A method to solve Tutte’s equations of higher degree
The case of constellations

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work in progress (Valentin Bonzom, Guillaume Chapuy, Elba Garcia-Failde, Wenjie Fang)

Journées de Combinatoire de Bordeaux
Laboratoire Bordelais de Recherche en Informatique
January 30th 2023
1 Definitions, motivation, context
   - Constellations: definitions
   - Motivation
   - Scope of the talk

2 Tutte/Loop equations

3 Solving higher loop equations
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   - Topological recursion
Definition (Map)

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Definition ($m$-constellations)

Let $m \geq 1$. An $m$-constellation is a bicolored map (gray/white faces) such that:

- each gray face (hyperedge) has degree $m$;
- each vertex has a color $c \in \{1, \ldots, m\}$, s.t. the colors around a hyperedge increase clockwise.

For instance ($m = 3$):

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{3} \\
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\]

Degree of a white face (hyperface): multiple of $m$.

Size of a constellation: #hyperedges.

Constellations of type $(g, n)$

Let $g \in \mathbb{Z} \geq 0$, $n \in \mathbb{Z} \geq 1$. $M$ is an $m$-constellations of type $(g, n)$ ($M \in \mathcal{C}_{g, n}$) if it is connected, of genus $g$, and has $n$ marked labelled rooted hyperfaces. Other hyperfaces: internal.

Roots: edges $1 - m$. 
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```plaintext
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\begin{array}{c}
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2 \\
3 \\
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3 \\
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3 \\
2 \\
1 \\
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\]

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**Weight**

Let $M \in \mathcal{C}_{g,n}$. Put a weight:

- $v$ per vertex;
- $t$ per hyperedge;
- $p_k$ per internal face of degree $k \cdot m$ ($1 \leq k \leq K$, bounded degree);
- $X - \ell_i - 1$ for the $i$th marked face of degree $\ell_i \cdot m$.

$$w(M) = X - 3_1 \cdot 2 \cdot t \cdot p_1 \cdot \ldots$$
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$$w(M) = v^{10}$$
Generating functions

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**Definition (Generating functions)**

The generating function of $m$-constellations of type $(g, n)$ is

$$C_{g,n}(X_1, \ldots, X_n; v, t, p_k) = \sum_{M \in \mathcal{C}_{g,n}} w(M) \in \mathbb{Q}[v, p_k, X_i^{-1}][[t]]$$

**Remarks:** No automorphism factor ($n \geq 1$).
Parameters of the model: $t, v, p_k$; variables: $X_1, \ldots, X_n$. 

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Goal: compute the $C_{g,n}$'s
Why should we compute the $C_{g,n}$’s?

**Constellations as Hurwitz numbers**

$M$: connected $m$-constellation, size $d$.

- Label the hyperedges from 1 to $d$;
- For $c \in \{1, \ldots, m\}$: vertices of color $c$ encode cycles of $\sigma_c \in S_d$ (turn clockwise);
- Edges “1−$m$” around hyperfaces (clockwise): cycles of $\sigma_0 \in S_d$.

Proposition

Labelled connected $m$-constellations of size $d$ are in bijection with $(m+1)$-tuples of permutations $\sigma_0, \ldots, \sigma_m \in S_d$ s.t.

$$\sigma_m \ldots \sigma_1 \cdot \sigma_0 = 1_d$$

and $\sigma_0, \ldots, \sigma_m$ act transitively on $\{1, \ldots, d\}$.

Definition (**Hurwitz numbers**)

Let $\mu(0), \mu(1), \ldots, \mu(m) \vdash d$. Hurwitz number:

$$H(\mu(0), \mu(1), \ldots, \mu(m)) := \# \{ \sigma_0, \ldots, \sigma_m \in S_d | \sigma_m \ldots \sigma_0 = 1_d, \sigma_c \text{ of type } \mu(c) \}$$

Counting constellations ⇔ computing Hurwitz numbers

Séverin Charbonnier (UniGe)  Tutte and constellations  JCB – LaBRI 30/01/2023  6 / 26
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\[ \sigma_1 = (12)(45), \quad \sigma_2 = (16)(23), \quad \sigma_3 = (143)(56), \]

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\begin{array}{c}
1 & 6 & 2 \\
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Counting constellations $\iff$ computing Hurwitz numbers
**Enumerative geometry**

Let \( d \in \mathbb{Z}_{\geq 1}, \mu^{(0)}, \ldots, \mu^{(m)} \vdash d \), and \( x_0, \ldots, x_m \in \mathbb{P}^1 \) distinct. \( H(\mu^{(0)}, \ldots, \mu^{(m)}) \) counts the (possibly disconnected) isomorphism classes of branched coverings of degree \( d \) of the sphere with ramification profile \( \mu^{(c)} \) over the point \( x_c \).
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**Integrable hierarchies**

\[
\tau(p_k, t, v) := 1 + \sum_{d \geq 1} \frac{t^d}{d!} \sum_{\mu^{(0)}, \ldots, \mu^{(m)} \vdash d} H(\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(m)}) p_{\mu^{(0)}} v^{\ell(\mu^{(1)}) + \cdots + \ell(\mu^{(m)})}
\]

is a $\tau$-function of the KP hierarchy. [Orlov–Sherbin’00].

The $C_{g,n}$’s are descendents of the $\tau$-function.
What do we aim for?

\[ C_{g,n}(X_1, \ldots, X_n) \in \mathbb{Q}[p_k, v, X_i^{-1}][[t]]. \]

For small enough \( t, X_1^{-1}, \ldots, X_n^{-1} \), \( C_{g,n} \) converges uniformly in \( p_k, v \).
What do we aim for?

\[ C_{g,n}(X_1, \ldots, X_n) \in \mathbb{Q}[p_k, \nu, X_i^{-1}][[t]]. \]

For small enough \( t, X_1^{-1}, \ldots, X_n^{-1}, C_{g,n} \) converges uniformly in \( p_k, \nu \).

