Symbolic dynamics and substitution shifts

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Overview

First part: Symbolic dynamics

- Symbolic dynamics
- Conjugacy. State splitting

Second part: Substitution shifts

- Substitution shifts
- S-adic shifts
- Recognizability
- Open problems

Symbolic dynamics

Let A be a finite alphabet.

Let $A^{\mathbb{Z}}$ be the set of two-sided infinite sequences (also called points), a compact metric space.

The *shift transformation* $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by

$$y = S(x) \text{ if } y_n = x_{n+1} \text{ for every } n \in \mathbb{Z}.$$

A *shift space* X on a finite alphabet A is a closed and shift invariant subset of $A^{\mathbb{Z}}$.

Symbolic dynamics

Let A be a finite alphabet and X a shift space.

The factors (or blocks) of X are the finite words u such that u appears in some point of X.

A point *avoids* a finite word u if u does not appear as factor in the point.

Shift spaces are also the set X_F of all points avoiding all the words of F, for some set of finite words F.

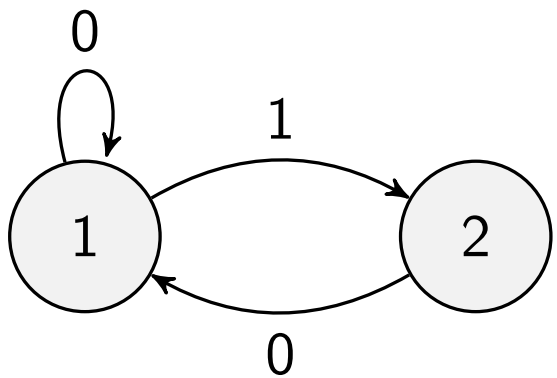
Examples of shift spaces

The *full shift* is $A^{\mathbb{Z}}$ (all points).

Shift spaces X_F where F is a finite set are called *shifts of finite type*.
Shift spaces X_F where F is a regular are called *sofic shifts*.

Example

X_F with $F = \{11\}$ is a shift of finite type called the golden mean shift.

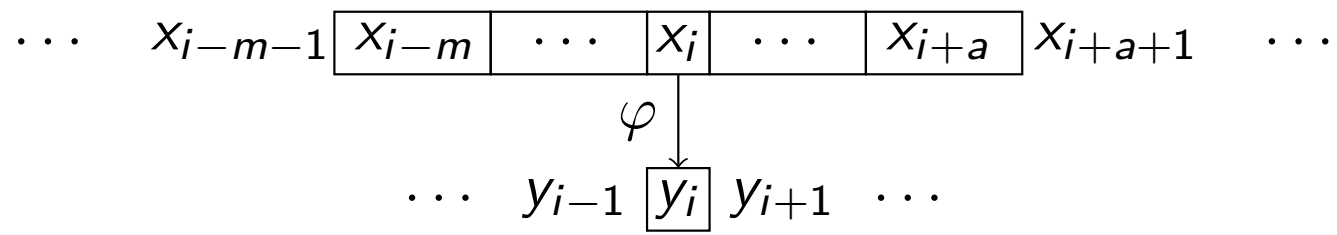


If a sofic shift is irreducible, *i.e.* presented by a finite automaton (labeled graph) which is strongly connected, it has a minimal deterministic presentation.

Conjugacy

A *block map* (or factor map) is a map $\phi\colon A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ for which there are nonnegative integers m, a and a map $\varphi\colon A^{m+1+a} \to B$ such that:

$$\phi(x) = y \text{ if and only if } y_i = \varphi(x_{[i-a,\ldots,i-1,i,i+1,\ldots i+a]}).$$



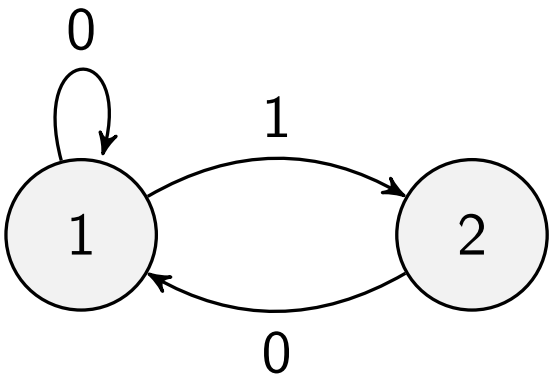
A *conjugacy* (or isomorphism) from a shift space X to a shift space Y is a block map which is invertible. In this case the inverse is also a block map.

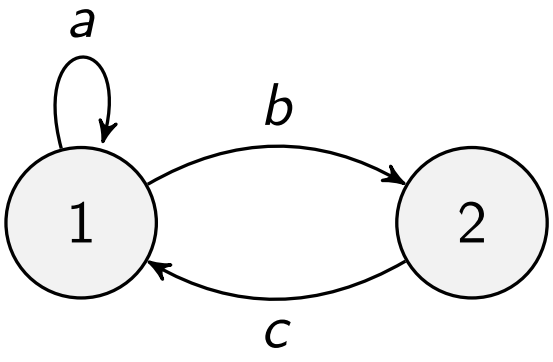
If there is a conjugacy from X to Y, we can think of Y as being a "recoded" version of X.

