

A method to solve Tutte's equations of higher degree

The case of constellations

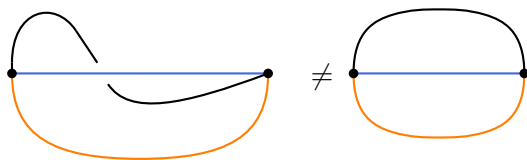
Séverin Charbonnier – Université de Genève
work in progress (Valentin Bonzom, Guillaume Chapuy, Elba Garcia-Failde, Wenjie Fang)

Journées de Combinatoire de Bordeaux
Laboratoire Bordelais de Recherche en Informatique
January 30th 2023

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 - Constellations : definitions
 - Motivation
 - Scope of the talk
- 2 Tutte/Loop equations
- 3 Solving higher loop equations
 - Spectral curve
 - Topological recursion

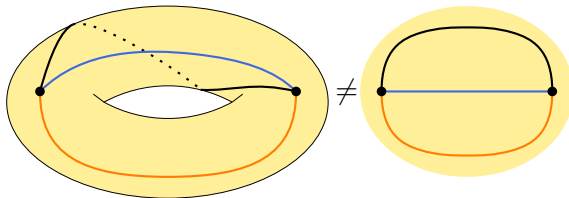
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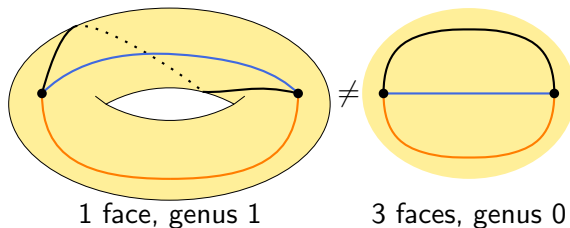
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The **genus** g of a connected map is given by Euler's formula:

$$\#\text{vertices} - \#\text{edges} + \#\text{faces} = 2 - 2g$$

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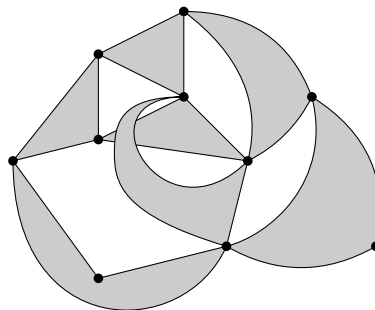
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For instance ($m = 3$):

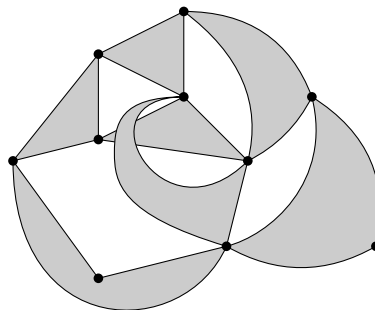


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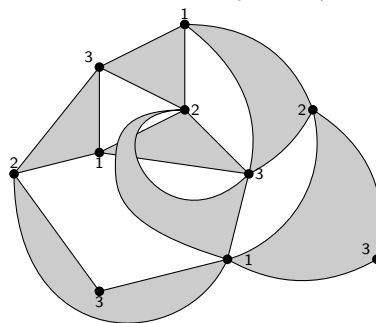


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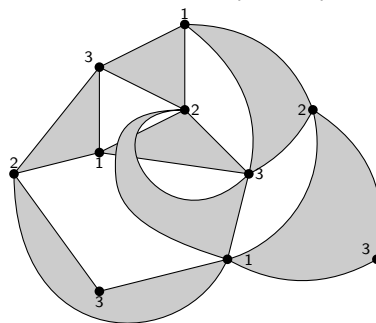


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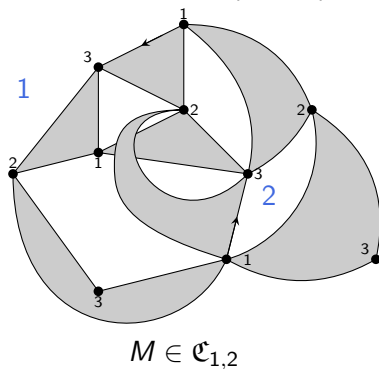
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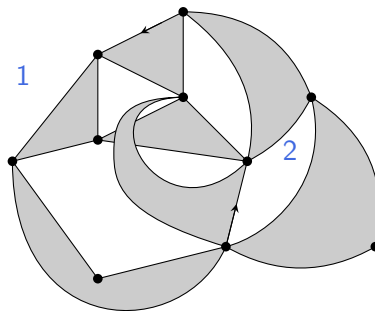
Constellations of type (g, n)

Let $g \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$. M is an m -constellation of type (g, n) ($M \in \mathfrak{C}_{g,n}$) if it is connected, of genus g , and has n marked **labelled rooted** hyperfaces. Other hyperfaces: **internal**.

Roots: edges $1 - m$

Weight

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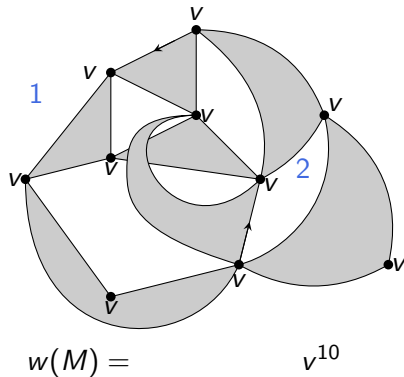


$$w(M) =$$

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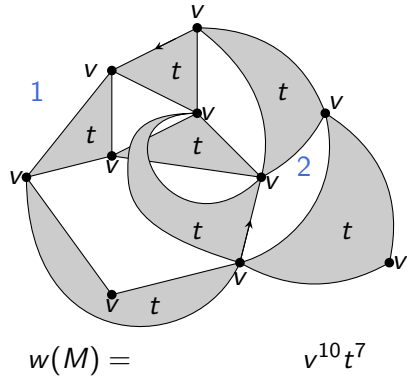
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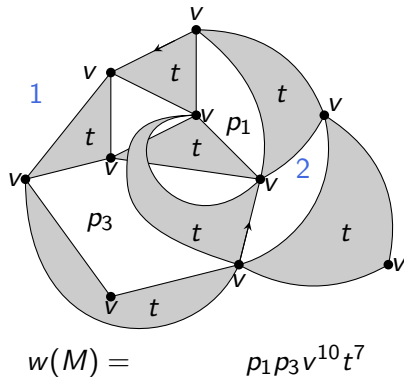
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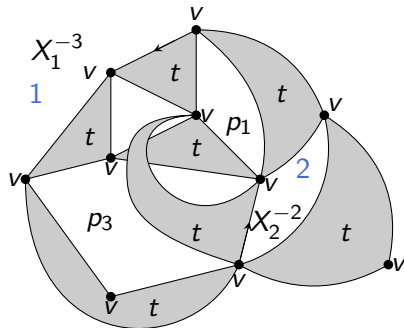
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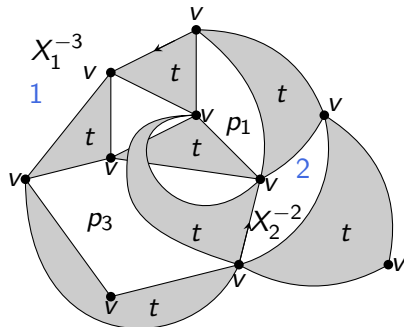


$$w(M) = X_1^{-3} X_2^{-2} p_1 p_3 v^{10} t^7$$

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Definition (Generating functions)

