

A Pitman's Theorem for a Brownian motion in an affine Weyl chamber

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I- A Pitman's Theorem (1975)

If $\{b_t, t \geq 0\}$ a standard real Brownian motion then

$$\{\mathcal{P}b_t := b_t - 2 \inf_{0 \leq s \leq t} b_s, t \geq 0\}$$

is a *BES(3)* process

I- A Pitman's Theorem (1975)

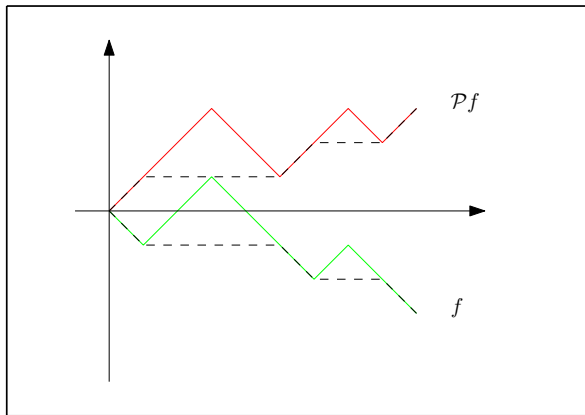


Figure – Pitman's transform $\mathcal{P}f(t) = f(t) - 2 \inf_{s \leq t} f(s)$

I- A Pitman's Theorem (1975)

BES(3) : Markov process with transition densities

$$q_t(x, y) = \frac{h(y)}{h(x)} p_t^0(x, y), \quad x, y > 0, t > 0,$$

$$p_t^0(x, y) = p_t(x, y) - p_t(x, -y), \quad h(x) = x.$$

I- A Pitman's Theorem (1975)

Remarks :

1. $BES(3)$ lives in \mathbb{R}_+
2. \mathbb{R}_+ is a fundamental domain for $W = \langle s_0 \rangle$.
3. $BES(3)$ is a Doob's h -process

II- A $\pi/3$ angle cone

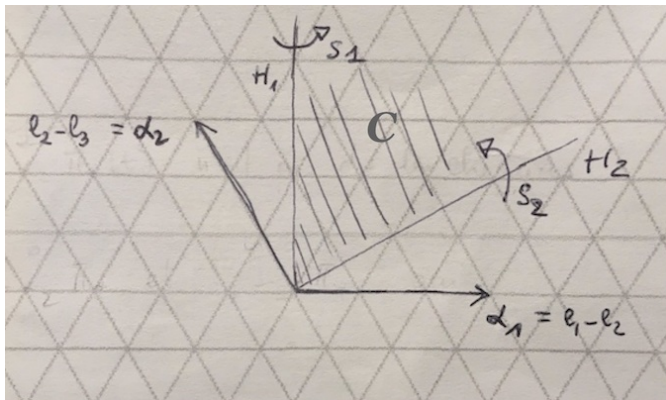


Figure – A $\pi/3$ angle cone

$W = \langle s_1, s_2 \rangle$. For any $y \in \mathbb{R}^2$, there exists a unique x in C s.t. $y \in W \cdot \{x\}$.

II- A $\pi/3$ angle cone

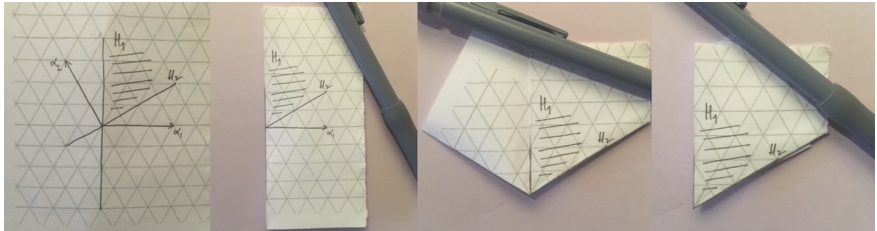


Figure – Fundamental domain - longest element

$$w_0 = s_1 s_2 s_1.$$

II- A $\pi/3$ angle cone

1. $W = \langle s_1, s_2 \rangle$ acting on \mathbb{R}^2 .
2. A fundamental domain \mathcal{C} , a $\pi/3$ angle cone.
3. $w_0 = s_1 s_2 s_1$.