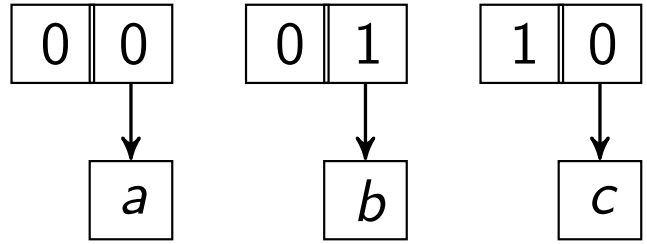
Conjugacy: example

Example

X_F with $F = \{11\}$ is a shift of finite type called the golden mean shift.
X_G with $G = \{ac, ba, bb, cc\}$ is a shift of finite.





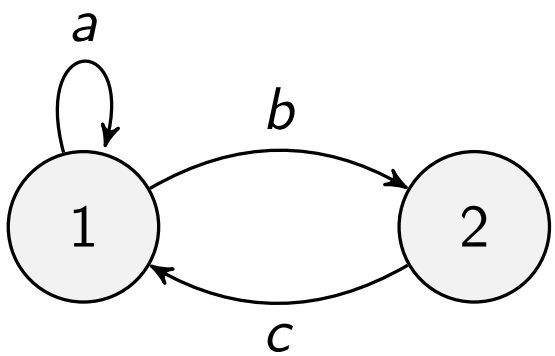


The decidability of conjugacy is an open problem

Open problem

The decidability of the conjugacy of two shifts of finite type is unknown, even for irreducible shifts of finite type.

Taking a minimal presentation of an irreducible shift of finite type, up to a conjugacy, one may assume that all labels are distinct and the shift can be defined by the transition matrix of the graph G of the presentation. Such shifts are called *edge shifts*.



$$M(G) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

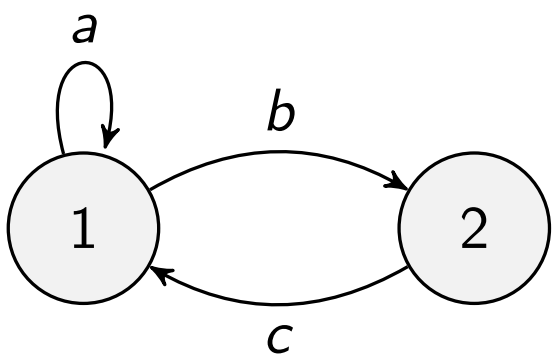
State splitting

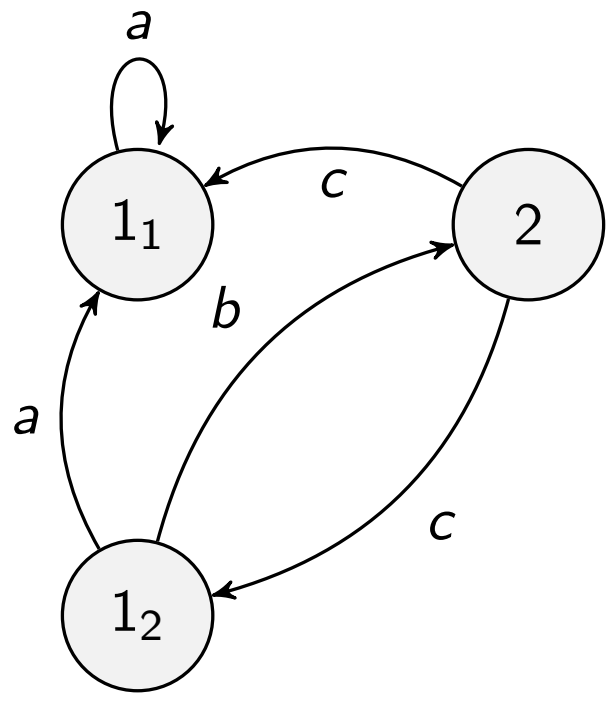
A state splitting is a transformation of an automaton or a graph.

An *output state splitting* of a single state s is obtained as follows:

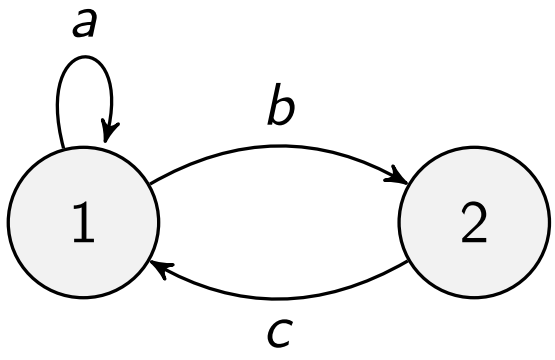
- one splits the output edges of s into two parts E_1 and E_2
- one replaces s with two copies s_1 and s_2
- the edges in E_1 become the output edges of s_1, the edges in E_2 become the output edges of s_2
- each input edge of s is changed into an input edge of s_1 and an input edge of s_2 (with the same labels).