The **generating function** of m -constellations of type (g, n) is

$$C_{g,n}(X_1, \dots, X_n; v, t, \mathbf{p}_k) = \sum_{M \in \mathfrak{C}_{g,n}} w(M) \in \mathbb{Q}[v, \mathbf{p}_k, X_i^{-1}][[t]]$$

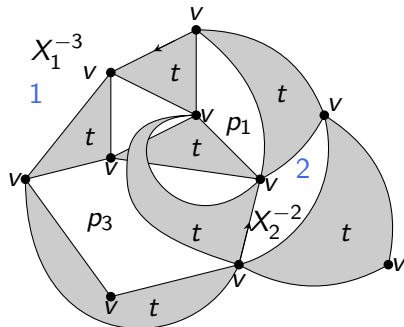
Remarks: No automorphism factor ($n \geq 1$).

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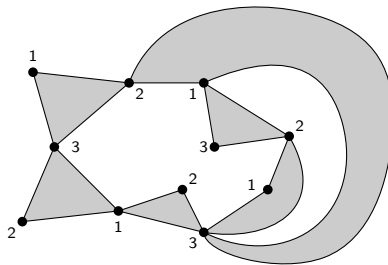
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Goal: compute the $C_{g,n}$'s

Why should we compute the $C_{g,n}$'s?

Constellations as Hurwitz numbers

M : connected m -constellation, size d .

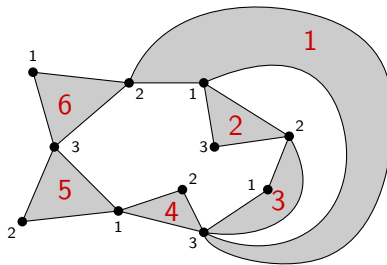


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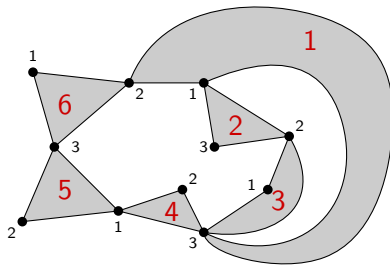


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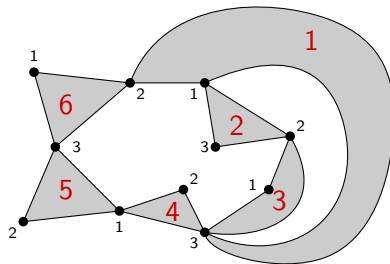
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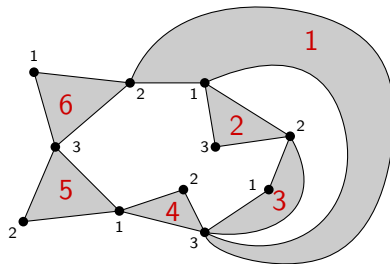
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Labelled m -constellations of size d are in bijection with $(m+1)$ -tuples of permutations $\sigma_0, \dots, \sigma_m \in \mathfrak{S}_d$ s.t.

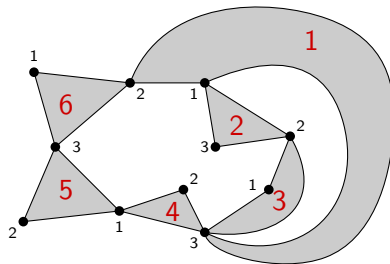
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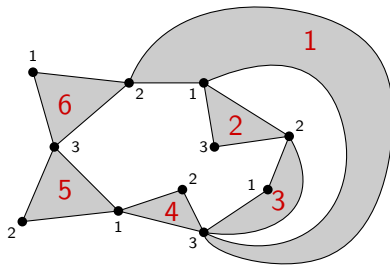
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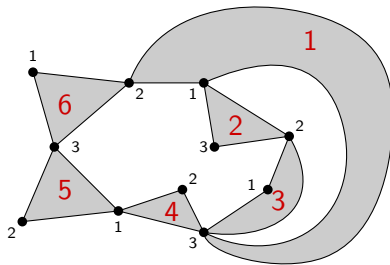
$$H(\mu^{(0)}, \mu^{(1)}, \dots, \mu^{(m)}) := \#\{\sigma_0, \dots, \sigma_m \in \mathfrak{S}_d \mid \sigma_m \dots \sigma_0 = \mathbf{1}_d, \sigma_c \text{ of type } \mu^{(c)}\}$$

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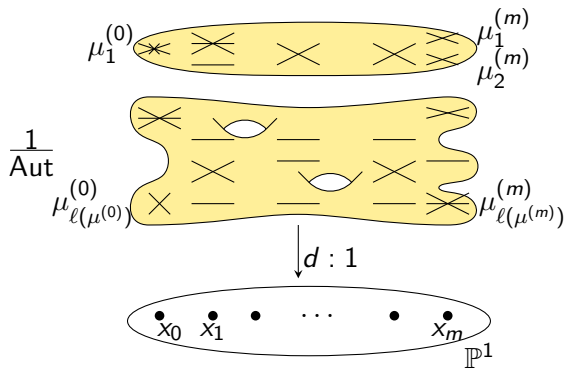
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Counting constellations \Leftrightarrow computing Hurwitz numbers

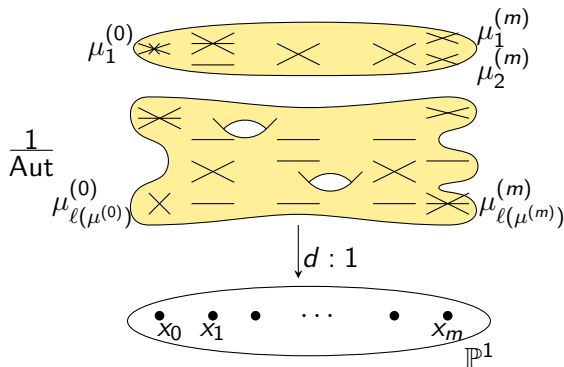
Enumerative geometry

Let $d \in \mathbb{Z}_{\geq 1}$, $\mu^{(0)}, \dots, \mu^{(m)} \vdash d$,
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 $H(\mu^{(0)}, \dots, \mu^{(m)})$ counts the
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Integrable hierarchies

$$\tau(\mathbf{p}_k, t, v) := 1 + \sum_{d \geq 1} \frac{t^d}{d!} \sum_{\mu^{(0)}, \dots, \mu^{(m)} \vdash d} H(\mu^{(0)}, \mu^{(1)}, \dots, \mu^{(m)}) \mathbf{p}_{\mu^{(0)}} v^{\ell(\mu^{(1)}) + \dots + \ell(\mu^{(m)})}$$

is a τ -function of the KP hierarchy. [Orlov–Sherbin '00].