II- A $\pi/3$ angle cone

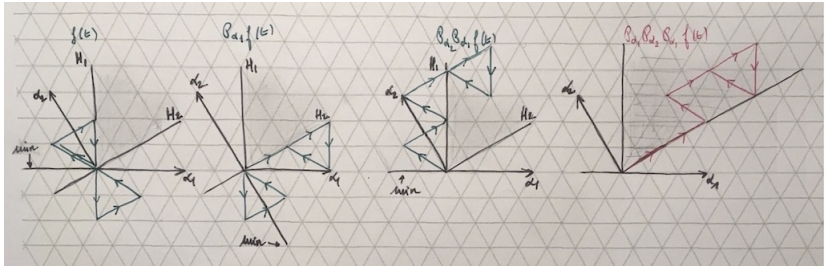


Figure – Pitman's transforms

$i \in \{1, 2\}$, $\mathcal{P}_{\alpha_i} f(t) = f(t) - \inf_{s \leq t} (f(s), \alpha_i) \alpha_i$.
 (\cdot, \cdot) standard inner product on \mathbb{R}^3 .

II- A $\pi/3$ angle cone

If $\{b_t, t \geq 0\}$ is a standard Brownian motion on \mathbb{R}^2 then

$$\{\mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_2} \mathcal{P}_{\alpha_1} b_t, t \geq 0\}$$

is a Doob's h-process in \mathcal{C} with transition densities

$$q_t(x, y) = \frac{h(y)}{h(x)} p_t^{\mathcal{C}}(x, y), \quad x, y \in \mathcal{C}, t > 0,$$

$h(x) = (\alpha_1, x)(\alpha_2, x)(\alpha_3, x)$, $\alpha_3 = \alpha_1 + \alpha_2$,
($p_t^{\mathcal{C}}, t \geq 0$) transition densities of b killed on $\partial\mathcal{C}$.