We want:

(i) Find a change of variables \((t, X) \leftrightarrow (S, z)\) s.t.
   - \( X = x(z) \) and \( C_{0,1}(x(z)) \) are rational in \( z \);
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- \( C_{g,n}(x(z_1), \ldots, x(z_n); p_k, \nu, t(S)) \) are \textit{analytic} for small enough \( S, z_i \).
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(ii) Find a universal formula using complex geometry to compute the \( C_{g,n} \)'s recursively on \( 2g - 2 + n \).
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(ii) Find a universal formula using complex geometry to compute the \( C_{g,n} \)'s recursively on \( 2g - 2 + n \).

Remark: once \( C_{g,n}(x(z_1), \ldots, x(z_n)) \) is known, one recovers:

\[ C_{g,n}(X_1, \ldots, X_n) = C_{g,n}(x(Z(X_1)), \ldots, x(Z(X_n))) \]

where \( Z(X) \) are formal series in \( t \) s.t. \( x(Z(X)) = X \).
**Topological Recursion (TR):** procedure developed by Chekhov–Eynard–Orantin (’07)

**Input**
Spectral curve
\[ S = (\Sigma, x, y, \omega_0, 2) \]

**Output**
Differentials \( (\omega_g, n)_{g \geq 0, n \geq 0} \)
recursion on \( 2g - 2 + n \)
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\( \Sigma : \text{Riemann surface;} \)
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\[
\omega_{g,n}(z_1, l) = \sum_{a \in \Sigma, dx(a) = 0} \text{Res}_{z=a} \left( \frac{1}{2} \int_{\sigma_a(z)}^z \omega_{0,2}(z_1, \cdot) \right) \left( \omega_{g-1,n+1}(z, \sigma_a(z), l) \right)
\]

\[ l = \{z_2, \ldots, z_n\}; \]
\[ \sigma_a : \Sigma \to \Sigma \text{ local involution around } a: \sigma_a(a) = a, x(\sigma_a(z)) = x(z). \]
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Various applications:
- Matrix models (hermitian, Kontsevich), map enumeration.
- Weil-Petersson volumes, intersection numbers (Witten–Kontsevich)
- Integrable hierarchies (KdV, KP)
- ...
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**Goal:** prove that constellations satisfy TR.

For constellations with \(2g - 2 + n > 0\):

\[ \omega_{g,n}(z_1, \ldots, z_n) := C_{g,n}(x(z_1), \ldots, x(z_n))dx(z_1)\ldots dx(z_n). \]
Scope of the talk

**Constellations and TR:**

- **[Alexandrov–Chapuy–Eynard–Harnad ’20]:** TR for constellations without internal face; weight $v_c$ for vertices of color $c$.

- **[Bonzom–Chapuy–C–Garcia-Failde ’22], [Bychkov–Dunin-Barkowski–Kazarian–Shadrin ’22]:** TR for constellations with internal faces; hyperedges of various degrees; weight $v_c$ for vertices of color $c$. (Even more general than that).

- **[BCCG-F]:** combinatorial techniques for the spectral curve (techniques of [Albenque–Bouttier]); deformation properties of TR for the other topologies.

- **[BD-BKS]:** semi-infinite wedge formalism + algebraic manipulations.

Scope of the talk:

- Prove TR for constellations with internal faces using Tutte’s equation (combinatorial techniques).

- The method can be applied to other models that satisfy higher degree loop equations/Tutte’s equation.

  Example: Ciliated maps **[Belliard–C–Eynard–Garcia-Failde ’21]**.
Scope of the talk

Constellations and TR:

- [Alexandrov–Chapuy–Eynard–Harnad '20]: TR for constellations without internal face; weight $v_c$ for vertices of color $c$.

- [Bonzom–Chapuy–C–Garcia-Failde '22], [Bychkov–Dunin-Barkowski–Kazarian–Shadrin '22]: TR for constellations with internal faces; hyperedges of various degrees; weight $v_c$ for vertices of color $c$. (Even more general than that).

- [BCCG-F]: combinatorial techniques for the spectral curve (techniques of Albenque–Bouttier); deformation properties of TR for the other topologies.

- [BD-BKS]: semi-infinite wedge formalism + algebraic manipulations.
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- Prove TR for constellations with internal faces using Tutte’s equation (combinatorial techniques).
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- Prove TR for constellations with internal faces using Tutte’s equation (combinatorial techniques).
- The method can be applied to other models that satisfy higher degree loop equations/Tutte’s equation.

Example: Ciliated maps [Belliard–C–Eynard–Garcia-Failde '21].
Definitions, motivation, context
- Constellations: definitions
- Motivation
- Scope of the talk

Tutte/Loop equations

Solving higher loop equations
- Spectral curve
- Topological recursion
Tutte’s equation for usual maps

**Principle:**

- Start with $m \in \mathcal{M}_{g,n}$ contributing to $M_{g,n}(X_1, \ldots, X_n)$.

Since an edge has only 2 ends, Tutte’s equation is at most quadratic in $M_{g,n}$.
Tutte’s equation for usual maps

**Principle:**
- Start with \( m \in \mathcal{M}_{g,n} \) contributing to \( \mathcal{M}_{g,n}(X_1, \ldots, X_n) \).
- Erase the root edge of the first marked face;

![Diagram](image-url)
Tutte’s equation for usual maps

**Principle:**

- Start with $m \in \mathcal{M}_{g,n}$ contributing to $\mathcal{M}_{g,n}(X_1, \ldots, X_n)$.
- Erase the root edge of the first marked face; reroot the resulting map(s). Obtain maps in $\mathcal{M}_{g',n'}$ with $2g' - 2 + n' \leq 2g - 2 + n$.