The $C_{g,n}$'s are descendants of the τ -function

What do we aim for?

$$C_{g,n}(X_1, \dots, X_n) \in \mathbb{Q}[\mathbf{p}_k, v, X_i^{-1}][[t]].$$

For small enough $t, X_1^{-1}, \dots, X_n^{-1}$, $C_{g,n}$ converges uniformly in \mathbf{p}_k, v .

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Remark: once $C_{g,n}(x(z_1), \dots, x(z_n))$ is known, one recovers:

$$C_{g,n}(X_1, \dots, X_n) = C_{g,n}(x(Z(X_1)), \dots, x(Z(X_n)))$$

where $Z(X)$ are formal series in t s.t. $x(Z(X)) = X$.

Topological Recursion

Topological Recursion (TR): procedure developed by Chekhov–Eynard–Orantin ('07)

Input

Spectral curve
 $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$

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Output

Differentials $(\omega_{g,n})_{g \geq 0, n \geq 0}$
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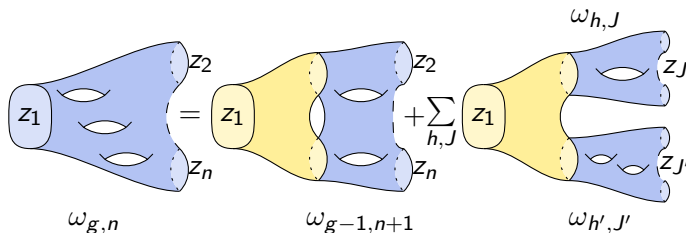
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$$\omega_{g,n}(z_1, l) = \sum_{a \in \Sigma, dx(a)=0} \operatorname{Res}_{z=a} \frac{\frac{1}{2} \int_{\sigma_a(z)} \omega_{0,2}(z_1, \cdot)}{(y(z) - y(\sigma_a(z))) dx(z)} \left(\omega_{g-1,n+1}(z, \sigma_a(z), l) + \sum_{\substack{\text{no } (0,1) \\ h+h'=g \\ J \sqcup J' = l}} \omega_{h,1+J}(z, J) \omega_{h',1+J'}(\sigma_a(z), J') \right)$$

$l = \{z_2, \dots, z_n\}$;

$\sigma_a : \Sigma \rightarrow \Sigma$ local involution around a : $\sigma_a(a) = a$, $x(\sigma_a(z)) = x(z)$.

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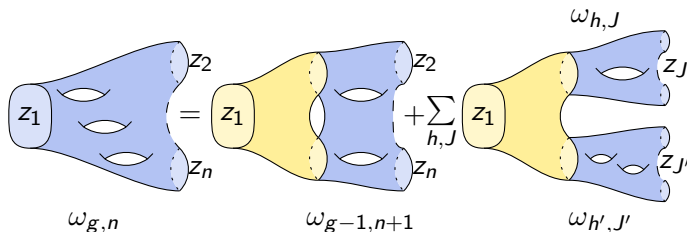


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Various applications :

- Matrix models (hermitian, Kontsevich), map enumeration.
- Weil-Petersson volumes, intersection numbers (Witten–Kontsevich)
- Integrable hierarchies (KdV, KP)
- ...

Topological Recursion (TR): procedure developed by Chekhov–Eynard–Orantin ('07)

Input

Spectral curve
 $\mathcal{S} = (\Sigma, x, y, \omega_{0,2})$

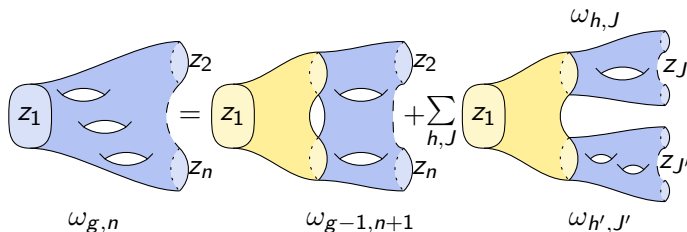


Output

Differentials $(\omega_{g,n})_{g \geq 0, n \geq 0}$
 recursion on $2g - 2 + n$

Spectral curve

Σ : Riemann surface;
 $x : \Sigma \rightarrow \mathbb{P}^1$ branched covering;
 $y : \Sigma \rightarrow \mathbb{P}^1$; $\omega_{0,2} \in H^0(\Sigma, K^{\otimes 2})$.



Various applications :

- Matrix models (hermitian, Kontsevich), map enumeration.
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Goal: prove that constellations satisfy TR.

For constellations with $2g - 2 + n > 0$:

$$\omega_{g,n}(z_1, \dots, z_n) := C_{g,n}(x(z_1), \dots, x(z_n)) dx(z_1) \dots dx(z_n).$$

Constellations and TR:

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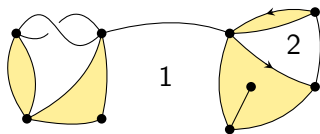
- Prove TR for constellations with internal faces using Tutte's equation (combinatorial techniques).
- The method can be applied to other models that satisfy higher degree loop equations/Tutte's equation.

Example: Ciliated maps [Belliard–C–Eynard–Garcia-Failde '21].

- 1 Definitions, motivation, context
 - Constellations : definitions
 - Motivation
 - Scope of the talk
- 2 Tutte/Loop equations
- 3 Solving higher loop equations
 - Spectral curve
 - Topological recursion

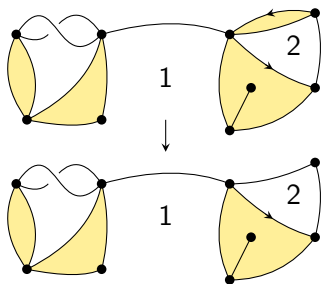
Principle:

- Start with $m \in \mathfrak{M}_{g,n}$ contributing to $M_{g,n}(X_1, \dots, X_n)$.



