

A quasisymmetric invariant for families of posets

Doriann Albertin

Université de Bordeaux

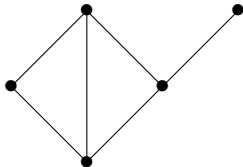
Laboratoire Bordelais de Recherche Informatique

doriann.albertin@labri.fr

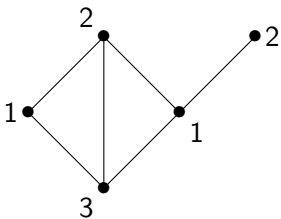
2023-01-30

Joint work with J. C. Aval & H. Mlodecki

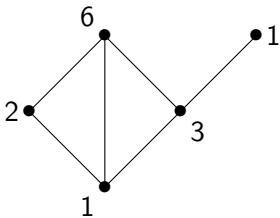
Context



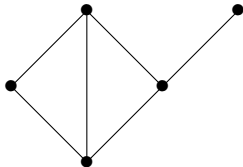
$$X_G =$$



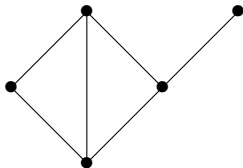
$$X_G = x_1^2 x_2^2 x_3^1 +$$



$$X_G = x_1^2 x_2^2 x_3^1 + x_1^2 x_2^1 x_3^1 x_6^1 + \dots$$



$$X_G = x_1^2 x_2^2 x_3^1 + x_1^2 x_2^1 x_3^1 x_6^1 \\ + \dots \\ \in \text{Sym}.$$



$$X_G = x_1^2 x_2^2 x_3^1 + x_1^2 x_2^1 x_3^1 x_6^1 + \dots$$

$$\in \text{Sym}.$$

Definition (Stanley '95)

Given a graph G , its **chromatic symmetric function** is:

$$X_G(\mathbf{x}) := \sum_{\kappa} \prod_{i \in \mathbb{N}} x_i^{\#\kappa^{-1}(i)} \in \text{Sym} \subsetneq \mathbb{Q}[[\mathbf{x}]],$$

where $\mathbf{x} := \{x_i \mid i \in \mathbb{N}\}$ is a commutative alphabet, and the sum ranges over all proper colorings of G .

Obvious fact: $G \sim H \Rightarrow X_G = X_H$.

Obvious fact: $G \sim H \Rightarrow X_G = X_H$.

Obvious non-fact: the converse is false.

Obvious fact: $G \sim H \Rightarrow X_G = X_H$.

Obvious non-fact: the converse is false.

Conjecture (Stanley '95)

Chromatic symmetric functions distinguish trees.

Obvious fact: $G \sim H \Rightarrow X_G = X_H$.

Obvious non-fact: the converse is false.

Conjecture (Stanley '95)

Chromatic symmetric functions distinguish trees.

Read "*the restriction of the map $G \mapsto X_G$ to trees is injective*".

Obvious fact: $G \sim H \Rightarrow X_G = X_H$.

Obvious non-fact: the converse is false.

Conjecture (Stanley '95)

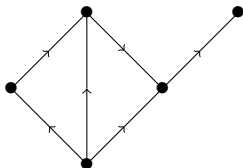
Chromatic symmetric functions distinguish trees.

Read "*the restriction of the map $G \mapsto X_G$ to trees is injective*".

This conjecture spanned a lot of interest and is still open.

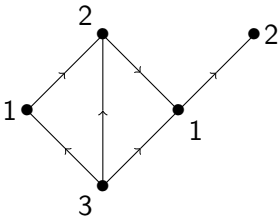
This is not its story.