Since an edge has only 2 ends, Tutte’s equation is at most quadratic in $\mathcal{M}_{g',n'}$. 

\[ \mathcal{M}_{g,n} \]
Tutte’s equation for usual maps

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\[
\begin{array}{c}
\text{1} \quad 2 \\
M_{g,n}
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{2} \\
\end{array}
\]

\[
\begin{array}{c}
\text{1} \quad 2 \\
\end{array}
\quad \quad \quad
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\[
\begin{align*}
\mathcal{M}_{g,n} & \quad \xrightarrow{\text{1}} \quad 2 \quad \mathcal{M}_{g,n} \\
\mathcal{M}_{g,n} & \quad \xrightarrow{\text{1}} \quad 2 \quad \mathcal{M}_{g,n}
\end{align*}
\]
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- Start with \( m \in \mathcal{M}_{g,n} \) contributing to \( \mathcal{M}_{g,n}(X_1, \ldots, X_n) \).
- Erase the root edge of the first marked face; reroot the resulting map(s). Obtain maps in \( \mathcal{M}_{g',n'} \) with \( 2g' - 2 + n' \leq 2g - 2 + n \).
- Get an equation (Tutte/Loop equation) between \( \mathcal{M}_{g,n} \) and terms involving \( \mathcal{M}_{g',n'} \) with \( 2g' - 2 + n' \leq 2g - 2 + n \).
Tutte’s equation for usual maps

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![Diagram showing marked maps](image1)

Since an edge has only 2 ends, Tutte’s equation is at most quadratic in $\mathcal{M}_{g',n'}$. 
**Principle** [Fang ’16]: start with $M \in \mathcal{C}_{g,n}$,
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**Principle** [Fang ’16]: start with $M \in \mathcal{C}_{g,n}$, erase the root hyperedge of the first marked hyperface and reroot the resulting constellations.

![Diagram of Tutte's equation for constellations]

1. $C_{g_1,n_1}$
2. $m$
3. $1$
4. $2$
5. $C_{g_2,n_2}$
6. $m-1$
7. $3$
8. $C_{g_3,n_3}$
**Principle** [Fang ’16]: start with $M \in C_{g,n}$, erase the root hyperedge of the first marked hyperface and reroot the resulting constellations.

⇒ Higher order terms (up to degree $m$) + many cases!
**Principle** [Fang ’16]: start with $M \in \mathcal{C}_{g,n}$, erase the root hyperedge of the first marked hyperface and reroot the resulting constellations.

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**Idea:** decompose the procedure into steps, erasing one vertex of the root hyperedge at a time.
Tutte’s equation for constellations

**Principle [Fang ’16]:** start with $M \in \mathcal{C}_{g,n}$, erase the root hyperedge of the first marked hyperface and reroot the resulting constellations.

⇒ Higher order terms (up to degree $m$) + many cases!

**Idea:** decompose the procedure into steps, erasing one vertex of the root hyperedge at a time.

⇒ Deform the model by allowing the root hyperedge to have smaller degree.
**Principle [Fang ’16]:** start with $M \in \mathcal{C}_{g,n}$, erase the root hyperedge of the first marked hyperface and reroot the resulting constellations.

$$C_{g_1,n_1}, \quad C_{g_2,n_2}, \quad C_{g_3,n_3}$$

$\Rightarrow$ Higher order terms (up to degree $m$) + many cases!

**Idea:** decompose the procedure into steps, erasing one vertex of the root hyperedge at a time.

$\Rightarrow$ **Deform** the model by allowing the root hyperedge to have smaller degree.

**Remark:** same problem for ciliated maps (erasing a vertex of degree $r$); same solution!
Deformation of the root hyperedge

Let \( g \in \mathbb{Z}_{\geq 0}, \ n \in \mathbb{Z}_{\geq 1}. \)

**Definition (twisted constellation)**

\( M \in \mathcal{G}_{g,n} \) is a **twisted constellation** of type \((g, n)\) if it is a constellation of type \((g, n)\) s.t. the first root hyperedge has degree \(1 \leq \ell \leq m - 1.\)

First root: edge \(1 - \ell.\)

\[ M \in \mathcal{G}_{0,2} \]
Deformation of the root hyperedge

Let \( g \in \mathbb{Z}_{\geq 0}, \ n \in \mathbb{Z}_{\geq 1}. \)

**Definition (twisted constellation)**

\( M \in \mathcal{S}_{g,n} \) is a twisted constellation of type \((g, n)\) if it is a constellation of type \((g, n)\) s.t. the first root hyperedge has degree \(1 \leq \ell \leq m - 1\).

First root: edge \(1 - \ell\).
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Let $g \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$.

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$M \in \mathcal{S}_{g,n}$ is a twisted constellation of type $(g, n)$ if it is a constellation of type $(g, n)$ s.t. the first root hyperedge has degree $1 \leq \ell \leq m - 1$.

First root: edge $1 - \ell$.

**Generating function**

Catalytic variable $u$: weight $tu^{m-1-\ell}$ for the first root hyperedge of degree $\ell$.

$1^{st}$ marked face, degree $km + q \rightarrow X_1^{-k-1}$.

$$H_{g,n}(u; X_1, X_2, \ldots, X_n) := \sum_{M \in \mathcal{S}_{g,n}} w(M)$$
Deformation of the root hyperedge

Let \( g \in \mathbb{Z}_{\geq 0} \), \( n \in \mathbb{Z}_{\geq 1} \).