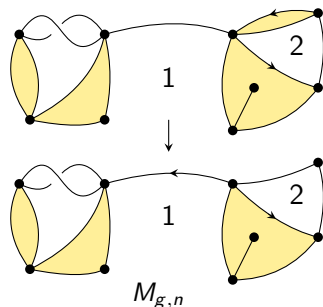
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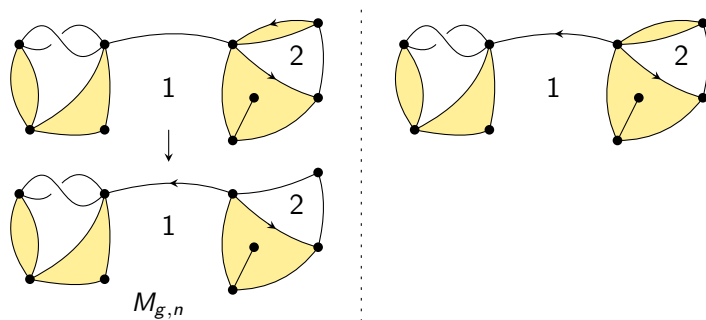
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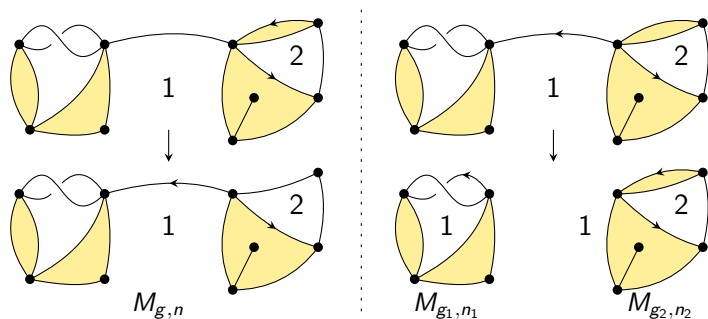
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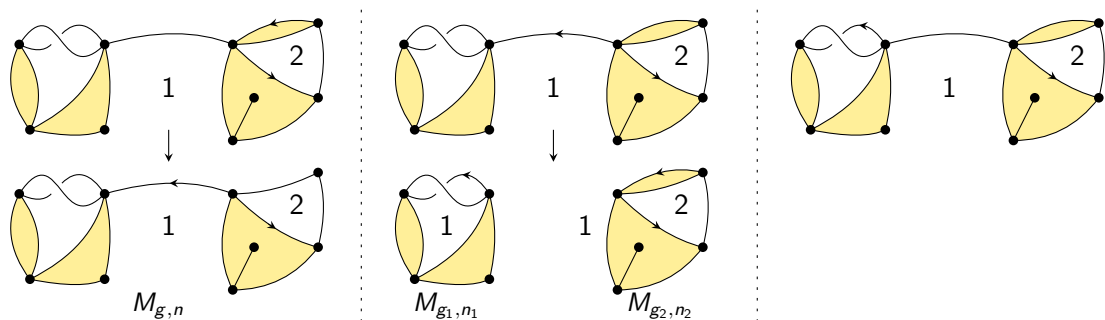
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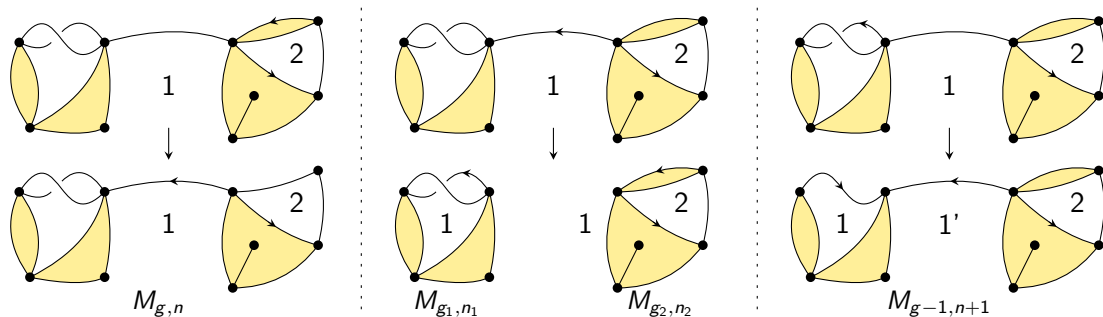
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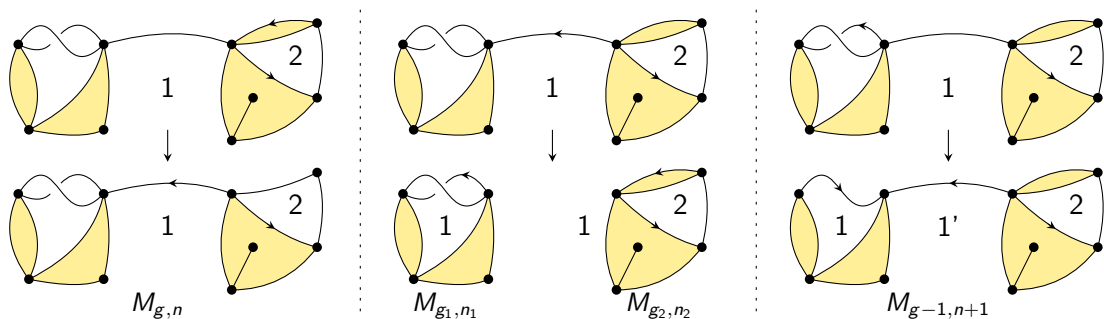
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Tutte's equation for usual maps

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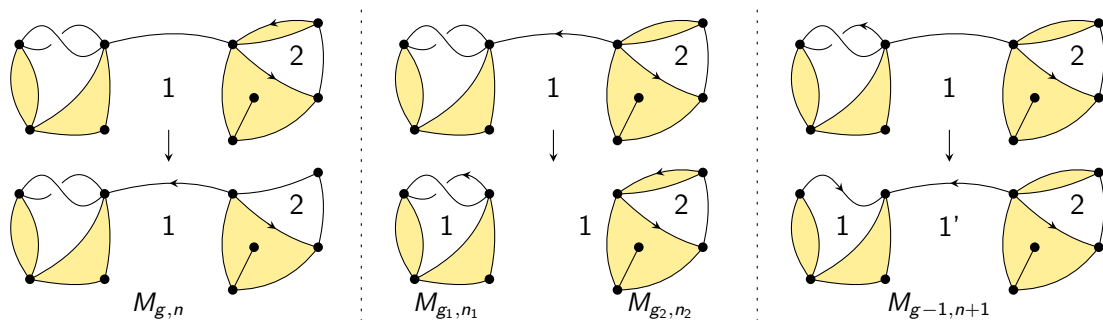
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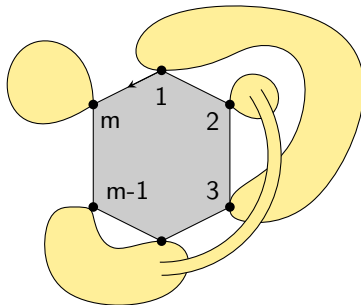
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- Get an equation (**Tutte/Loop** equation) between $M_{g,n}$ and terms involving $M_{g',n'}$ with $2g' - 2 + n' \leq 2g - 2 + n$.