Oriented graphs



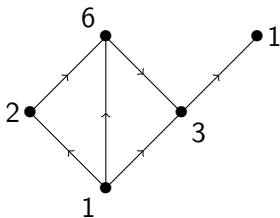
$$X_G =$$

Oriented graphs



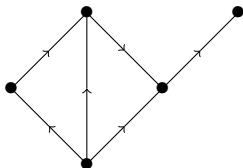
$$X_G = t^2 \cdot x_1^2 x_2^2 x_3^1 +$$

Oriented graphs

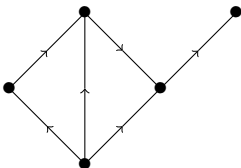


$$X_G = t^2 \cdot x_1^2 x_2^2 x_3^1 + t^4 \cdot x_1^2 x_2^1 x_3^1 x_6^1 + \dots$$

Oriented graphs



$$\begin{aligned}
 X_G &= t^2 \cdot x_1^2 x_2^2 x_3^1 + \\
 &\quad t^4 \cdot x_1^2 x_2^1 x_3^1 x_6^1 + \dots \\
 &\in \text{QSym}[t].
 \end{aligned}$$



$$\begin{aligned}
 X_G &= t^2 \cdot x_1^2 x_2^2 x_3^1 + \\
 &\quad t^4 \cdot x_1^2 x_2^1 x_3^1 x_6^1 + \dots \\
 &\in \text{QSym}[t].
 \end{aligned}$$

Definition (Shareshian & Wachs '16, Ellzey '17)

Given an oriented \vec{G} with no cycle, its **chromatic quasisymmetric function** is:

$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{\kappa} t^{\text{asc}(\kappa)} \prod_{i \in \mathbb{N}} x_i^{\#\kappa^{-1}(i)} \in \text{QSym}[t] \subsetneq \mathbb{Q}[[\mathbf{x}]] [t],$$

where $\text{asc}(\kappa)$ denotes the number of *naturally labeled* edges in \vec{G} , meaning that the orientation respects the natural order on the colors.

Conjecture (Alexandersson & Sulzgruber '21)

Chromatic quasisymmetric functions distinguish directed trees.

Conjecture (Alexandersson & Sulzgruber '21)

Chromatic quasisymmetric functions distinguish directed trees.

Observation

This conjecture is neither stronger nor weaker than Stanley's!

Conjecture (Alexandersson & Sulzgruber '21)

Chromatic quasisymmetric functions distinguish directed trees.

Observation

This conjecture is neither stronger nor weaker than Stanley's!

We will focus on a stronger conjecture, and look at the leading coefficient of the chromatic quasisymmetric functions.

Conjecture (TBD)

The leading coefficient of the chromatic quasisymmetric function distinguishes some family of oriented graphs.

Conjecture (TBD)

The leading coefficient of the chromatic quasisymmetric function distinguishes some family of oriented graphs.

What does the leading coefficient in t of $X_{\vec{G}}(\mathbf{x}, t)$ look like?

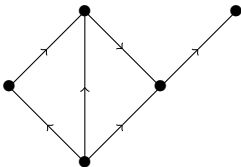
$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{\kappa} t^{\text{asc}(\kappa)} \prod_{i \in \mathbb{N}} x_i^{\#\kappa^{-1}(i)},$$

Conjecture (TBD)

The leading coefficient of the chromatic quasisymmetric function distinguishes some family of oriented graphs.

What does the leading coefficient in t of $X_{\vec{G}}(\mathbf{x}, t)$ look like?

$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{\kappa} t^{\text{asc}(\kappa)} \prod_{i \in \mathbb{N}} x_i^{\#\kappa^{-1}(i)},$$

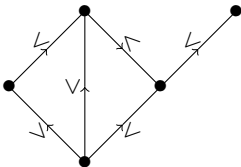


Conjecture (TBD)

The leading coefficient of the chromatic quasisymmetric function distinguishes some family of oriented graphs.

What does the leading coefficient in t of $X_{\vec{G}}(\mathbf{x}, t)$ look like?

$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{\kappa} t^{\text{asc}(\kappa)} \prod_{i \in \mathbb{N}} x_i^{\#\kappa^{-1}(i)},$$

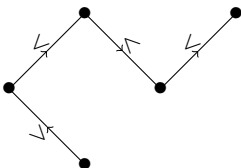


Conjecture (TBD)

The leading coefficient of the chromatic quasisymmetric function distinguishes some family of oriented graphs.