**Definition (twisted constellation)**

\( M \in \mathcal{F}_{g,n} \) is a **twisted constellation** of type \((g, n)\) if it is a constellation of type \((g, n)\) s.t. the first root hyperedge has degree \( 1 \leq \ell \leq m - 1 \).

First root: edge \( 1 - \ell \).

**Generating function**

Catalytic variable \( u \): weight \( tu^{m-1-\ell} \) for the first root hyperedge of degree \( \ell \).

1st marked face, degree \( km + q \rightarrow X_1^{-k-1} \).

\[
H_{g,n}(u; X_1, X_2, \ldots, X_n) := \sum_{M \in \mathcal{F}_{g,n}} w(M)
\]

**Lemma (constellations from twisted constellations)**

\[
tC_{g,n}(X_1, \ldots, X_n) = [u^{m-2}] H_{g,n}(u; X_1, X_2, \ldots, X_n)
\]
**Principle:** start from $M \in \mathcal{S}_{g,n}$;

![Diagram](image)

5 cases:

- **Case 1**

- **Case 2**
**Principle:** start from $M \in \mathcal{S}_{g,n}$; remove the vertex of biggest color in the first root hyperedge
Tutte’s equation for constellations

**Principle:** start from $M \in \mathcal{H}_{g,n}$; remove the vertex of biggest color in the first root hyperedge and reroot.
Tutte’s equation for constellations

**Principle:** start from $M \in \mathcal{S}_{g,n}$; remove the vertex of biggest color in the first root hyperedge and reroot.

5 cases:

Case 1
Tutte’s equation for constellations

**Principle:** start from $M \in \mathcal{S}_{g,n}$; remove the vertex of biggest color in the first root hyperedge and reroot.

\[ x_1^\ell \ell - 1 \rightarrow \]

5 cases:
\[ I = \{X_2, \ldots, X_n\} \]

**Case 1**

Contribution: \[ \frac{v}{u} H_{g,n}(u; X_1; I) \]
Tutte’s equation for constellations

**Principle:** start from $M \in \mathcal{S}_{g,n}$; remove the vertex of biggest color in the first root hyperedge and reroot.

![Diagram of Tutte’s equation for constellations]

5 cases:
$I = \{X_2, \ldots, X_n\}$

**Case 1**

**Case 2**

Contribution: $\frac{v}{u} H_{g,n}(u; X_1; I)$
Tutte’s equation for constellations

**Principle:** start from $M \in \mathcal{S}_{g,n}$; remove the vertex of biggest color in the first root hyperedge and reroot.

**5 cases:**

$I = \{X_2, \ldots, X_n\}$

**Case 1**

**Contribution:** \( \frac{v}{u} H_{g,n}(u; X_1; I) \)

\[
\frac{1}{u} \sum_{k=1}^{K} p_k \left\{ X_1^k H_{g,n}(u; X_1; I) \right\} < 0 \text{ (in $X_1$)}
\]
Case 3

\[ I = \{ X_2, \ldots, X_n \}, \quad I_j = \{ X_j \} \]

\[ x_1^{\ell - 1} x_j^{1} \]

\[ X_1^{u} X_j^{D} X_1^{H} g, n-1 (u; X_1^{H}; I_j) - X_j^{H} g, n-1 (u; X_j; I_j) \]
\[ I = \{ X_2, \ldots, X_n \}, \quad l_j = l \setminus \{ X_j \} \]

\[
\frac{1}{u} \sum_{j=2}^{n} \frac{d}{dX_j} \frac{X_1 H_{g,n-1}(u; X_1; l_j) - X_j H_{g,n-1}(u; X_j; l_j)}{X_1 - X_j}
\]
$I = \{X_2, \ldots, X_n\}$, $I_j = I \setminus \{X_j\}$

**Case 3**

$$\frac{1}{u} \sum_{j=2}^{n} \frac{d}{dX_j} \frac{X_1 H_{g,n-1}(u; X_1; I_j) - X_j H_{g,n-1}(u; X_j; I_j)}{X_1 - X_j}$$

**Case 4**
\( I = \{X_2, \ldots, X_n\}, \ I_j = I \setminus \{X_j\} \)

Case 3

\[
\frac{1}{u} \sum_{j=2}^{n} \frac{d}{dX_j} \frac{X_1 H_{g,n-1}(u; X_1; I_j) - X_j H_{g,n-1}(u; X_j; I_j)}{X_1 - X_j}
\]

Case 4

\[
\frac{X_1}{u} \sum_{\substack{h+h' = g \\ J \sqcup J' = l}} H_{h,1+\#J}(u; X_1; J) C_{h',1+\#J'}(X_1, J')
\]
$l = \{X_2, \ldots, X_n\}$, $l_j = l \setminus \{X_j\}$

\[
\frac{1}{u} \sum_{j=2}^{n} \frac{d}{dX_j} \frac{X_1 H_{g, n-1}(u; X_1; l_j) - X_j H_{g, n-1}(u; X_j; l_j)}{X_1 - X_j}
\]

Case 3

Case 4

Case 5

\[
\frac{X_1}{u} \sum_{h+h' = g \atop J \sqcup J' = l} H_{h, 1+\#J(u; X_1; J)} C_{h', 1+\#J'(X_1, J')}
\]
\[ I = \{X_2, \ldots, X_n\}, \quad l_j = I \setminus \{X_j\} \]