Since an edge has only 2 ends, Tutte's equation is **at most quadratic** in $M_{g',n'}$.

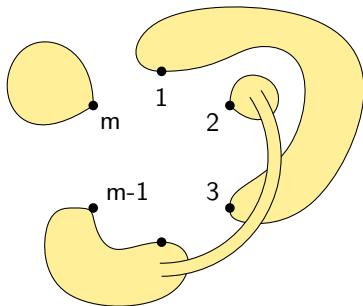
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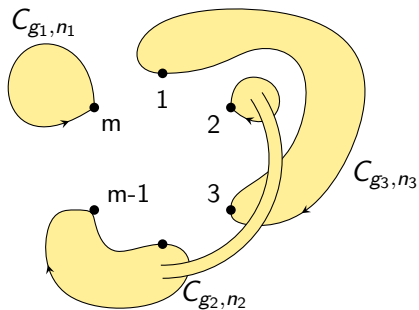
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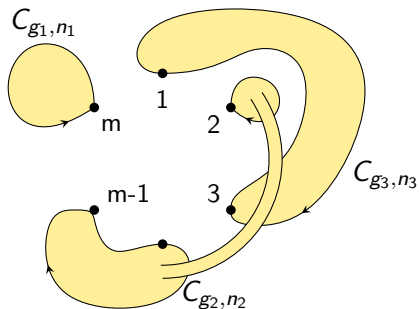
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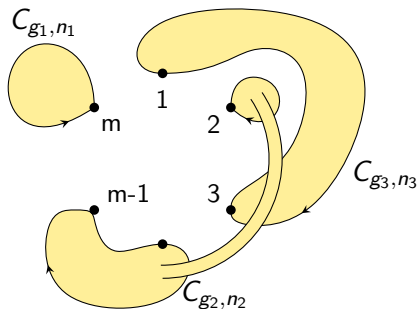
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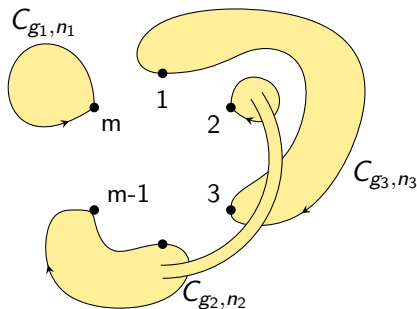
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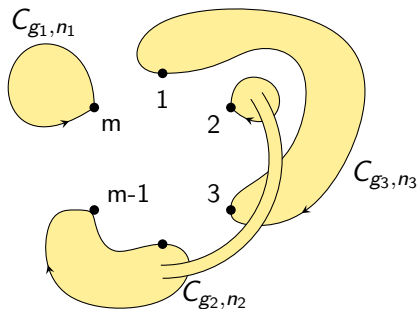


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Remark: same problem for **ciliated maps** (erasing a vertex of degree r); same solution!

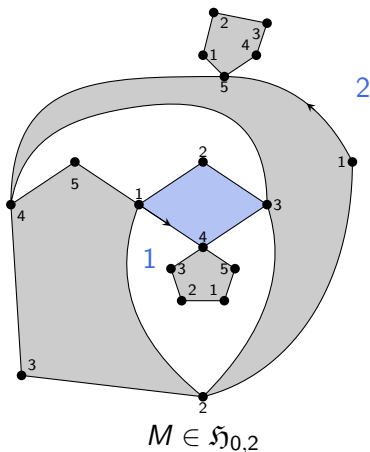
Deformation of the root hyperedge

Let $g \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$.

Definition (**twisted constellation**)

$M \in \mathfrak{H}_{g,n}$ is a **twisted constellation** of type (g, n) if it is a constellation of type (g, n) s.t. the first root hyperedge has degree $1 \leq \ell \leq m - 1$.

First root: edge $1 - \ell$.



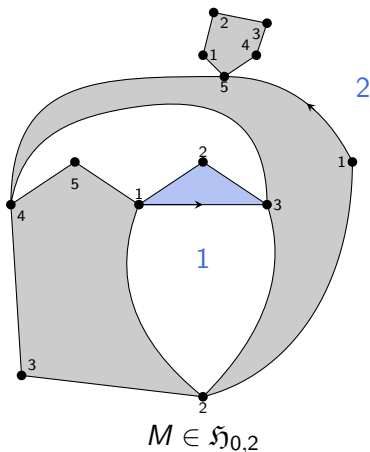
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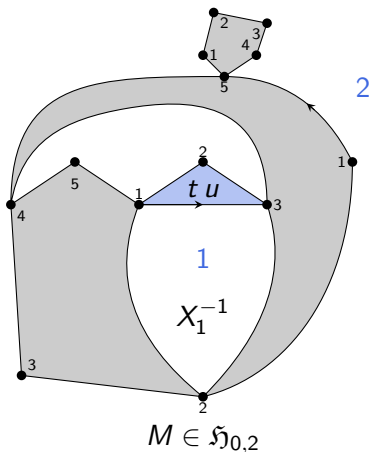
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Generating function

Catalytic variable u : weight $t u^{m-1-\ell}$ for the first root hyperedge of degree ℓ .