What does the leading coefficient in t of $X_{\vec{G}}(\mathbf{x}, t)$ look like?

$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{\kappa} t^{\text{asc}(\kappa)} \prod_{i \in \mathbb{N}} x_i^{\#\kappa^{-1}(i)},$$

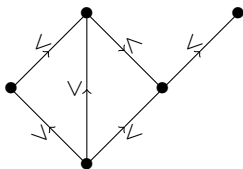


Conjecture (TBD)

The leading coefficient of the chromatic quasisymmetric function distinguishes some family of oriented graphs.

What does the leading coefficient in t of $X_{\vec{G}}(\mathbf{x}, t)$ look like?

$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{\kappa} t^{\text{asc}(\kappa)} \prod_{i \in \mathbb{N}} x_i^{\#\kappa^{-1}(i)},$$

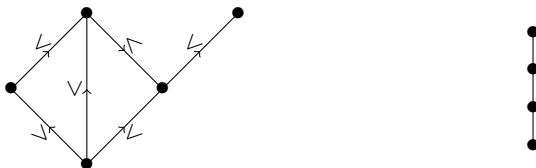


Conjecture (TBD)

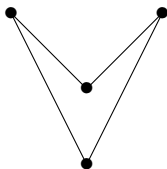
The leading coefficient of the chromatic quasisymmetric function distinguishes some family of oriented graphs.

What does the leading coefficient in t of $X_{\vec{G}}(\mathbf{x}, t)$ look like?

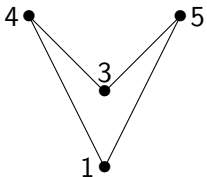
$$X_{\vec{G}}(\mathbf{x}, t) := \sum_{\kappa} t^{\text{asc}(\kappa)} \prod_{i \in \mathbb{N}} x_i^{\#\kappa^{-1}(i)},$$



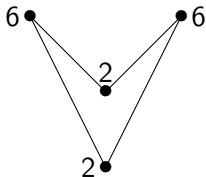
$$[t^6]X_{\vec{G}}(\mathbf{x}, t) = \sum_{1 \leq i < j < k < l} x_i x_j x_k x_l$$



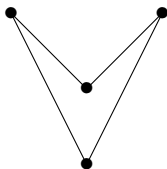
$$\bar{K}_P(\mathbf{x}) =$$



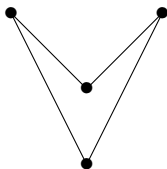
$$\bar{K}_P(\mathbf{x}) = x_1^1 x_3^1 x_4^1 x_5^1$$



$$\begin{aligned}\bar{K}_P(\mathbf{x}) &= x_1^1 x_3^1 x_4^1 x_5^1 \\ &\quad + x_2^2 x_6^2 \\ &\quad + \dots\end{aligned}$$



$$\begin{aligned}
 \bar{K}_P(\mathbf{x}) &= x_1^1 x_3^1 x_4^1 x_5^1 \\
 &\quad + x_2^2 x_6^2 \\
 &\quad + \dots \\
 &\in \text{QSym}.
 \end{aligned}$$



$$\begin{aligned}\bar{K}_P(\mathbf{x}) &= x_1^1 x_3^1 x_4^1 x_5^1 \\ &\quad + x_2^2 x_6^2 \\ &\quad + \dots \\ &\in \text{QSym}.\end{aligned}$$

Definition

Given a poset P , a **strict P -partition** is a map $f : P \mapsto \mathbb{N}$ such that $i \leq_P j \Rightarrow f(i) < f(j)$.

The strict P -partition enumerator is:

$$\bar{K}_P(\mathbf{x}) := \sum_f \prod_{i \in \mathbb{N}} x_i^{\#f^{-1}(i)},$$

where the sum ranges over all P -partitions

Conjecture (Hasebe & Tsujie '17)

Strict partition enumerators distinguish trees.

