

# A Pitman's Theorem for a Brownian motion in an affine Weyl chamber

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# I- A Pitman's Theorem (1975)

If  $\{b_t, t \geq 0\}$  a standard real Brownian motion then

$$\{\mathcal{P}b_t := b_t - 2 \inf_{0 \leq s \leq t} b_s, t \geq 0\}$$

is a  $BES(3)$  process

# I- A Pitman's Theorem (1975)

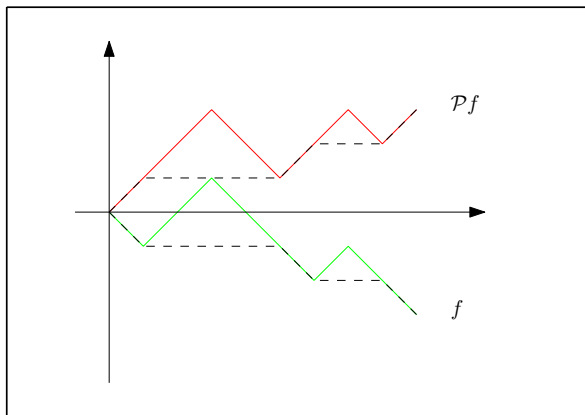


Figure – Pitman's transform  $\mathcal{P}f(t) = f(t) - 2 \inf_{s \leq t} f(s)$

## I- A Pitman's Theorem (1975)

*BES*(3) : Markov process with transition densities

$$q_t(x, y) = \frac{h(y)}{h(x)} p_t^0(x, y), \quad x, y > 0, t > 0,$$

$$p_t^0(x, y) = p_t(x, y) - p_t(x, -y), \quad h(x) = x.$$

# I- A Pitman's Theorem (1975)

Remarks :

1.  $BES(3)$  lives in  $\mathbb{R}_+$
2.  $\mathbb{R}_+$  is a fundamental domain for  $W = \langle s_0 \rangle$ .
3.  $BES(3)$  is a Doob's  $h$ -process

## II- A $\pi/3$ angle cone

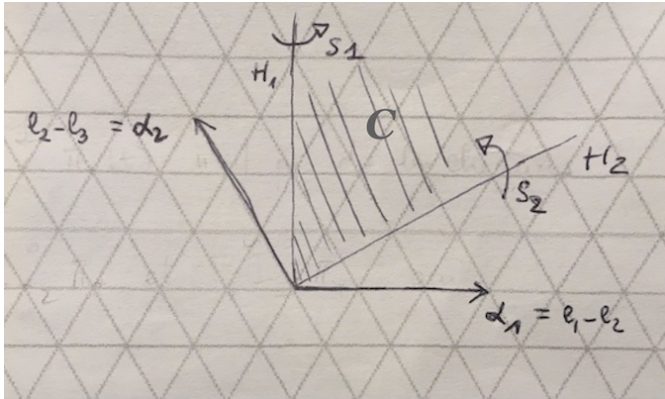


Figure – A  $\pi/3$  angle cone

$W = \langle s_1, s_2 \rangle$ . For any  $y \in \mathbb{R}^2$ , there exists a unique  $x$  in  $C$  s.t.  $y \in W.\{x\}$ .

## II- A $\pi/3$ angle cone

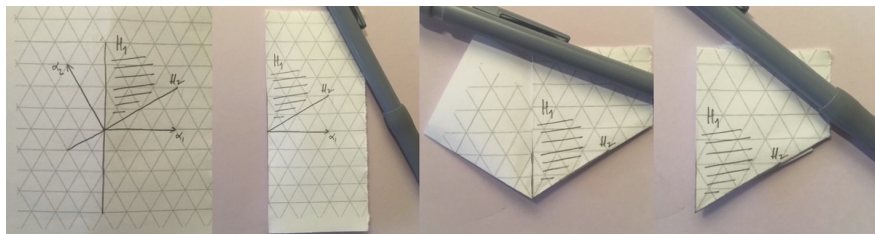


Figure – Fundamental domain - longest element

$$w_0 = s_1 s_2 s_1.$$

## II- A $\pi/3$ angle cone

1.  $W = \langle s_1, s_2 \rangle$  acting on  $\mathbb{R}^2$ .
2. A fundamental domain  $\mathcal{C}$ , a  $\pi/3$  angle cone.
3.  $w_0 = s_1 s_2 s_1$ .