Case 3

\[
\frac{1}{u} \sum_{j=2}^{n} \frac{d}{dX_j} \frac{X_1 H_{g,n-1}(u; X_1; l_j) - X_j H_{g,n-1}(u; X_j; l_j)}{X_1 - X_j}
\]

Case 4

Case 5

\[
\frac{X_1}{u} \sum_{\substack{h+h'=g \\text{ and} \ J \cup J' = l}} H_{h,1+\# J}(u; X_1; J) C_{h',1+\# J'}(X_1, J')
\]

\[
\frac{X_1}{u} H_{g-1,n+1}(u; X_1; X_1, l)
\]
Tutte's equation

\[ H_{g,n}(u; X_1; l) + \frac{X_1}{u} C_{g,n}(X_1, \ldots, X_n) = \frac{v}{u} H_{g,n}(u; X_1; l) + \frac{1}{u} \sum_{k=1}^{K} p_k \left\{ X_1^k H_{g,n}(u; X_1; l) \right\} \]

\[ + \frac{1}{u} \sum_{j=2}^{n} \frac{d}{dX_j} \left( \frac{X_1 H_{g,n-1}(u; X_1; l_j) - X_j H_{g,n-1}(u; X_j; l_j)}{X_1 - X_j} \right) \]

\[ + \frac{X_1}{u} \sum_{\substack{h+h'=g \\ J \sqcup J' = I}} H_{h,1+\#J}(u; X_1; J) C_{h',1+\#J'}(X_1, J') + \frac{X_1}{u} H_{g-1,n+1}(u; X_1; X_1; l) \]
Tutte’s equation

\[ H_{g,n}(u; X_1; l) + \frac{X_1}{u} C_{g,n}(X_1, \ldots, X_n) = \frac{\nu}{u} H_{g,n}(u; X_1; l) + \frac{1}{u} \sum_{k=1}^{K} p_k \left\{ X_1^k H_{g,n}(u; X_1; l) \right\} < 0 \]

\[ + \frac{1}{u} \sum_{j=2}^{n} \frac{d}{dX_j} \frac{X_1 H_{g,n-1}(u; X_1; l_j)}{X_1 - X_j} - X_j H_{g,n-1}(u; X_1; l_j) \]

\[ + \frac{X_1}{u} \sum_{h+h' = g \atop J \sqcup J' = l} H_{h,1+\#J}(u; X_1; J) C_{h',1+\#J'}(X_1, J') + \frac{X_1}{u} H_{g-1,n+1}(u; X_1; X_1, l) \]

Set \( \sum \) discarding terms with topology \((0, 1)\), and:

\[ \tilde{H}_{g,n}(u; X_1, l) := H_{g,n}(u; X_1, l) - \delta_{g,0} \delta_{n,1} \]

\[ \tilde{C}_{g,n}(X_1, l) := C_{g,n}(X_1, l) + \frac{\delta_{g,0} \delta_{n,2}}{(X_1 - X_2)^2} \]

\[ Y(X_1) := \frac{\nu}{X_1} + \sum_{k=1}^{K} p_k X_1^{k-1} + C_{0,1}(X_1) \]
Tutte’s equation

**Tutte’s equation for constellations** \((2g - 2 + n > 0)\)

\[
\left( \frac{u}{X_1} - Y(X_1) \right) \tilde{H}_{g,n}(u; X_1, I) = \tilde{H}_{0,1}(u; X_1) \tilde{C}_{g,n}(X_1, I)
\]

\[
+ \sum_{\begin{array}{c} h + h' = g \\ J \sqcup J' = I \end{array}} \tilde{H}_{h,1+\#J}(u; X_1; J) \tilde{C}_{h',1+\#J'}(X_1, J') + \tilde{H}_{g-1,n+1}(u; X_1; X_1, I)
\]

\[
- \frac{1}{X_1} \sum_{k=1}^{K} p_k \left\{ X_1^k \tilde{H}_{g,n}(u; X_1; I) \right\} \geq 0 - \frac{1}{X_1} \sum_{j=2}^{n} \frac{d}{dX_j} \frac{X_j \tilde{H}_{g,n-1}(u; X_j; I_j)}{X_1 - X_j}
\]

where \(\sum_{\text{no (0,1)}}\) discards terms with topology \((0, 1)\), and:

\[
\tilde{H}_{g,n}(u; X_1, I) := H_{g,n}(u; X_1, I) - \delta_{g,0}\delta_{n,1}
\]

\[
\tilde{C}_{g,n}(X_1, I) := C_{g,n}(X_1, I) + \frac{\delta_{g,0}\delta_{n,2}}{(X_1 - X_2)^2}
\]

\[
Y(X_1) := \frac{v}{X_1} + \sum_{k=1}^{K} p_k X_1^{k-1} + C_{0,1}(X_1)
\]
1 Definitions, motivation, context
   - Constellations: definitions
   - Motivation
   - Scope of the talk

2 Tutte/Loop equations

3 Solving higher loop equations
   - Spectral curve
   - Topological recursion
Theorem (Spectral curve) [23+], [Fang ’16]

Carry out the change of variables $(t, X) \leftrightarrow (S, z)$:

$$
T = \frac{S}{\left( \nu + \sum_{k=1}^{K} p_k S^k \binom{m-1}{k} \right)^{m-1}}; \quad X(z) = \frac{S (1 + z)^m}{z}
$$
Theorem (Spectral curve) [23+, Fang ’16]

Carry out the change of variables \((t, X) \leftrightarrow (S, z)\):

\[
t = \frac{S}{\left( v + \sum_{k=1}^{K} p_k S^k \left( \binom{m k - 1}{k} \right) \right)^{m-1}} ; \quad X(z) = \frac{S(1 + z)^m}{z}
\]