III- Discrete models

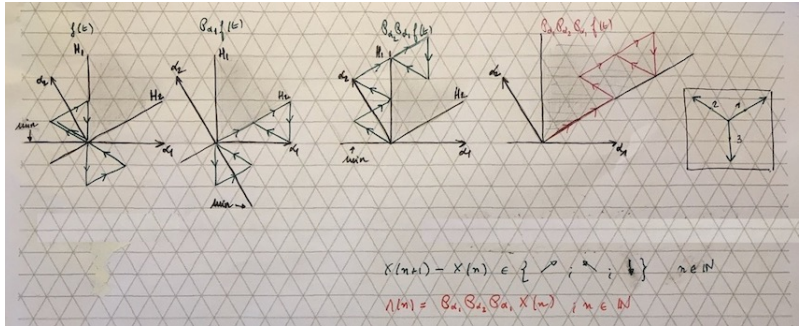


Figure – Random walk

III- Discrete models

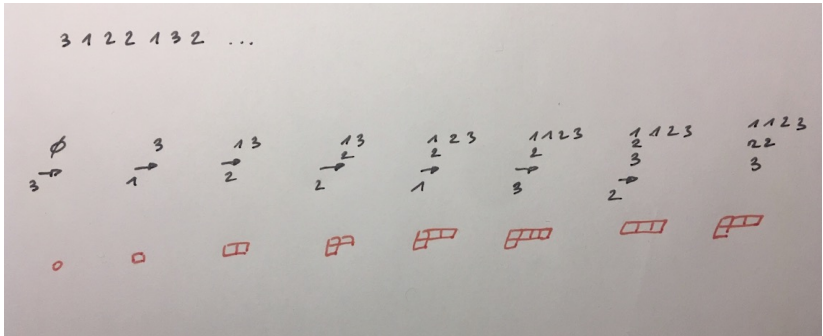


Figure – Young diagram

III- Discrete models

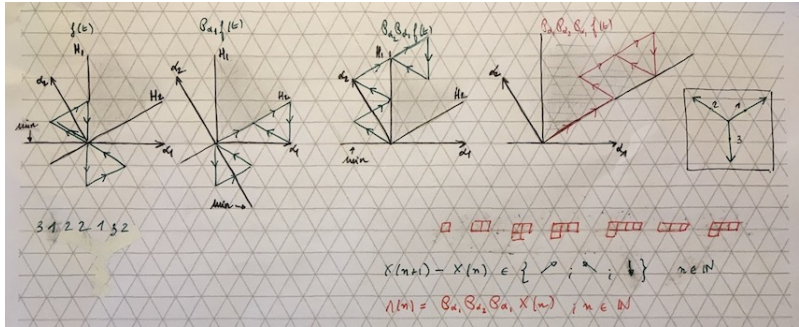


Figure – Random walk and Young diagram

III- Discrete models

1. Discrete models \leftrightarrow Representation theory
2. Simple random walk \leftrightarrow Complex representations of $sl_2(\mathbb{C})$
3. Two dimensional random walk in \mathbb{R}^2 \leftrightarrow Complex representations of $sl_3(\mathbb{C})$

III- Discrete models

$$sl_3(\mathbb{C}) = \{M \in M_3(\mathbb{C}) : \text{tr}(M) = 0\}$$

Standard representation $V = \mathbb{C}^3$

$$he_i = h_{ii}e_i = \gamma_i(h)e_i, \quad i \in \{1, 2, 3\}$$

$$\gamma_1 = \nearrow, \quad \gamma_2 = \nwarrow, \quad \gamma_3 = \downarrow$$

$$X(n+1) - X(n) \in \{\gamma_1, \gamma_2, \gamma_3\}$$

IV- A Brownian motion in an affine Weyl chamber

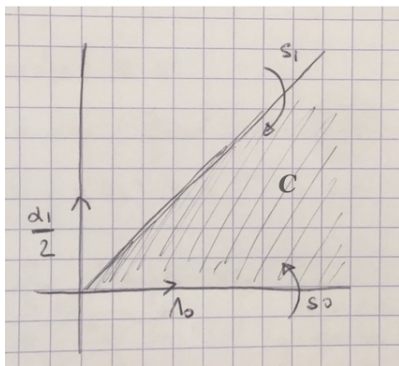


Figure – Affine Weyl chamber

$W = \langle s_0, s_1 \rangle$ acting on $\mathbb{R}_+^* \times \mathbb{R} \cup \{(0,0)\}$, $C = \{x\lambda_0 + y\alpha_1/2 : 0 \leq y \leq x\}$
 $|W| = \infty$.

IV- A Brownian motion in an affine Weyl chamber

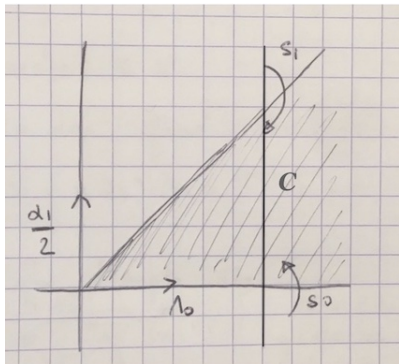


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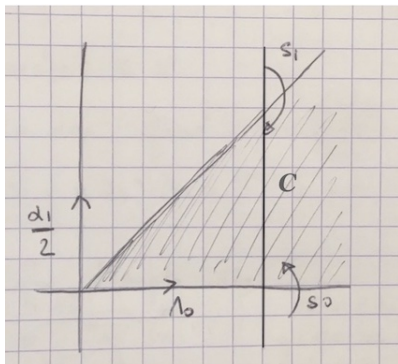


Figure – Affine Weyl chamber

$\{X_t, t \geq 0\}$?,
 $\dots \mathcal{P}_1 \mathcal{P}_0 \dots \mathcal{P}_1 \mathcal{P}_0 X_t = ?$

IV- A Brownian motion in an affine Weyl chamber

W , Weyl group of an Affine Lie algebra A_1^1

V an integrable Highest-Weight Module

Set of weights $P(V) \subset \{l\Lambda_0 + k\alpha_1/2 : k \in \mathbb{Z}\}$, $l \in \mathbb{Z}_+^*$.