1st marked face, degree $km + q \rightarrow X_1^{-k-1}$.

$$H_{g,n}(u; X_1; X_2, \dots, X_n) := \sum_{M \in \mathfrak{H}_{g,n}} w(M)$$



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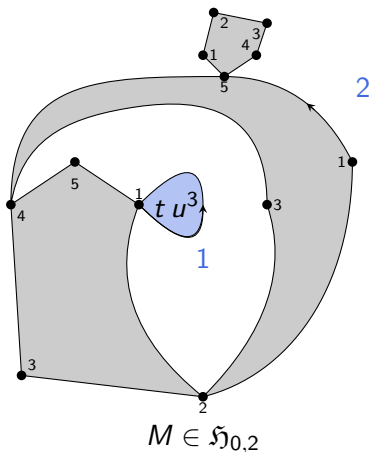
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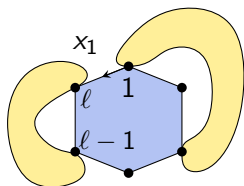


Lemma (constellations from twisted constellations)

$$tC_{g,n}(X_1, \dots, X_n) = [u^{m-2}] H_{g,n}(u; X_1; X_2, \dots, X_n)$$

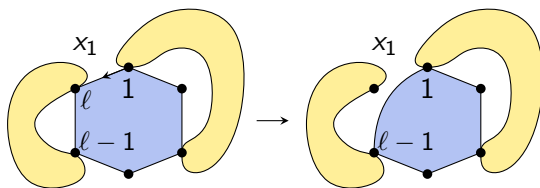
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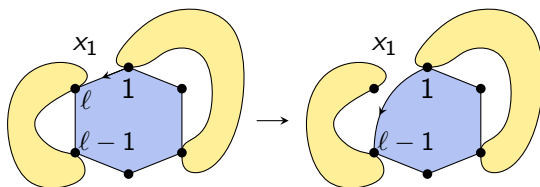
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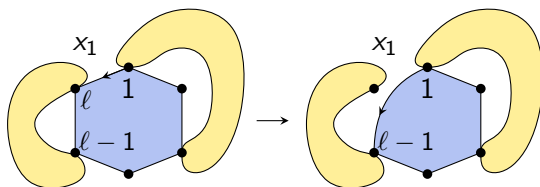
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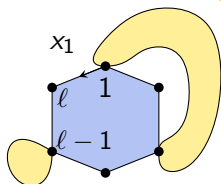
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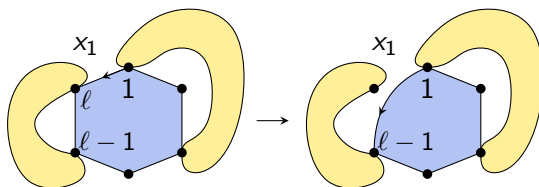
5 cases:

Case 1



Tutte's equation for constellations

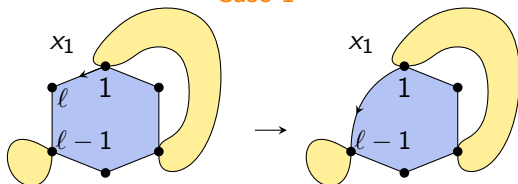
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5 cases:

$$I = \{X_2, \dots, X_n\}$$

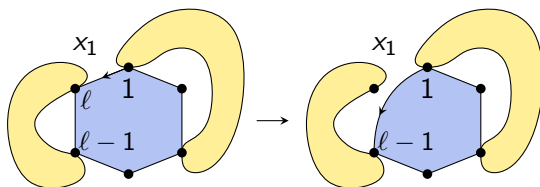
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Contribution: $\frac{v}{u} H_{g,n}(u; X_1; I)$

Tutte's equation for constellations

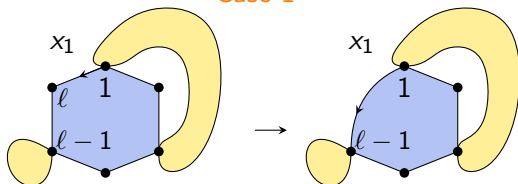
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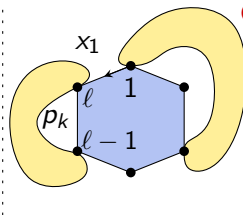
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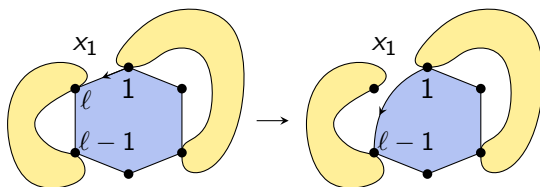
Case 2



Contribution: $\frac{v}{u} H_{g,n}(u; X_1; I)$

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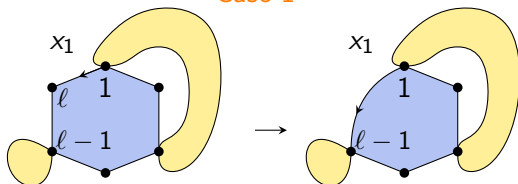
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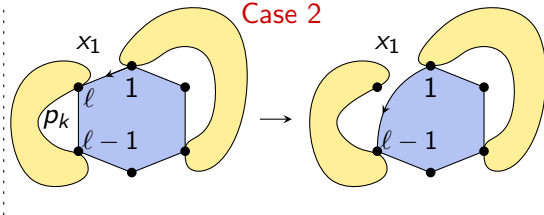
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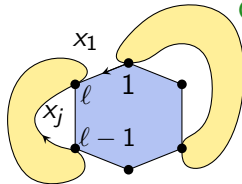
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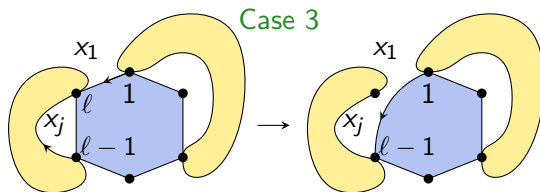
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$$\frac{1}{u} \sum_{k=1}^K p_k \left\{ X_1^k H_{g,n}(u; X_1; I) \right\}_{<0 \text{ (in } X_1)}$$

Case 3

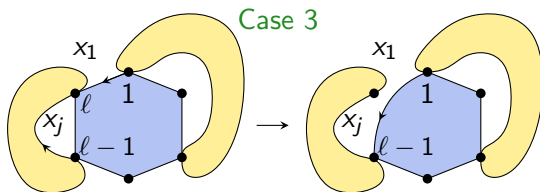


$$I = \{X_2, \dots, X_n\}, I_j = I \setminus \{X_j\}$$

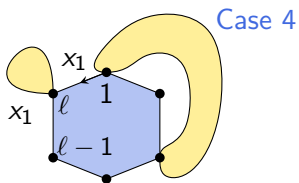


$$\frac{1}{u} \sum_{j=2}^n \frac{d}{dX_j} \frac{X_1 H_{g,n-1}(u; X_1; I_j) - X_j H_{g,n-1}(u; X_j; I_j)}{X_1 - X_j}$$