Then:

\[
Y(X(z)) = \frac{z}{S(1 + z)^{m-1}} \left( v + \sum_{k=1}^{K} p_k S^k \sum_{\ell=0}^{k} \binom{m k - 1}{k - \ell} z^{-\ell} \right)
\]
Theorem (Spectral curve) [23+, Fang ’16]

Carry out the change of variables \((t, X) \leftrightarrow (S, z)\):

\[
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\]

\[
\widetilde{C}_{0,2}(X(z_1), X(z_2)) = \frac{1}{X'(z_1)X'(z_2)(z_1 - z_2)^2}
\]
Theorem \((\textbf{Spectral curve})\)\cite{23+, Fang'16}

Carry out the change of variables \((t, X) \leftrightarrow (S, z)\):

\[ t = \frac{S}{\left( v + \sum_{k=1}^{K} p_k S^k \binom{m}{k} \right)^{m-1}} \quad ; \quad X(z) = \frac{S(1 + z)^m}{z} \]

Then:

\[ Y(X(z)) = \frac{z}{S(1 + z)^{m-1}} \left( v + \sum_{k=1}^{K} p_k S^k \sum_{\ell=0}^{k} \binom{mk - 1}{k - \ell} z^{-\ell} \right) \]

\[ \tilde{C}_{0,2}(X(z_1), X(z_2)) = \frac{1}{X'(z_1)X'(z_2)(z_1 - z_2)^2} \]

\textbf{Strategy of the proof}

- \(Y\): guess and check. Tools: Tutte’s equation and harmonizing operator.
- \(\tilde{C}_{0,2}\): careful analysis of the poles. Tools: Tutte’s equation.
Harmonizing operators

[Bousquet-Mélou–Chapuy–Préville-Ratelle ’13], [Fang ’16].

Set $x, y$ to be the rational functions given on previous slide.
Harmonizing operators

[Bousquet-Mélou–Chapuy–Préville-Ratelle ’13], [Fang ’16].

Set $x, y$ to be the rational functions given on previous slide.

For $z \in \mathbb{C}$, let $Z_0, \ldots, Z_K$ be the solutions of $xy(Z) = xy(z)$.

**Sharp operator $\#$**

Let $f$ be a function.

$$f^\#(z) = \sum_{i=0}^{K} \frac{f(Z_i)}{\prod_{j \neq i} (x(Z_i) - x(Z_j))}$$
Harmonizing operators

[Bousquet-Mélou–Chapuy–Préville-Ratelle ’13], [Fang ’16].

Set $x, y$ to be the rational functions given on previous slide.

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**Sharp operator $\#$**

Let $f$ be a function.

$$f^\#(z) = \sum_{i=0}^{K} \frac{f(Z_i)}{\prod_{j \neq i} (x(Z_i) - x(Z_j))}$$

**Lemma**

Let $Q$ be a polynomial in $x$ of degree $\leq K - 1$:

$$Q^\# = 0, \ (x^K)^\# = 1, \ (x^{-1})^\# = -\frac{p_K}{t(xy)^m}$$
Harmonizing operators

[Bousquet-Mélou–Chapuy–Préville-Ratelle '13], [Fang '16].

Set \( x, y \) to be the rational functions given on previous slide.

For \( z \in \mathbb{C} \), let \( Z_0, \ldots, Z_K \) be the solutions of \( xy(Z) = xy(z) \).

**Sharp operator \( \# \)**

Let \( f \) be a function.

\[
 f^\#(z) = \sum_{i=0}^{K} \frac{f(Z_i)}{\prod_{j \neq i} (x(Z_i) - x(Z_j))} \]

**Flat operator \( \flat \)**

Let \( f \) be a function.

\[
 f^\flat(z) = \sum_{i=0}^{m-1} \frac{f(z(i))}{\prod_{j \neq i} (y(z(i)) - y(z(j)))} \]

**Lemma**

Let \( Q \) be a polynomial in \( x \) of degree \( \leq K - 1 \):

\[
 Q^\# = 0, \quad (x^K)^\# = 1, \quad (x^{-1})^\# = -\frac{p_K}{t(xy)^m} \]

**Lemma**

Let \( Q \) be a polynomial in \( y \) of degree \( \leq m - 2 \):

\[
 Q^\flat = 0 \]
Harmonizing operators

[Bousquet-Mélou–Chapuy–Préville-Ratelle ’13], [Fang ’16].

Set $x$, $y$ to be the rational functions given on previous slide.

For $z \in \mathbb{C}$, let $Z_0, \ldots, Z_K$ be the solutions of $xy(Z) = xy(z)$.

For $z \in \mathbb{C}$, let $z^{(0)}, \ldots, z^{(m-1)}$ be the solutions of $x(Z) = x(z)$.

**Sharp operator $\#$**

Let $f$ be a function.

$$f^\#(z) = \sum_{i=0}^{K} \frac{f(Z_i)}{\prod_{j \neq i} (x(Z_i) - x(Z_j))}$$

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Let $Q$ be a polynomial in $y$ of degree $\leq m - 2$:

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The harmonizing operators allow to get rid of some polynomial terms in $x$, $y$!
Proof for $Y(X(z))$

Tutte’s equation for $(g, n) = (0, 1)$:

$$(u - XY(X))\tilde{H}_{0,1}(u; X) = v - u + \frac{tu^m}{X} - X \sum_{k=1}^{K} p_k \left\{ X^k \tilde{H}_{0,1}(u; X) \right\} \geq 0$$
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\]

Setting $u = XY(X)$, get the algebraic equation: $P(X, XY(X)) = 0$.