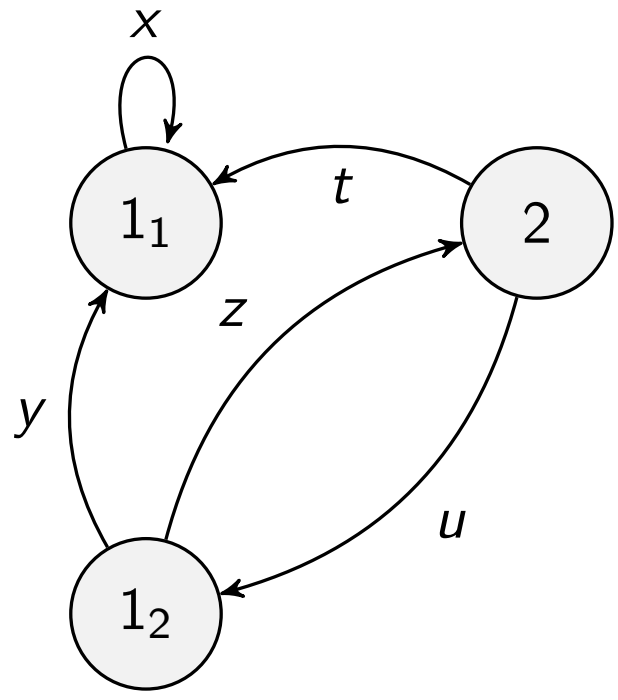
State splitting





State splitting





Proposition

If X is an edge shift defined by a graph G and Y is the edge shift defined by the split graph, then X and Y are conjugate.

State splitting

An *essential graph* is a graph such that each state has at least one incoming edge and at least one outgoing edge.

Theorem (Decomposition Theorem, R. Williams 1973)

Two edge shifts X and Y defined by essential graphs G and H are conjugate if and only there is sequence of (input and output) state splittings and (input and output) state mergings from G to H.

String shift equivalence

Two nonnegative integer matrices M, N are *elementary equivalent* if there are, possibly nonsquare, matrices R, S such that

$$M = RS, N = SR.$$

Two nonnegative integer matrices M, N are *strong shift equivalent* if there is a sequence of elementary equivalences from M to N:

$$\begin{aligned} M &= R_0 S_0, S_0 R_0 = M_1, \\ M_1 &= R_1 S_1, S_1 R_1 = M_2, \\ &\vdots \\ M_\ell &= R_\ell S_\ell, S_\ell R_\ell = N. \end{aligned}$$

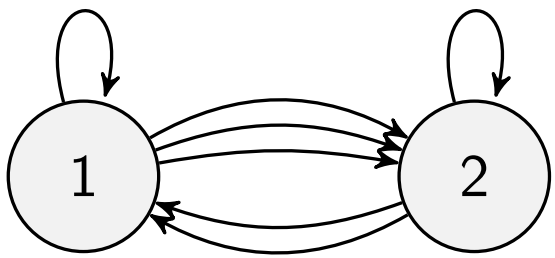
Theorem (Classification Theorem, R. Williams 1973)

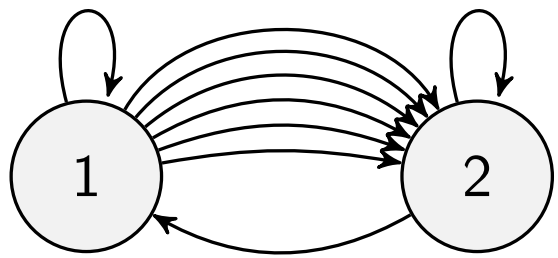
Two edge shifts defined by matrices M and N are conjugate if and only if M and N are strong shift equivalent.

Conjugacy: examples

Let X and Y be the edge shifts defined by the graphs given by the matrices M and N respectively:

$$M = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 6 \\ 1 & 1 \end{bmatrix}$$





The shifts X and Y are conjugate (K. Baker, using computer research to prove strong shift equivalence).

Conjugacy: examples

It is not known whether X and Y defined by M_k and N_k are conjugate for $k \geq 4$.

$$M_k = \begin{bmatrix} 1 & k \\ k-1 & 1 \end{bmatrix}, \quad N_k = \begin{bmatrix} 1 & k(k-1) \\ 1 & 1 \end{bmatrix}$$

Some conjugacy invariants

- *(Topological) entropy* :

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Card}(\mathcal{L}_n(X)),$$

where $\mathcal{L}_n(X)$ is the set of factors of length n of points of X.
For instance, $h(A^{\mathbb{Z}}) = \log \mathrm{Card}(A)$.

Some conjugacy invariants

- *Zeta function*: determines the numbers of periodic sequences of all periods.

$$\zeta_X(z) = \exp \sum_{n \geq 1}^{\infty} \frac{p_n(X)}{n} z^n,$$

where $p_n(X)$ is the number of points x such that $S^n(x) = x$.

The zeta function of an edge shift is $\frac{1}{\det(\mathrm{Id} - M)}$ (R. Bowen, O. E. Lanford 1970).

The zeta function of a sofic shift is $\mathbb{Z}$-rational (A. Manning 1971, R. Bowen 1978). It is further $\mathbb{N}$-rational (C. Reutenauer 1997).

Some conjugacy invariants

- Invariants of strong shift equivalence obtained from the category point of vue (E. Jeandel 2021).

Algebraic conjugacy invariants

For an edge shift defined by a matrix M of size $r \times r$.

- The Bowen-Franks group is $\mathrm{BF}(M) = \mathbb{Z}^r / \mathbb{Z}^r(\mathrm{Id} - M)$, where $\mathbb{Z}^r(\mathrm{Id} - M)$ is the image of $\mathbb{Z}^r$ under the matrix $\mathrm{Id} - M$ acting on the right.
- The eventual range $\mathcal{R}_M$ is $\bigcap_{k=1}^{\infty} \mathbb{Q}^r M^k$.
- $\Delta_M = \{v \in \mathcal{R}_M \mid vM^k \in \mathbb{Z}^r \text{ for some } k \geq 0\}$.
- The automorphism δ_M of M is the restriction of M to Δ_M, so that $\delta_M(v) = vM$ for $v \in \Delta_M$.
- $\Delta_M^+ = \{v \in \mathcal{R}_M \mid vM^k \in (\mathbb{Z}^+)^r \text{ for some } k \geq 0\}$.
- The dimension triple $(\Delta_M, \Delta_M^+, \delta_M)$, up to group isomorphisms, is a conjugacy invariant.