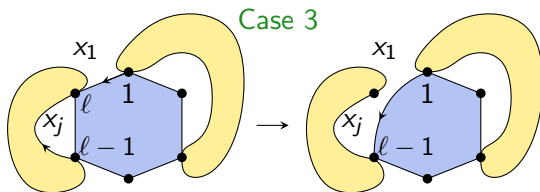
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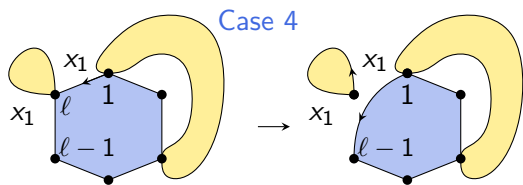
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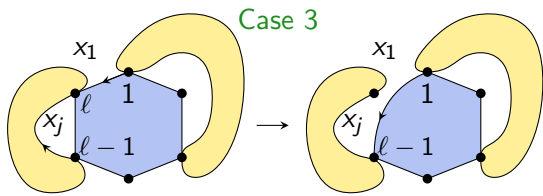


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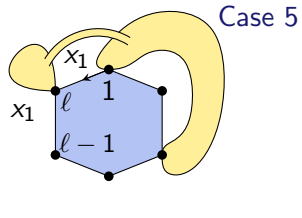
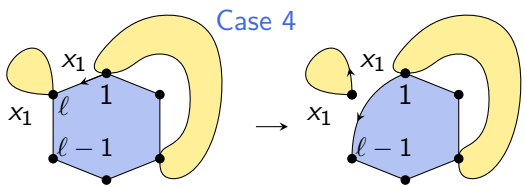


$$\frac{X_1}{u} \sum_{\substack{h+h'=g \\ J \sqcup J' = I}} H_{h,1+\#J}(u; X_1; J) C_{h',1+\#J'}(X_1, J')$$

$$I = \{X_2, \dots, X_n\}, I_j = I \setminus \{X_j\}$$

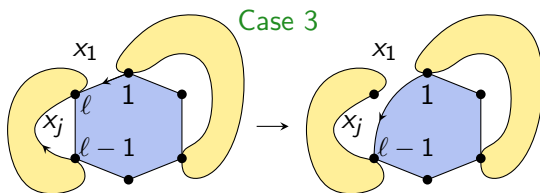


$$\frac{1}{u} \sum_{j=2}^n \frac{d}{dX_j} \frac{X_1 H_{g,n-1}(u; X_1; I_j) - X_j H_{g,n-1}(u; X_j; I_j)}{X_1 - X_j}$$

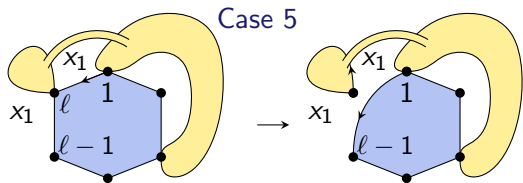
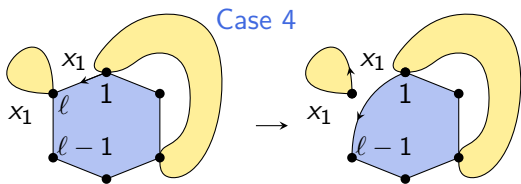


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$$\begin{aligned}
 H_{g,n}(u; X_1; l) + \frac{X_1}{u} C_{g,n}(X_1, \dots, X_n) &= \frac{v}{u} H_{g,n}(u; X_1; l) + \frac{1}{u} \sum_{k=1}^K p_k \left\{ X_1^k H_{g,n}(u; X_1; l) \right\}_{<0} \\
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Set $\sum^{\text{no}(0,1)}$ discarding terms with topology $(0, 1)$, and:

$$\tilde{H}_{g,n}(u; X_1, l) := H_{g,n}(u; X_1, l) - \delta_{g,0} \delta_{n,1}$$

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Tutte's equation for constellations ($2g - 2 + n > 0$)

$$\begin{aligned} \left(\frac{u}{X_1} - Y(X_1)\right) \tilde{H}_{g,n}(u; X_1, l) &= \tilde{H}_{0,1}(u; X_1) \tilde{C}_{g,n}(X_1, l) \\ &+ \sum_{\substack{\text{no } (0,1) \\ h+h'=g \\ J \sqcup J'=l}} \tilde{H}_{h,1+\#J}(u; X_1; J) \tilde{C}_{h',1+\#J'}(X_1, J') + \tilde{H}_{g-1,n+1}(u; X_1; X_1, l) \\ &- \frac{1}{X_1} \sum_{k=1}^K p_k \left\{ X_1^k \tilde{H}_{g,n}(u; X_1; l) \right\}_{\geq 0} - \frac{1}{X_1} \sum_{j=2}^n \frac{d}{dX_j} \frac{X_j \tilde{H}_{g,n-1}(u; X_j; l_j)}{X_1 - X_j} \end{aligned}$$

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- 1 Definitions, motivation, context
 - Constellations : definitions
 - Motivation
 - Scope of the talk
- 2 Tutte/Loop equations
- 3 Solving higher loop equations
 - Spectral curve
 - Topological recursion

Theorem (**Spectral curve**) [23+], [Fang '16]

Carry out the change of variables $(t, X) \leftrightarrow (S, z)$:

$$t = \frac{S}{\left(v + \sum_{k=1}^K p_k S^k \binom{m k - 1}{k}\right)^{m-1}} \quad ; \quad X(z) = \frac{S(1+z)^m}{z}$$

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Strategy of the proof

- Y : **guess and check**. Tools: **Tutte's equation** and **harmonizing operator**.
- $\tilde{C}_{0,2}$: careful analysis of the poles. Tools: **Tutte's equation**.

Harmonizing operators

[Bousquet-Mélou–Chapuy–Préville-Ratelle '13], [Fang '16].

Set x , y to be the rational functions given on previous slide.

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For $z \in \mathbb{C}$, let Z_0, \dots, Z_K be the solutions of $xy(Z) = xy(z)$.

Sharp operator \sharp

Let f be a function.

$$f^\sharp(z) = \sum_{i=0}^K \frac{f(Z_i)}{\prod_{j \neq i} (x(Z_i) - x(Z_j))}$$

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Let Q be a polynomial in x of degree $\leq K - 1$:

$$Q^\sharp = 0, (x^K)^\sharp = 1, (x^{-1})^\sharp = -\frac{p_K}{t(xy)^m}$$

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Let Q be a polynomial in y of degree $\leq m - 2$:

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The harmonizing operators allow to get rid of some polynomial terms in x, y !

Tutte's equation for $(g, n) = (0, 1)$:

$$(u - XY(X))\tilde{H}_{0,1}(u; X) = v - u + \frac{tu^m}{X} - X \overbrace{\sum_{k=1}^K p_k \{X^k \tilde{H}_{0,1}(u; X)\}}^{P(X,u)} \Big|_{\geq 0}$$

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Remark: there is a unique series $Y(X)$ in $\mathbb{Q}[X^{-1}, \mathbf{p}_k, v][[t]]$ s.t.