## II- A $\pi/3$ angle cone

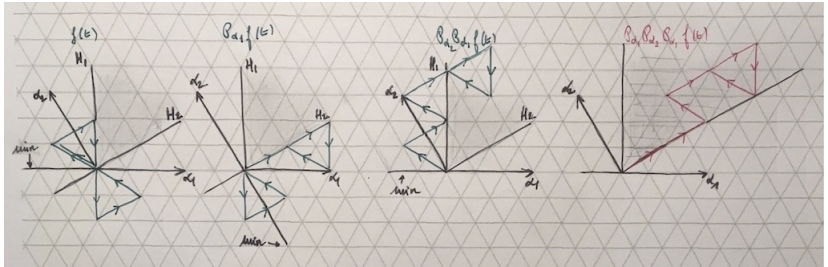


Figure – Pitman's transforms

$i \in \{1, 2\}$ ,  $\mathcal{P}_{\alpha_i} f(t) = f(t) - \inf_{s \leq t} (f(s), \alpha_i) \alpha_i$ .  
 $(\cdot, \cdot)$  standard inner product on  $\mathbb{R}^3$ .

## II- A $\pi/3$ angle cone

If  $\{b_t, t \geq 0\}$  is a standard Brownian motion on  $\mathbb{R}^2$  then

$$\{\mathcal{P}_{\alpha_1} \mathcal{P}_{\alpha_2} \mathcal{P}_{\alpha_1} b_t, t \geq 0\}$$

is a Doob's h-process in  $\mathcal{C}$  with transition densities

$$q_t(x, y) = \frac{h(y)}{h(x)} p_t^{\mathcal{C}}(x, y), \quad x, y \in \mathcal{C}, t > 0,$$

$h(x) = (\alpha_1, x)(\alpha_2, x)(\alpha_3, x)$ ,  $\alpha_3 = \alpha_1 + \alpha_2$ ,  
( $p_t^{\mathcal{C}}, t \geq 0$ ) transition densities of  $b$  killed on  $\partial\mathcal{C}$ .