Classes of shifts for which conjugacy is decidable

- A *one-sided shift space* is the set of right-infinite sequences that appear in a shift space X.
  One-sided conjugacies are now block maps with no memory.
  Conjugacy between two one-sided shifts of finite type is decidable (R. Williams 1973).
- Conjugacy between minimal substitution shifts is decidable (F. Durand, J. Leroy 2022).
  The problem is open for non-minimal substitution shifts.

Substitution shifts

Let A, B be finite alphabets.
A *substitution* is a morphism $\sigma\colon A^* \to B^*$.

When $B = A$ one may iterate σ.
It is *non-erasing* if $\sigma(a)$ is nonempty for every $a \in A$.

The *language* of σ, denoted $\mathcal{L}(\sigma)$ is the set of factors of the words $\sigma^n(a)$ for $n \geq 0$.

The *substitution shift defined* by σ, denoted by $X(\sigma)$ is the set of bi-infinite sequences with all their factors in $\mathcal{L}(\sigma)$.

The *language* of $X(\sigma)$ is the set $\mathcal{L}(X(\sigma))$ of factors of points in $X(\sigma)$.

A morphism is *primitive* if there is an integer n such that for each letters a, b, the letter b appears in $\sigma^n(a)$.

Substitution shifts

Example

The *Fibonacci shift* is defined by the morphism $\sigma : a \mapsto ab, b \mapsto a$.

$$\cdots abaab \cdot abaababa \cdots$$

Recognizability

The recognizability is a form of injectivity of σ that allows one to uniquely *desubstitute* y to another sequence x, with x traditionally required to be in $X(\sigma)$, *i.e.* express y as a concatenation of substitution words dictated by the letters in x.

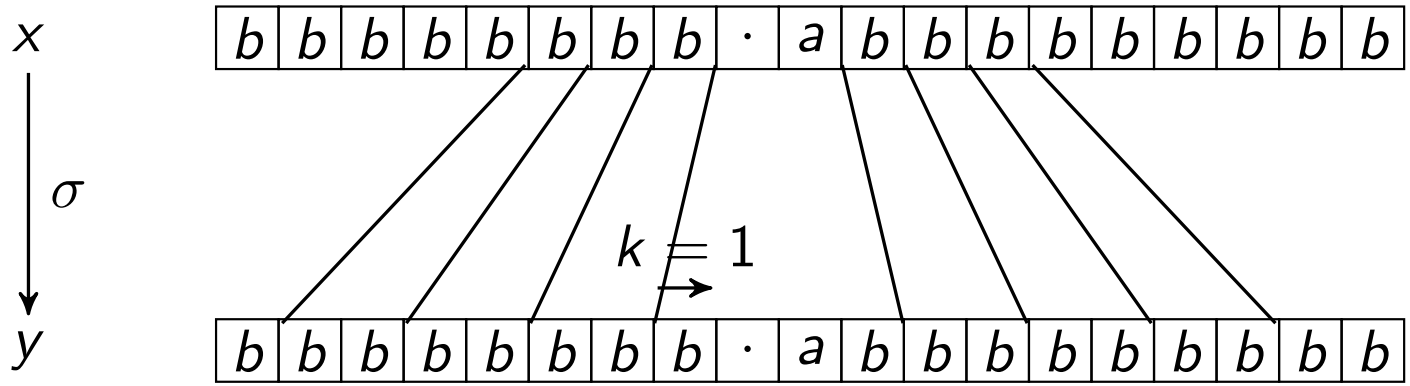
Recognizability of morphisms

Definition

Let $\sigma\colon A^* \to B^*$. A *centered σ-representation* of $y \in B^{\mathbb{Z}}$ is a pair (x, k) of a sequence $x \in A^{\mathbb{Z}}$ and an integer k with $0 \leq k < |\sigma(x_0)|$ such that $y = S^k(\sigma(x))$.

Example

Let $\sigma\colon a \mapsto bab, b \mapsto bb$.



Recognizability of morphisms

Definition

Let X be a shift space on A. A morphism $\sigma\colon A^* \to B^*$ is *recognizable* in X (respectively recognizable in X for aperiodic points) if for every point $y \in B^{\mathbb{Z}}$ (respectively every aperiodic point $y \in B^{\mathbb{Z}}$) there is at most one centered σ-representation (x, k) of y with $x \in X$.

Definition

Let X be a shift space on A. A morphism $\sigma\colon A^* \to B^*$ is *fully recognizable* (respectively fully recognizable for aperiodic points) if it is recognizable in $A^{\mathbb{Z}}$ (respectively recognizable in $A^{\mathbb{Z}}$ for aperiodic points).

Recognizability of morphisms

Example

The Fibonacci morphism $\sigma : a \mapsto ab, b \mapsto a$ is fully recognizable.