**Remark:** there is a unique series $Y(X)$ in $\mathbb{Q}[X^{-1}, p_k, v][[t]]$ s.t.

(i) $Y(X) = \frac{v}{X} + \sum_{k=1}^{K} p_k X^{k-1} + O(t)$

(ii) $P(X, XY(X)) = 0$. 

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(i) $Y(X) = \frac{v}{X} + \sum_{k=1}^{K} p_k X^{k-1} + O(t)$

(ii) $P(X, XY(X)) = 0$.

Strategy: check that the guessed rational functions $x(z), y(z)$ satisfy (i) and (ii).

(i): “routine” check.
Harmonizing operator business

\[ P(X, u) = v - u + \frac{tu^m}{X} - X \sum_{k=1}^{K} p_k \left\{ X^k \tilde{H}_{0,1}(u; X) \right\} \geq 0 \]

(ii) Observation:

\[ P(x(z), u) = \frac{tu^m}{x(z)} + p_K x(z)^K + \text{pol. in } x(z) \text{ of degree } K - 1. \]
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Apply the sharp operator to $$P(x(z), u)$$, using the lemma:

$$\left( P(x(z), u) \right)^\# = \frac{p_K}{x(z)^m y(z)^m} (x(z)^m y(z)^m - u^m)$$

$$\Rightarrow \left( P(x(z), x(z) y(z)) \right)^\# = 0. \text{ Does this imply } P(x(z), x(z) y(z)) = 0?$$
Harmonizing operator business

\[ P(X, u) = v - u + \frac{tu^m}{X} - X \sum_{k=1}^K p_k \left\{ X^k \tilde{H}_{0,1}(u; X) \right\} \geq 0 \]

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Apply the \textbf{sharp operator} to \( P(x(z), u) \), using the lemma:

\[ (P(x(z), u))^\# = \frac{p_K}{x(z)^m y(z)^m} (x(z)^m y(z)^m - u^m) \]

\[ \Rightarrow (P(x(z), x(z) y(z)))^\# = 0. \] Does this imply \( P(x(z), x(z) y(z)) = 0 ? \)

\[ \text{Reconstruction lemma [Fang '16]} \]

If \( A \in SzQ[z, p_k, v][[S]], [S^i]A \) is polynomial in \( z \) of degree \( \leq i \), and \( A^\# = 0 \), then \( A = 0. \)

Upshot: \( P(x(z), x(z) y(z)) = 0. \)
Harmonizing operator business

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If \( A \in Sz\mathbb{Q}[z, p_k, v][[S]], [S^i]A \) is polynomial in \( z \) of degree \( \leq i \), and \( A^\# = 0 \), then \( A = 0 \).

**Upshot:** \( P(x(z), x(z) y(z)) = 0. \)

**Corollary**

\[ \tilde{H}_{0,1}(u; x(z)) = \frac{t}{x(z)} \prod_{i=1}^{m-1} \left( u - x(z)y(z^{(i)}) \right) \]
For constellations, TR boils down to this result:

**Theorem (Topological Recursion for constellations) [23+]**

For $g \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$ s.t. $2g - 2 + n > 0$:

$$
\tilde{C}_{g,n}(x(z_1), l)dx(z_1) = \sum_{b \text{ s.t. } dx(b)=0} \text{Res } d\xi(z) \int_{\xi=0}^{z} \frac{dz_1 d\xi}{(z_1 - \xi)^2} \left( \tilde{H}_{g-1,n+1}(xy; x, x, l) 
+ \sum_{h+h'=g \atop J \sqcup J' = l} \tilde{H}_{h,1+\#J}(xy; x, J) \tilde{C}_{h',1+\#J'}(x, J') \right)
$$

where $l = \{x(z_2), \ldots, x(z_n)\}$, $x = x(z)$ and $y = y(z)$. 

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Fastforward proof

Tutte:

\[
\left( \frac{u}{X_1} - Y(X_1) \right) \tilde{H}_{g,n}(u; X_1, I) = \tilde{H}_{0,1}(u; X_1) \tilde{C}_{g,n}(X_1, I) \\
+ \sum_{\substack{h+h'=g \no (0,1) \\ J\sqcup J'=I}} \tilde{H}_{h,1+\#J}(u; X_1; J) \tilde{C}_{h',1+\#J'}(X_1, J') + \tilde{H}_{g-1,n+1}(u; X_1; X_1, I) \\
- \frac{1}{X_1} \sum_{k=1}^{K} p_k \left\{ X_1^k \tilde{H}_{g,n}(u; X_1; I) \right\}_{\geq 0} - \frac{1}{X_1} \sum_{j=2}^{n} \frac{d}{dX_j} \frac{X_j \tilde{H}_{g,n-1}(u; X_j; I_j)}{X_1 - X_j}
\]
Tutte:

\[
\left( \frac{u}{X_1} - Y(X_1) \right) \tilde{H}_{g,n}(u; X_1, I) = \tilde{H}_{0,1}(u; X_1) \tilde{C}_{g,n}(X_1, I)
\]

\[
+ \sum_{\text{no } (0,1)} \tilde{H}_{h,1+\#J}(u; X_1; J) \tilde{C}_{h',1+\#J'}(X_1; J') + \tilde{H}_{g-1,n+1}(u; X_1; X_1, I)
\]

\[
+ P_{g,n}(u; X_1; I)
\]
Fastforward proof

\[ u = X_1 Y(\lambda_1): \]

\[
\left( \frac{u}{X_1} - Y(\lambda_1) \right) \tilde{H}_{g,n}(u; X_1, I) = \tilde{H}_{0,1}(u; X_1) \tilde{C}_{g,n}(X_1, I)
\]