# III- Discrete models

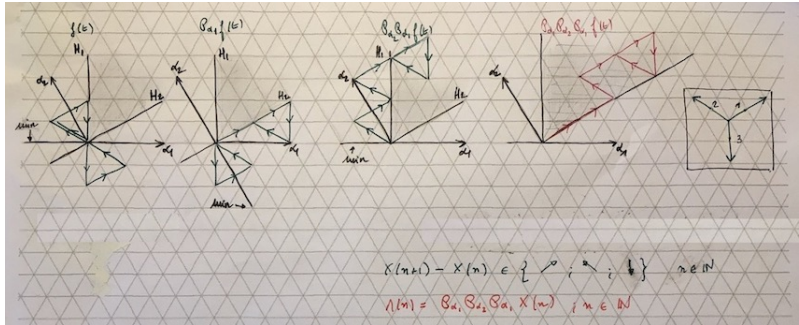


Figure – Random walk

# III- Discrete models

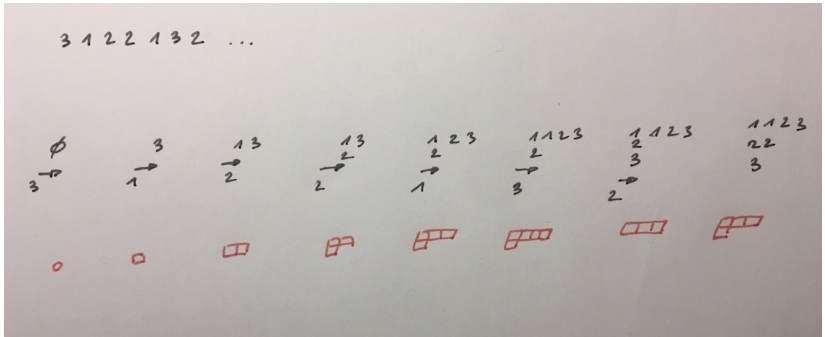


Figure – Young diagram

# III- Discrete models

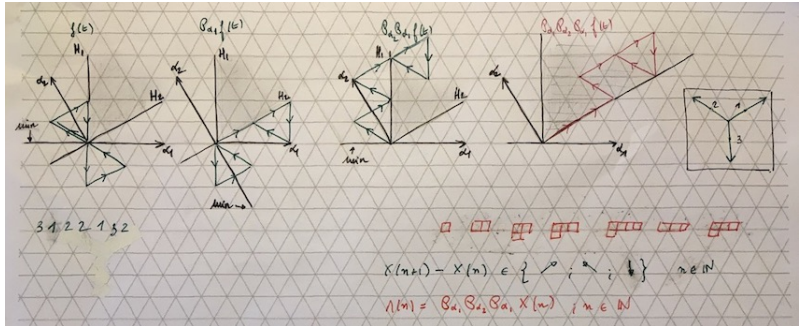


Figure – Random walk and Young diagram

## III- Discrete models

1. Discrete models  $\leftrightarrow$  Representation theory
2. Simple random walk  $\leftrightarrow$  Complex representations of  $sl_2(\mathbb{C})$
3. Two dimensional random walk in  $\mathbb{R}^2$   $\leftrightarrow$  Complex representations of  $sl_3(\mathbb{C})$

### III- Discrete models

$$sl_3(\mathbb{C}) = \{M \in M_3(\mathbb{C}) : \text{tr}(M) = 0\}$$

Standard representation  $V = \mathbb{C}^3$

$$he_i = h_{ii}e_i = \gamma_i(h)e_i, \quad i \in \{1, 2, 3\}$$

$$\gamma_1 = \nearrow, \quad \gamma_2 = \nwarrow, \quad \gamma_3 = \downarrow$$

$$X(n+1) - X(n) \in \{\gamma_1, \gamma_2, \gamma_3\}$$

## IV- A Brownian motion in an affine Weyl chamber

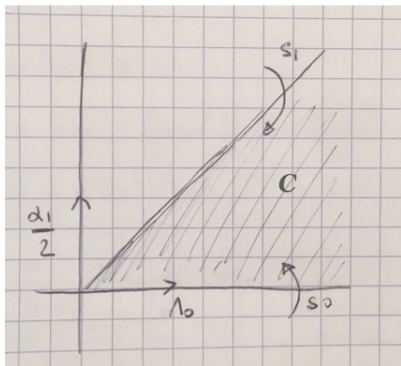


Figure – Affine Weyl chamber

$W = \langle s_0, s_1 \rangle$  acting on  $\mathbb{R}_+^* \times \mathbb{R} \cup \{(0, 0)\}$ ,

$C = \{x\Lambda_0 + y\alpha_1/2 : 0 \leq y \leq x\}$

$|W| = \infty$ .



## IV- A Brownian motion in an affine Weyl chamber

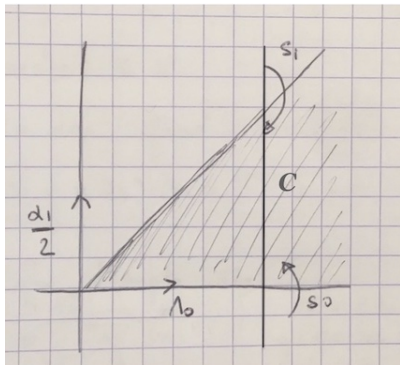


Figure – Affine Weyl chamber

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## IV- A Brownian motion in an affine Weyl chamber

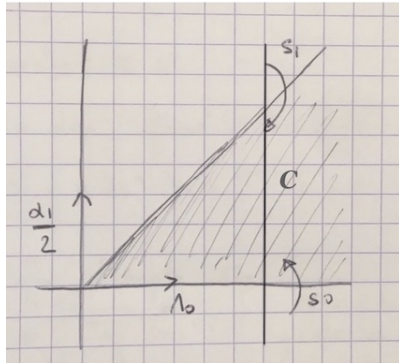


Figure – Affine Weyl chamber

$$\{X_t, t \geq 0\}?,$$
$$\dots P_1 P_0 \dots P_1 P_0 X_t = ?$$

## IV- A Brownian motion in an affine Weyl chamber

$W$ , Weyl group of an Affine Lie algebra  $A_1^1$

$V$  an integrable Highest-Weight Module

Set of weights  $P(V) \subset \{l\Lambda_0 + k\alpha_1/2 : k \in \mathbb{Z}\}$ ,  $l \in \mathbb{Z}_+^*$ .