Conjecture (Hasebe & Tsujie '17)

Strict partition enumerators distinguish trees.

This conjecture is stronger than Alexandersson & Sulzgruber's.

Conjecture (Hasebe & Tsujie '17)

Strict partition enumerators distinguish trees.

This conjecture is stronger than Alexandersson & Sulzgruber's.

We know strict partition enumerators distinguish:

- bowtie and N -free posets (Hasebe & Tsujie '17),
- width 2 posets (Liu & Weselcouch '20),
- rooted trees (Hasebe & Tsujie '17, Zhou '20),
- series-parallel posets (Liu & Weselcouch '21),
- ...

Definition

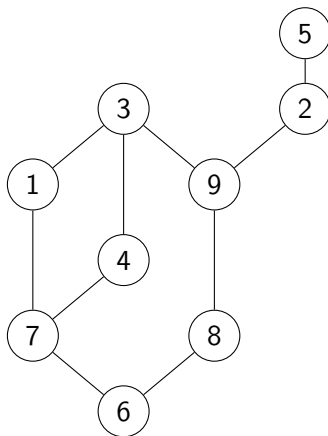
Given a *labeled* poset (P, ω) , a (P, ω) -**partition** is a map $f : P \mapsto \mathbb{N}$ such that:

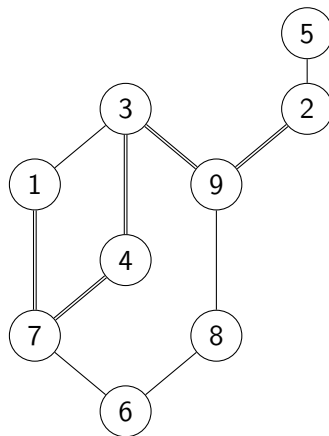
$$\forall i \leq_P j, \begin{cases} \omega(i) < \omega(j) \Rightarrow f(i) \leq f(j) \\ \omega(i) > \omega(j) \Rightarrow f(i) < f(j) \end{cases}$$

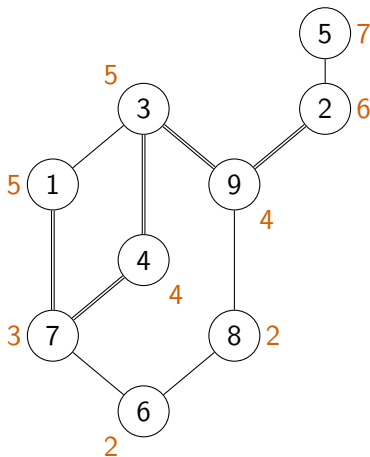
The (P, ω) -partition enumerator is:

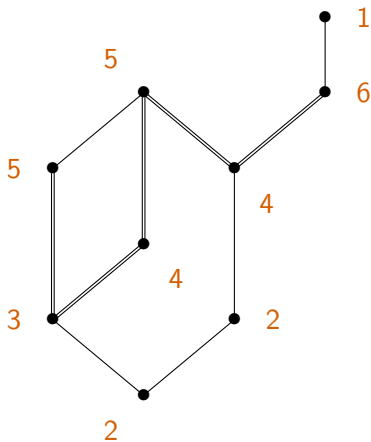
$$K_{(P, \omega)}(\mathbf{x}) := \sum_f \prod_{i \in \mathbb{N}} x_i^{\#f^{-1}(i)},$$

where the sum ranges over all (P, ω) -partitions





Mixed P -partitions


Mixed P -partitions

Partition enumerators don't distinguish trees: $K_{\vee} = K_{\wedge}$.

Partition enumerators don't distinguish trees: $K_{\vee} = K_{\wedge}$.

Conjecture (Hasebe & Tsujie '17)

Partition enumerators distinguish rooted trees.

Partition enumerators don't distinguish trees: $K_{\vee} = K_{\wedge}$.

Conjecture (Hasebe & Tsujie '17)

Partition enumerators distinguish rooted trees.