V_0 , the fundamental representation $P(V_0) \subset \{\Lambda_0 + k\alpha_1/2 : k \in \mathbb{Z}\}$.

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$B_t = t\Lambda_0 + b_t\alpha_1/2$, $t \geq 0$.

IV- A Brownian motion in an affine Weyl chamber

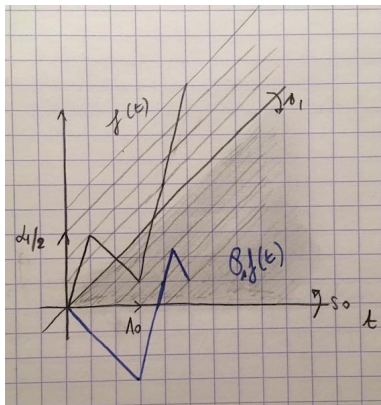


Figure – Pitman's transform corresponding to s_1

$$\begin{aligned} \mathcal{P}_0 \eta(t) &= (t, f(t) - 2 \inf_{s \leq t} (f(s))), \\ \mathcal{P}_1 \eta(t) &= (t, f(t) + 2 \inf_{s \leq t} (s - f(s))), \quad \eta(t) = (t, f(t)). \end{aligned}$$

IV- A Brownian motion in an affine Weyl chamber

Theorem (Ph. Bougerol, M. D. 2022)

If $\{B_t = (t, b_t), t \geq 0\}$ is a standard space-time Brownian motion then for every $t \in \mathbb{R}_+$ the limit

$$\lim_k \mathcal{P}_k \dots \mathcal{P}_0 B_t + (0, 2(-1)^k)$$

exists almost surely. The limiting process is a space-time Brownian motion conditioned in Doob's sense to remain in \mathcal{C} .

$$\mathcal{P}_{2k} = \mathcal{P}_0, \mathcal{P}_{2k+1} = \mathcal{P}_1.$$

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Remark :

$$\lim_k \mathcal{P}_k \dots \mathcal{P}_0 B_t + (0, 2(-1)^k) = \lim_k \mathcal{L}_{k+1} \mathcal{P}_k \dots \mathcal{P}_0 B_t$$

$$\mathcal{L}_0 B_t = (t, b_t + \inf_{s \leq t} (s - b_s)), \mathcal{L}_1 B_t = (t, b_t - \inf_{s \leq t} (b_s)).$$

IV- A Brownian motion in an affine Weyl chamber

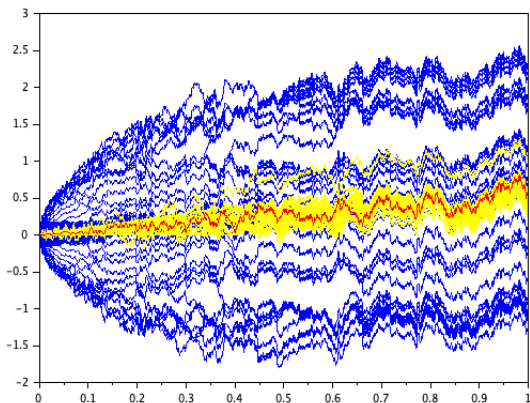


Figure – Successive Pitman transformations

V- Idea of the proofs

Let $\{B_t = (t, b_t + t/2), t \geq 0\}$ be a standard space-time Brownian motion with a drift. For $t \geq 0$, let $(x_k(t))_{k \geq 0}$ be a sequence of real numbers defined by

$$\mathcal{P}_k \dots \mathcal{P}_0 B_t = B_t + (0, \sum_{i=0}^k (-1)^{i-1} x_k(t)).$$

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We set $x_k = \lim_{t \rightarrow \infty} x_k(t)$.

$x_0 \geq 0$, $\frac{x_k}{k} \geq \frac{x_{k+1}}{k+1} \geq 0$, $k \geq 1$. (Anti-lecture hall composition)