(i) $Y(X) = \frac{v}{X} + \sum_{k=1}^K p_k X^{k-1} + O(t)$

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Strategy: check that the guessed rational functions $x(z)$, $y(z)$ satisfy (i) and (ii).

(i): "routine" check.

$$P(X, u) = v - u + \frac{tu^m}{X} - X \sum_{k=1}^K p_k \left\{ X^k \tilde{H}_{0,1}(u; X) \right\}_{\geq 0}$$

(ii) Observation:

$$P(x(z), u) = \frac{tu^m}{x(z)} + p_K x(z)^K + \text{pol. in } x(z) \text{ of degree } K - 1.$$

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Apply the **sharp operator** to $P(x(z), u)$, using the lemma:

$$(P(x(z), u))^{\sharp} = \frac{p_K}{x(z)^m y(z)^m} (x(z)^m y(z)^m - u^m)$$

$\Rightarrow (P(x(z), x(z)y(z)))^{\sharp} = 0$. Does this imply $P(x(z), x(z)y(z)) = 0$?

Harmonizing operator business

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Reconstruction lemma [Fang '16]

If $A \in \text{Sz}\mathbb{Q}[z, \mathbf{p}_k, v][[S]]$, $[S^i]A$ is polynomial in z of degree $\leq i$, and $A^{\sharp} = 0$, then $A = 0$.

Upshot: $P(x(z), x(z)y(z)) = 0$. \square

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Corollary

$$\tilde{H}_{0,1}(u; x(z)) = t/x(z) \prod_{i=1}^{m-1} (u - x(z)y(z^{(i)}))$$

For constellations, TR boils down to this result:

Theorem (Topological Recursion for constellations) [23+]

For $g \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$ s.t. $2g - 2 + n > 0$:

$$\begin{aligned} \tilde{C}_{g,n}(x(z_1), l) dx(z_1) = & \sum_{b \text{ s.t. } dx(b)=0} \operatorname{Res}_{z=b} dx(z) \frac{\int_{\zeta=0}^z \frac{dz_1 d\zeta}{(z_1 - \zeta)^2}}{\tilde{H}_{0,1}(xy; x)} \left(\tilde{H}_{g-1, n+1}(xy; x, x, l) \right. \\ & \left. + \sum_{\substack{\text{no } (0,1) \\ h+h'=g \\ J \sqcup J' = l}} \tilde{H}_{h, 1+\#J}(xy; x, J) \tilde{C}_{h', 1+\#J'}(x, J') \right) \end{aligned}$$

where $l = \{x(z_2), \dots, x(z_n)\}$, $x = x(z)$ and $y = y(z)$.

Tutte:

$$\begin{aligned}
\left(\frac{u}{X_1} - Y(X_1)\right) \tilde{H}_{g,n}(u; X_1, I) &= \tilde{H}_{0,1}(u; X_1) \tilde{C}_{g,n}(X_1, I) \\
&+ \sum_{\substack{\text{no } (0,1) \\ h+h'=g \\ J \sqcup J'=I}} \tilde{H}_{h,1+\#J}(u; X_1; J) \tilde{C}_{h',1+\#J'}(X_1, J') + \tilde{H}_{g-1,n+1}(u; X_1; X_1, I) \\
&- \frac{1}{X_1} \sum_{k=1}^K p_k \left\{ X_1^k \tilde{H}_{g,n}(u; X_1; I) \right\}_{\geq 0} - \frac{1}{X_1} \sum_{j=2}^n \frac{d}{dX_j} \frac{X_j \tilde{H}_{g,n-1}(u; X_j; I_j)}{X_1 - X_j}
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$u = X_1 Y(X_1)$:

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Cauchy formula

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Deformation of contours:

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Flat operator: $P_{g,n}^b(xy, x) = 0 \rightarrow$ the last term does not contribute.

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Ciliated maps [Belliard–C–Eynard–Garcia-Failde '21], related to:

- **Combinatorics:** **fully simple maps** [Borot–C–Garcia-Failde '21].
- **Enumerative geometry:** intersection numbers of r -spin **Witten class** on $\overline{\mathcal{M}}_{g,n}$.
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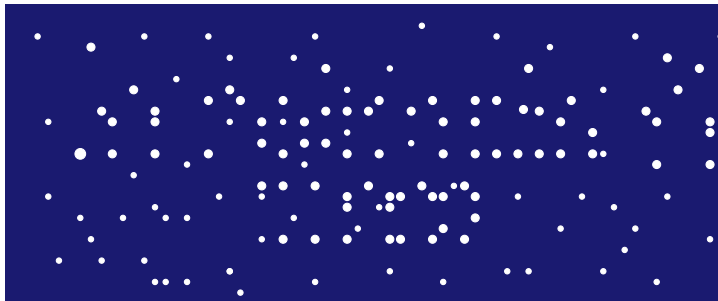
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To be done:

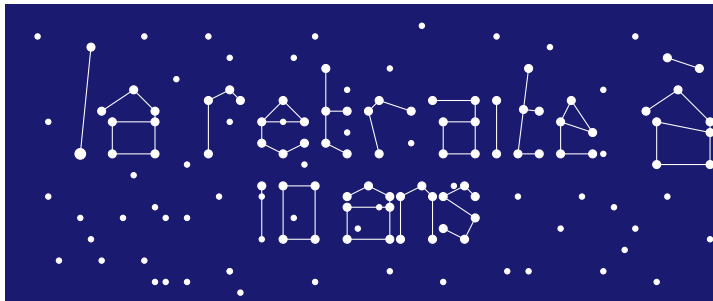
- Write the paper...
- Apply the same method to other models, e.g. the **Brézin–Gross–Witten** matrix model

$$Z^{\text{BGW}} = \int_{H_N} dM e^{-\hbar^{-\frac{1}{2}} \text{Tr} \left(\frac{M^{1-r}}{1-r} + \hbar^{\frac{1}{2}} N \log(M) + MY \right)}$$

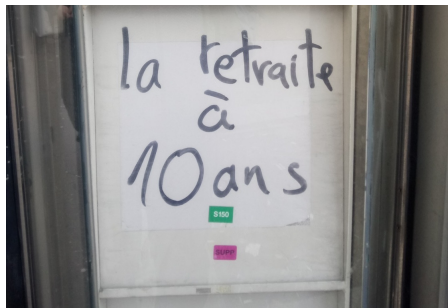
- Would it be useful for m -**Tamari intervals** [Bousquet–Mélou–Chapuy–Préville–Ratelle '13]?



THANKS FOR YOUR ATTENTION!



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Higher TR

Cylinder