Example

The Thue-Morse morphism $\sigma : a \mapsto ab, b \mapsto ba$ is not fully recognizable since $(ab)^\infty$ can be obtained as $\sigma(a^\infty)$ and as $S(\sigma(b^\infty))$. However, it is fully recognizable for aperiodic points since any sequence containing aa or bb has at most one factorization in $\{ab, ba\}$.

Recognizability of morphisms

Proposition

The family of morphisms which are fully recognizable for aperiodic points is closed under composition.

Elementary morphisms

Definition (A. Ehrenfeucht, G. Rozenberg 1978)

A morphism $\sigma\colon A^* \to C^*$ is *elementary* if for every alphabet B and every pair of morphisms $\beta\colon A^* \to B^*$ and $\alpha\colon B^* \to C^*$ such that $\sigma = \alpha \circ \beta$, one has $\mathrm{Card}(B) \geq \mathrm{Card}(A)$.

If $\sigma\colon A^* \to C^*$ is elementary, one has in particular $\mathrm{Card}(C) \geq \mathrm{Card}(A)$ and moreover σ is non-erasing.

Example

The Thue-Morse morphism $\sigma\colon a \mapsto ab, b \mapsto ba$ is elementary. Indeed, if $\sigma = \alpha \circ \beta$ with $\beta\colon \{a, b\}^* \to c^*$, then $ab = \alpha(c^i)$ and $ba = \alpha(c^j)$ which is impossible.

Elementary morphisms

Theorem (J. Karhumäki, J. Maňuch, W. Plandowski 2003)

An elementary morphism is fully recognizable for aperiodic points.

Other proofs by V. Berthé, W. Steiner, J. Thuswaldner, R. Yassawi 2019 with a stronger hypothesis, and B., D. Perrin, A. Restivo 2022

Recognizability of one morphism

- Mossé's Theorem (B. Mossé 1992, B. Mosse 1996): every aperiodic primitive morphism σ is recognizable in the shift $X(\sigma)$.
- Generalization by S. Bezuglyi, J. Kwiatkowski, K. Medynets 2009, who proved that every aperiodic non-erasing morphism σ is recognizable in $X(\sigma)$.
- Generalisation by V. Berthé, W. Steiner, J. Thuswaldner, R. Yassawi 2019, who proved that every non-erasing morphism σ is recognizable in $X(\sigma)$ for aperiodic points.
- Extension to possibly erasing morphisms by B., D. Perrin, A. Restivo 2022.

Theorem (B., D. Perrin, A. Restivo 2022)

Any morphism σ is recognizable for aperiodic points in $X(\sigma)$.

S-adic shifts

Let $\sigma = (\sigma_n)_{n\geq 0}$ be a sequence of morphisms $\sigma_n \colon (A_{n+1})^* \to (A_n)^*$, where A_n are finite alphabets:

$$\cdots A_{n+1} \xrightarrow{\sigma_n} A_n \xrightarrow{\sigma_{n-1}} A_{n-1} \cdots A_1 \xrightarrow{\sigma_0} A_0.$$

For $0 \leq n \leq m$ we define

$$\sigma_{[n,m)} = \sigma_n \circ \sigma_{n+1} \circ \cdots \circ \sigma_{m-1}.$$

- $\mathcal{L}^{(n)}(\sigma)$ is the set of factors of the words $\sigma_{[n,m)}(a)$ for some $a \in A_m$ and some $m > n$.
- The shift $X^{(n)}(\sigma)$ is the set of sequences with all their factors in $\mathcal{L}^{(n)}(\sigma)$.
- The *S-adic* shift defined by σ is $X^{(0)}(\sigma)$.
- A sequence of morphisms $\sigma = (\sigma_n)_{n\geq 0}$ is *non-erasing* if all σ_n are non-erasing.

Representability of S-adic shifts

A sequence of morphisms σ is *representable at level n* if every point y in $X^{(n)}(\sigma)$ has at least one σ_n-representation in $X^{(n+1)}(\sigma)$(*i.e.* a point x with $y = S^i(\sigma_n(x))$).

A sequence of morphisms is *representable* if it is representable at each level.

It is *eventually representable* if there is an integer M such that it is representable at each level at least equal to M.

Representability of S-adic shifts

Erasing sequences of morphisms may not be representable:

$$\sigma_n \colon x \mapsto bax, a \mapsto aa, b \mapsto b, \text{ for } n \geq 1,$$
$$\sigma_0 \colon a \mapsto \varepsilon, b \mapsto b, x \mapsto \varepsilon.$$

One can check that the point $y = \cdots bbbb \cdots$ belongs to $X^{(0)}(\sigma)$ and there is no point x in $X^{(1)}(\sigma)$ such that $\sigma_0(x) = y$.

Representability of S-adic shifts

Nevertheless each non-erasing sequence of morphisms is representable (V. Berthé, W. Steiner, J. Thuswaldner, R. Yassawi 2019).

Proposition (B., D. Perrin, A. Restivo, W. Steiner, 2023 preprint)

Let σ be a sequence of morphisms with alphabets A_n such that $\liminf_{n>0} \mathrm{Card}(A_n)$ is finite. Then σ is eventually representable.

Recognizability of S-adic shifts

A sequence of morphisms σ is *recognizable at level n* (respectively *recognizable at level n for aperiodic points*) in $X^{(n+1)}(\sigma)$ if σ_n is recognizable (respectively recognizable for aperiodic points) in $X^{(n+1)}(\sigma)$.