$V_0$ , the fundamental representation

$P(V_0) \subset \{\Lambda_0 + k\alpha_1/2 : k \in \mathbb{Z}\}$ .

$X(n+1) - X(n) \in P(V_0)$ ,  $X(n) = n\Lambda_0 + x(n)\alpha_1/2$ .

$B_t = t\Lambda_0 + b_t\alpha_1/2$ ,  $t \geq 0$ .

## IV- A Brownian motion in an affine Weyl chamber

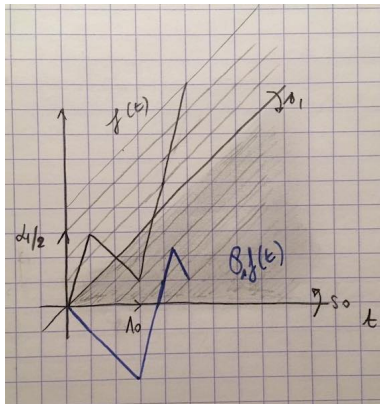


Figure – Pitman's transform corresponding to  $s_1$

$$\mathcal{P}_0 \eta(t) = (t, f(t) - 2 \inf_{s \leq t} (f(s))),$$

$$\mathcal{P}_1 \eta(t) = (t, f(t) + 2 \inf_{s < t} (s - f(s)), \quad \eta(t) = (t, f(t)).$$

## IV- A Brownian motion in an affine Weyl chamber

Theorem (Ph. Bougerol, M. D. 2022)

If  $\{B_t = (t, b_t), t \geq 0\}$  is a standard space-time Brownian motion then for every  $t \in \mathbb{R}_+$  the limit

$$\lim_k \mathcal{P}_k \dots \mathcal{P}_0 B_t + (0, 2(-1)^k)$$

exists almost surely. The limiting process is a space-time Brownian motion conditioned in Doob's sense to remain in  $\mathcal{C}$ .

$$\mathcal{P}_{2k} = \mathcal{P}_0, \mathcal{P}_{2k+1} = \mathcal{P}_1.$$

Remark :

$$\lim_k \mathcal{P}_k \dots \mathcal{P}_0 B_t + (0, 2(-1)^k) = \lim_k \mathcal{L}_{k+1} \mathcal{P}_k \dots \mathcal{P}_0 B_t$$

$$\mathcal{L}_0 B_t = (t, b_t + \inf_{s \leq t} (s - b_s)), \mathcal{L}_1 B_t = (t, b_t - \inf_{s \leq t} (b_s)).$$

## IV- A Brownian motion in an affine Weyl chamber

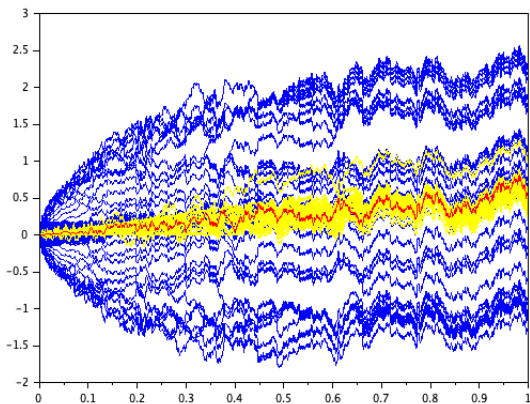


Figure – Successive Pitman transformations

## V- Idea of the proofs

Let  $\{B_t = (t, b_t + t/2), t \geq 0\}$  be a standard space-time Brownian motion with a drift. For  $t \geq 0$ , let  $(x_k(t))_{k \geq 0}$  be a sequence of real numbers defined by

$$\mathcal{P}_k \dots \mathcal{P}_0 B_t = B_t + (0, \sum_{i=0}^k (-1)^{i-1} x_k(t)).$$

We set  $x_k = \lim_{t \rightarrow \infty} x_k(t)$ .

$x_0 \geq 0$ ,  $\frac{x_k}{k} \geq \frac{x_{k+1}}{k+1} \geq 0$ ,  $k \geq 1$ . (Anti-lecture hall composition)