\[
\text{no } (0,1)
\]

\[
+ \sum_{\begin{array}{c}
    h + h' = g \\
    J \sqcup J' = I
\end{array}} \tilde{H}_{h,1+\#J}(u; X_1; J) \tilde{C}_{h',1+\#J'}(X_1; J') + \tilde{H}_{g-1,n+1}(u; X_1; X_1, I)
\]

\[
+ P_{g,n}(u; X_1; I)
\]
Fastforward proof

\[ u = X_1 Y(X_1): \]

\[ 0 = \tilde{H}_{0,1}(X_1 Y_1; X_1) \tilde{C}_{g,n}(X_1, I) \]

\[ \text{no (0,1)} \]

\[ + \sum_{h+h'=g, J\sqcup J'=I} \tilde{H}_{h,1+\#J}(X_1 Y_1; X_1; J) \tilde{C}_{h',1+\#J'}(X_1, J') + \tilde{H}_{g-1,n+1}(X_1 Y_1; X_1; X_1, I) \]

\[ + P_{g,n}(X_1 Y_1; X_1; I) \]
Cauchy formula

\[ 0 = \tilde{H}_{0,1}(X_1, Y_1; X_1) \tilde{C}_{g,n}(X_1, I) \]

\[ + \sum_{\substack{h+h' = g \\cap J \cap J' = I}} \tilde{H}_{h,1+\#J}(X_1, Y_1; X_1; J) \tilde{C}_{h',1+\#J'}(X_1; J') + \tilde{H}_{g-1,n+1}(X_1, Y_1; X_1; X_1, I) \]

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Cauchy formula

\[
\tilde{C}_{g,n}(x(z_1), l) dx(z_1) = - \text{Res} \left[ z = z_1 \right] \frac{\int_{\zeta = 0}^{z} \frac{dz_1 d\zeta}{(z_1 - \zeta)^2}}{H_{0,1}(xy; x)} \left( \tilde{H}_{g-1,n+1}(xy; x; x, l) \right.

\left. + \sum_{\text{no (0,1)}} \tilde{H}_{h,1} + \# J(xy; x; J) \tilde{C}_{h',1} + \# J'(x, J') + P_{g,n}(xy; x; l) \right)
\]
Deformation of contours:

\[
\tilde{C}_{g,n}(x(z_1), I) dx(z_1) = - \text{Res}_{z=z_1} dx(z) \frac{\int_{\zeta=0}^{z} \frac{dz_1 d\zeta}{(z_1 - \zeta)^2}}{H_{0,1}(xy; x)} \left( \tilde{H}_{g-1,n+1}(xy; x; x, I) \right. \\
+ \sum_{\text{no (0,1)}} \tilde{H}_{h,1+\#J}(xy; x; J) \tilde{C}_{h',1+\#J'}(x, J') + P_{g,n}(xy; x; I) \\
\left. + \sum_{h+h'=g \atop J \sqcup J' = I} \tilde{H}_{h,1+\#J}(xy; x; J) \tilde{C}_{h',1+\#J'}(x, J') + P_{g,n}(xy; x; I) \right)
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+ \sum_{h + h' = g \atop J \sqcup J' = l} \tilde{H}_{h,1+\#J}(xy; x; J) \tilde{C}_{h',1+\#J'}(x, J') + P_{g,n}(xy; x; l)
\]
Flat operator: $P_{g,n}^b(xy,x) = 0 \rightarrow$ the last term does not contribute.

$$\tilde{C}_{g,n}(x(z_1), I)dx(z_1) = \sum_{b, dx(b)=0} \text{Res}_b \int_{\tilde{z}=b} d(x(z)) \frac{dz_1 d\zeta}{\tilde{H}_{0,1}(xy;x)} \left( \tilde{H}_{g-1,n+1}(xy;x;x,x,I) ight)$$

$$+ \sum_{h+h'=g \atop J \sqcup J' = l} \tilde{H}_{h,1 + \# J}(xy;x;J) \tilde{C}_{h',1 + \# J'}(x,J') + P_{g,n}(xy;x;I)$$
Flat operator: $P_{g,n}^b(xy,x) = 0 \rightarrow$ the last term does not contribute.

\[
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\]

\[+ \sum_{h+h'=g \atop J \cup J'=l} \tilde{H}_{h,1+\#J}(xy;x;J) \tilde{C}_{h',1+\#J'}(x,J') \]
Take away:

- For higher degree loop equations: allow to deform a bit the model with a catalytic variable $\rightarrow$ get quadratic equation.
- Use the harmonizing operators to get rid of annoying polynomial terms.

Other applications:
- Ciliated maps [Belliard–C–Eynard–Garcia-Failde '21], related to:
  - Combinatorics: fully simple maps [Borot–C–Garcia-Failde '21].
  - Enumerative geometry: intersection numbers of $r$-spin Witten class on $M_{g,n}$.
  - Integrable hierarchies: $r$th reduction of KP hierarchy.

To be done:
- Write the paper...
- Apply the same method to other models, e.g. the Brézin–Gross–Witten matrix model $Z_{BGW} = Z_{HN} d M e^{-\frac{1}{2} Tr M^1 - r Tr + \frac{1}{2} N \log(M) + MY}$
- Would it be useful for $m$-Tamari intervals [Bousquet-Mélou–Chapuy–Prévile-Ratelle '13]?

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Conclusion

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$$Z^{BGW} = \int_{H_N} dMe^{-\frac{1}{2} Tr\left( \frac{M^1-r}{1-r} + h \frac{1}{2} N \log(M) + MY \right)}$$

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THANKS FOR YOUR ATTENTION!
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Higher TR
Cylinder