minor-universal for the graphs with \textit{polygonal embedding} \( P_{\sigma,k}(N', 0) \).
step 2/2: grid-like minor-universal graph

There is a graph with polygonal embedding $P_{\sigma,k}(N', (kN')^2)$ minor-universal for the graphs with polygonal embedding $P_{\sigma,k}(N', 0)$. 
There is a graph with polygonal embedding $P_{\sigma,k}(N', \frac{(kN')^2}{2})$ minor-universal for the graphs with polygonal embedding $P_{\sigma,k}(N', 0)$.

step 2/2: grid-like minor-universal graph
Conclusion

For all $n, g \geq 1$, there is a graph with genus $g$ and $O(g^2(n + g)^2)$ vertices minor-universal for the $n$-graphs with genus $g$.

Relies on the fact that graphs with oriented genus $g$ have polygonal embeddings with same parameters.
Conclusion

C. Gavoille, C. H., 2023+

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Relies on the fact that graphs with oriented genus $g$ have polygonal embeddings with same parameters.

More generally:
We construct a minor-universal graph for any family of graphs with polygonal embedding having same parameters.
What about non-orientable surfaces?

F. Lazarus, M. Pocchiola, G. Vegter, A. Verrous, 2001 (version E. Colin de Verdière)

Let $G$ be an $n$-graph embedded on an orientable surface $S$ of genus $g \geqslant 1$.
There is a canonical polygonal schema $P$ of $S$ such that each border of $P$ intersects at most 4 times each edge of $G$.

N. Fuladi, A. Hubard, A. de Mesmay, 2023+

Let $G$ be an $n$-graph embedded on a non-orientable surface $S$ of genus $g \geqslant 1$.
There is a canonical polygonal schema of $S$ such that each border intersects at most 30 times each edge of $G$. 

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What about lower bounds?

Open even for planar graphs!

Open problem

Is there a sub-quadratic planar graph containing all the planar $n$-graphs as minor?
Thank you!