A sequence of morphisms σ is *recognizable* (respectively *recognizable for aperiodic points*) in its shift spaces if it is recognizable (respectively recognizable for aperiodic points) in its shift spaces at each level.

A sequence of morphisms is *eventually recognizable* (respectively *eventually recognizable for aperiodic points*) in its shift spaces if there is a nonnegative integer i such is recognizable in its shift spaces at each level at least i.

Recognizability of S-adic shifts

A simple example of a sequence of non-erasing morphisms which is eventually recognizable in its shift spaces for aperiodic points, but not recognizable in its shift spaces for aperiodic points.

Example

Let σ be the sequence of morphisms such that $\sigma_n = \tau$ for $n \geq 1$, $\sigma_0 = \phi$ with

$$\tau\colon a \mapsto bab, a' \mapsto b'a'b', b \mapsto b, b' \mapsto b',$$
$$\phi\colon a \mapsto a, a' \mapsto a, b \mapsto b, b' \mapsto b.$$

Then $\cdots bbbbb \cdot abbbbb \cdots$ is an aperiodic point of $X^{(0)}(\sigma)$ which has the two centered representation $\cdots bbbbb \cdot abbbbb \cdots$ and $\cdots b'b'b'b' \cdot a'b'b'b'b'b' \cdots$ in $X^{(1)}(\sigma)$.

Berthé *et al.* give another example which is moreover primitive.

Recognizability of S-adic shifts

A simple example of a sequence of non-erasing morphisms which is not eventually recognizable in its shift spaces.

Example

Let σ be the sequence of morphisms such that $A_n = \{a_0, \dots a_{n+1}\}$ and $\sigma_n \colon (A_{n+1})^* \to (A_n)^*$ is defined by

$$\begin{aligned} \sigma_n \colon a_0 &\mapsto a_0, \\ a_1 &\mapsto a_0 a_1 a_0, \cdots \\ a_{n+1} &\mapsto a_0 a_{n+1} a_0, \\ a_{n+2} &\mapsto a_{n+1}. \end{aligned}$$

Then for each $n \geq 0$, $\cdots a_0 a_0 a_0 a_0 a_0 \cdot a_{n+1} a_0 a_0 a_0 a_0 \cdots$ is an aperiodic point of $X^{(n)}(\sigma)$ which has the two centered representation $\cdots a_0 a_0 a_0 a_0 a_0 \cdot a_{n+1} a_0 a_0 a_0 a_0 \cdots$ and $\cdots a_0 a_0 a_0 a_0 a_0 \cdot a_{n+2} a_0 a_0 a_0 a_0 \cdots$ in $X^{(n+1)}(\sigma)$.

Recognizability of S-adic shifts

Theorem (V. Berthé, W. Steiner, J. Thuswaldner, R. Yassawi 2019)

Let $\sigma = (\sigma_n)_{n\geq 0}$ with $\sigma_n \colon (A_{n+1})^* \to (A_n)^*$ be a non-erasing sequence of morphisms such that $\liminf_{n\geq 0} \mathrm{Card}(A_n)$ is finite and $\mathrm{Card}(\{\mathcal{L}_x \mid x \in X^{(n)}(\sigma)\})$ is bounded. Then σ is eventually recognizable for aperiodic points in its shift spaces.

Recognizability of S-adic shifts

Theorem (B., D. Perrin, A. Restivo, W. Steiner, 2023 preprint)

Let $\sigma = (\sigma_n)_{n\geq 0}$ with $\sigma_n \colon (A_{n+1})^* \to (A_n)^*$ be a sequence of morphisms such that $\liminf_{n\geq 0} \mathrm{Card}(A_n)$ is finite. Then σ is eventually recognizable for aperiodic points in its shift spaces.

Further,

- The proof is very simple. It uses the full recognizability of elementary morphisms for aperiodic points.
- We obtain a bound of $\liminf_{n\geq 0} \mathrm{Card}(A_n) - 2$ for the number of levels at which σ is not recognizable in its shift spaces for aperiodic points (improving the bound of Berthé *et al.*).
- This bound is optimal.

Recognizability of one morphism

We obtain a simple proof of:

Theorem (B., D. Perrin, A. Restivo 2022)

Any morphism σ is recognizable for aperiodic points in $X(\sigma)$.

proof B., D. Perrin, A. Restivo, W. Steiner, 2023.

- Let $\sigma : A^* \to A^*$ and $K = \mathrm{Card}(A)$. Assume that σ is not recognizable for aperiodic points in $X(\sigma)$.
- Then σ it is not elementary. Choose a decomposition $\sigma = \alpha_1 \circ \beta_1$ with $\beta_1 : A^* \to (B_1)^*$ and $\alpha_1 : (B_1)^* \to A^*$ and $\mathrm{Card}(B_1)$ minimal.

Recognizability of one morphism

We obtain a simple proof of:

Theorem (B., D. Perrin, A. Restivo 2022)

Any morphism σ is recognizable for aperiodic points in $X(\sigma)$.

proof B., D. Perrin, A. Restivo, W. Steiner, 2023.

- Then $\sigma \circ \alpha_1$ is not elementary.
  Indeed, let x be an aperiodic point in $X(\sigma)$ with two centered σ-representations (y, k), (y', k') of x in $X(\sigma)$. Let (z, ℓ), (z', ℓ') be two centered σ-representations of y and y' in $X(\sigma)$. This gives two centered $\sigma \circ \alpha_1$-representations $(\beta_1(z), m)$ and $(\beta_1(z'), m')$ of x.

Recognizability of one morphism

We obtain a simple proof of:

Theorem (B., D. Perrin, A. Restivo 2022)

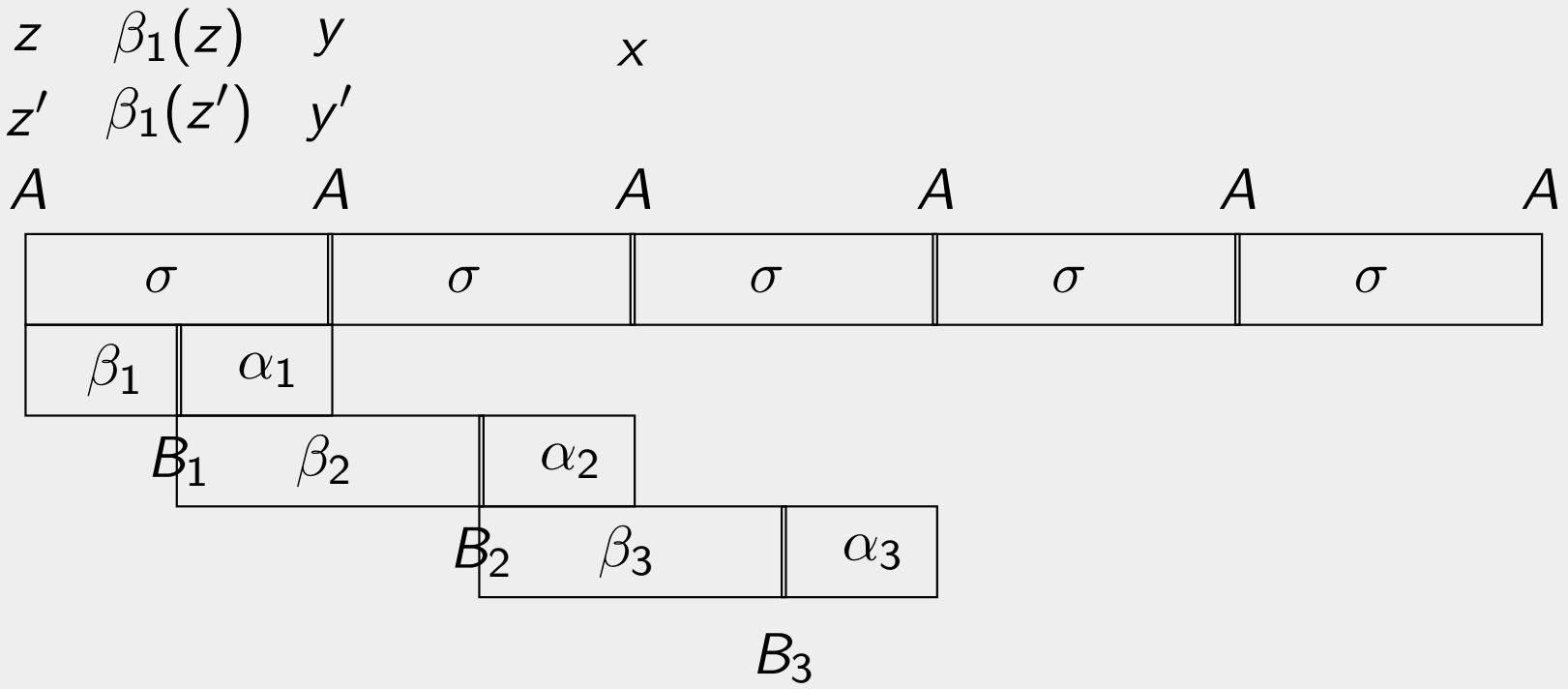
Any morphism σ is recognizable for aperiodic points in $X(\sigma)$.

proof B., D. Perrin, A. Restivo, W. Steiner, 2023.

- Choose a decomposition $\sigma \circ \alpha_1 = \alpha_2 \circ \beta_2$ with $\beta_2 \colon (B_1)^* \to (B_2)^*$ and $\alpha_2 \colon (B_2)^* \to A^*$ and $\mathrm{Card}(B_2)$ minimal.
  One has $\mathrm{Card}(B_2) < \mathrm{Card}(B_1) < \mathrm{Card}(A)$.

Recognizability of one morphism

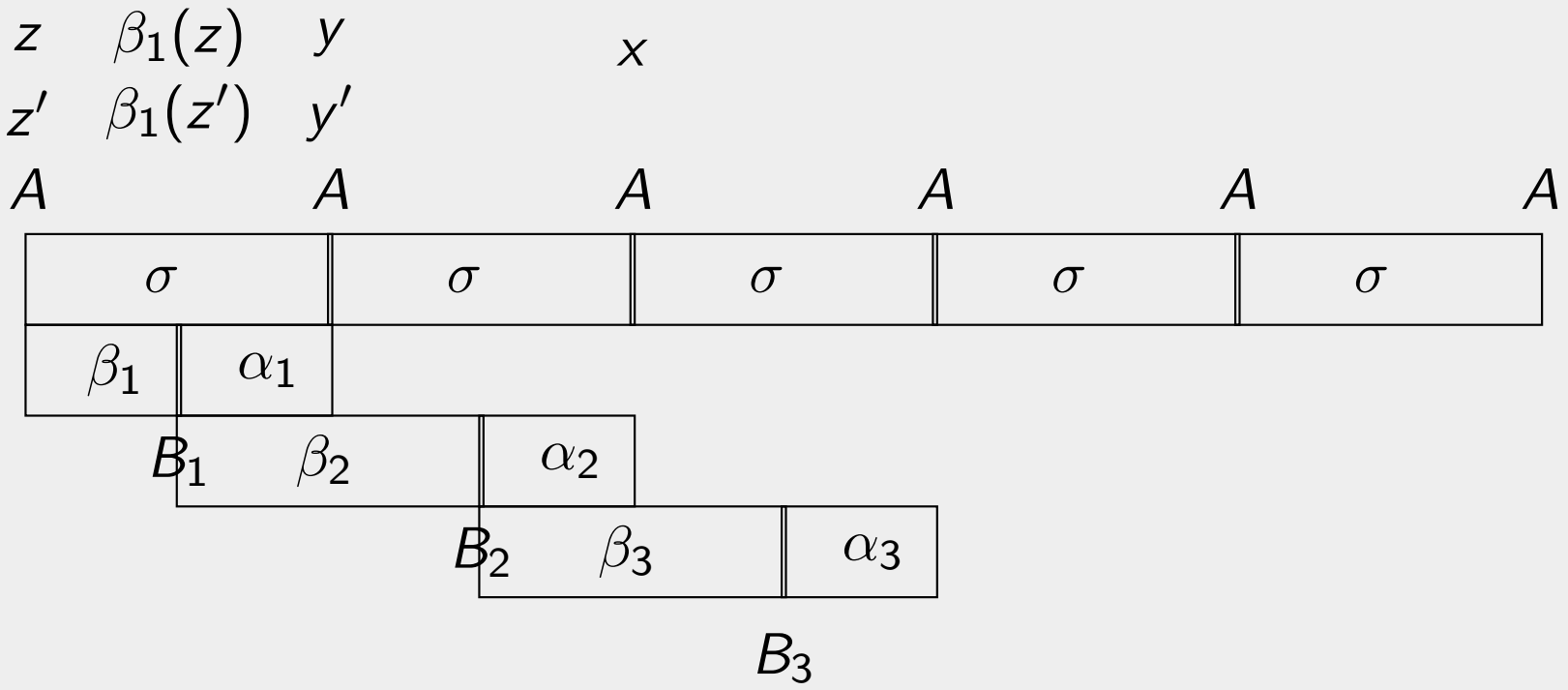
proof B., D. Perrin, A. Restivo, W. Steiner, 2023.



- For each $2 \leq k \leq K-1$, Then $\sigma \circ \alpha_{k-1}$ is not elementary. Choose a decomposition $\sigma \circ \alpha_{k-1} = \alpha_k \circ \beta_k$ with $\beta_{k-1} \colon (B_{k-1})^* \to (B_k)^*$ and $\alpha_k \colon (B_k)^* \to A^*$ and $\mathrm{Card}(B_k)$ minimal.

Recognizability of one morphism

proof B., D. Perrin, A. Restivo, W. Steiner, 2023.



- $\mathrm{Card}(B_{K-1}) < \mathrm{Card}(B_{K-2}) < \cdots < \mathrm{Card}(B_1) < K = \mathrm{Card}(A)$.
- One may assume that $\mathrm{Card}(B_k) \geq 2$ (otherwise σ is recognizable for aperiodic points in $X(\sigma)$). Hence the contradiction.

Recognizability in dimension 2

Considered by S. Mozes in 1989 where the recognizability property is used to prove that some 2D substitution shift are sofic.

A generalization of Mossé's Theorem is proved for self-similar tilings by B. Solomyak in 1998.

A proof of the recognizability in the 2D case has been obtained by M. Sablik (unpublished).

The property is also used by N. Aubrun and M. Sablik in 2013 and 2014 to simulate an effective subshift by a sofic.

Decidable properties of substitution shifts

Proposition

It can be checked in quadratic time in the size of $\sigma(A)$ whether an injective morphism $\sigma : A^* \to B^*$ is recognizable on $A^{\mathbb{Z}}$ for aperiodic points.

Proposition

Let σ be a morphism. It is decidable in linear time whether a letter belongs to $\mathcal{L}(X(\sigma))$.

Proposition

For every morphism $\sigma : A^* \to A^*$, the language $\mathcal{L}(X(\sigma))$ is decidable.

Decidable properties of substitution shifts

Theorem (J. J. Pansiot 1983, T. Harju and M. Linna 1983, for primitive morphisms)

It is decidable whether a morphism is aperiodic. The set of periodic points in $X(\sigma)$ is finite and their periods are effectively bounded in terms of σ.

Theorem

It is decidable whether a morphism is periodic (formed of periodic points).

A morphism $\sigma\colon A^* \to A^*$ is *minimal* if $X(\sigma)$ is a minimal shift space.

Theorem

It is decidable whether a morphism is minimal.

Open problems

Theorem (F. Durand, J. Leroy 2021)

It is decidable whether two minimal substitution shifts are conjugate.

Is conjugay decidable for all substitution shifts?

Is the inclusion problem $X(\sigma) \subseteq X(\tau)$ decidable?
It is decidable for automatic shifts, *i.e.* substitution shifts defined by a constant-length morphism (I. Fagnot 1997).
It is decidable for minimal substitution shifts (F. Durand, J. Leroy 2021).

What is the situation of all decidability results in 2D?

Thank you for your attention