Question

What other families of posets are characterized by their partition enumerators?

QSym & **FQSym**

Definition

$QSym \subsetneq \mathbb{Q}[[X]]$ is the Hopf algebra of formal power series over the ordered commutative alphabet X such that for all $i_1 < \dots < i_k$, $j_1 < \dots < j_k$ and $\alpha = (\alpha_1, \dots, \alpha_k) \models n$, $\prod_{\ell \in [k]} x_{i_\ell}^{\alpha_\ell}$ and $\prod_{\ell \in [k]} x_{j_\ell}^{\alpha_\ell}$ have equal coefficients.

Definition

$QSym \subsetneq \mathbb{Q}[[X]]$ is the Hopf algebra of formal power series over the ordered commutative alphabet X such that for all $i_1 < \dots < i_k$, $j_1 < \dots < j_k$ and $\alpha = (\alpha_1, \dots, \alpha_k) \models n$, $\prod_{\ell \in [k]} x_{i_\ell}^{\alpha_\ell}$ and $\prod_{\ell \in [k]} x_{j_\ell}^{\alpha_\ell}$ have equal coefficients.

$$QSym \ni x_1^1 x_2^3 x_3^1 + \dots$$

Definition

$QSym \subsetneq \mathbb{Q}[[X]]$ is the Hopf algebra of formal power series over the ordered commutative alphabet X such that for all $i_1 < \dots < i_k$, $j_1 < \dots < j_k$ and $\alpha = (\alpha_1, \dots, \alpha_k) \models n$, $\prod_{\ell \in [k]} x_{i_\ell}^{\alpha_\ell}$ and $\prod_{\ell \in [k]} x_{j_\ell}^{\alpha_\ell}$ have equal coefficients.

$$QSym \ni x_1^1 x_2^3 x_3^1 + x_2^1 x_4^3 x_5^1 + \dots$$

Definition

$QSym \subsetneq \mathbb{Q}[[X]]$ is the Hopf algebra of formal power series over the ordered commutative alphabet X such that for all $i_1 < \dots < i_k$, $j_1 < \dots < j_k$ and $\alpha = (\alpha_1, \dots, \alpha_k) \models n$, $\prod_{\ell \in [k]} x_{i_\ell}^{\alpha_\ell}$ and $\prod_{\ell \in [k]} x_{j_\ell}^{\alpha_\ell}$ have equal coefficients.

$$QSym \ni x_1^1 x_2^3 x_3^1 + x_2^1 x_4^3 x_5^1 + x_{12}^1 x_{42}^3 x_{77}^1 + \dots$$

Definition

$QSym \subsetneq \mathbb{Q}[[X]]$ is the Hopf algebra of formal power series over the ordered commutative alphabet X such that for all $i_1 < \dots < i_k$, $j_1 < \dots < j_k$ and $\alpha = (\alpha_1, \dots, \alpha_k) \models n$, $\prod_{\ell \in [k]} x_{i_\ell}^{\alpha_\ell}$ and $\prod_{\ell \in [k]} x_{j_\ell}^{\alpha_\ell}$ have equal coefficients.

$$QSym \ni x_1^1 x_2^3 x_3^1 + x_2^1 x_4^3 x_5^1 + x_{12}^1 x_{42}^3 x_{77}^1 + \dots =: M_{131}$$

Proposition

QSym is spanned by the monomial basis, indexed by integer compositions:

$$\forall \alpha \models n, \quad M_\alpha := \sum_{i_1 < \dots < i_k} \prod_{\ell \in [k]} x_{i_\ell}^{\alpha_\ell}.$$

Proposition

QSym is spanned by the monomial basis, indexed by integer compositions:

$$\forall \alpha \models n, \quad M_\alpha := \sum_{i_1 < \dots < i_k} \prod_{\ell \in [k]} x_{i_\ell}^{\alpha_\ell}.$$

Proposition

QSym is spanned by the fundamental basis, indexed by integer compositions:

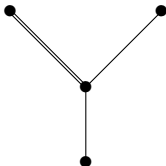
$$\forall \alpha \models n, \quad F_\alpha := \sum_{\beta \leq \alpha} M_\beta.$$

Theorem (Stanley '71)

$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{\text{des}(\sigma)}.$$

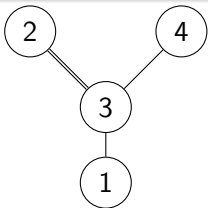
Theorem (Stanley '71)

$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{\text{des}(\sigma)}.$$



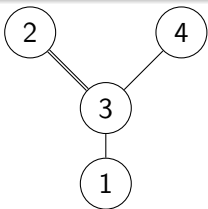
Theorem (Stanley '71)

$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{\text{des}(\sigma)}.$$



Theorem (Stanley '71)

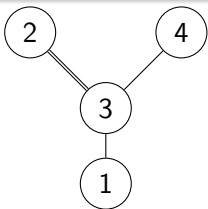
$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{\text{des}(\sigma)}.$$



$$\mathcal{L}(P,\omega) = \{1324, 1342\}$$

Theorem (Stanley '71)

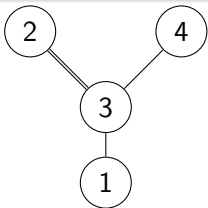
$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{\text{des}(\sigma)}.$$



$$\mathcal{L}(P,\omega) = \{13|24, 134|2\}$$

Theorem (Stanley '71)

$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{\text{des}(\sigma)}.$$

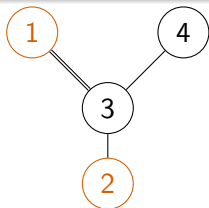


$$\mathcal{L}(P,\omega) = \{13|24, 134|2\}$$

$$K_{(P,\omega)} = F_{22} + F_{31}$$

Theorem (Stanley '71)

$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{\text{des}(\sigma)}.$$

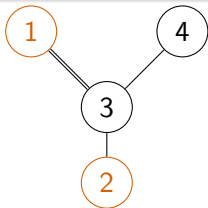


$$\mathcal{L}(P, \omega') = \{23|14, 234|1\}$$

$$K_{(P,\omega')} = F_{22} + F_{31}$$

Theorem (Stanley '71)

$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{\text{des}(\sigma)}.$$



$$\mathcal{L}(P, \omega') = \{23|14, 234|1\}$$

$$K_{(P,\omega')} = F_{22} + F_{31}$$

Theorem (Hazewinkel '01, Lam & Pylyavskyy '08)

QSym is a unique factorization domain.

Proposition

The Hopf algebra **FQSym** $\subsetneq \mathbb{Q}[[A]]$ admits a basis \mathbb{F} indexed by permutations with:

$$\mathbb{F}_\sigma := \sum_{\text{std}(w)=\sigma} \prod_{i \in [n]} a_{w_i},$$

$$\mathbb{F}_\sigma \cdot \mathbb{F}_\tau = \sum_{\nu \in \sigma \sqcup \tau} \mathbb{F}_\nu,$$

where A is a non-commutative alphabet.

$$12 \sqcup 21 = 12 \sqcup 43 = 1243 + 1423 + 4123 + 1432 + 4132 + 4312.$$

Proposition

$QSym \hookrightarrow \mathbf{FQSym}$ through the map

$$F_\alpha \mapsto \sum_{\text{des}(\sigma)=\alpha} \mathbb{F}_\sigma.$$

Proposition

$QSym \hookrightarrow \mathbf{FQSym}$ through the map

$$F_\alpha \mapsto \sum_{\text{des}(\sigma)=\alpha} \mathbb{F}_\sigma.$$

This is nice, because otherwise, computing products in $QSym$ is hard:

$$F_\alpha \cdot F_\beta = K_{P_\alpha} \cdot K_{P_\beta} = K_{P_\alpha \sqcup P_\beta}.$$

Proposition

$QSym \hookrightarrow \mathbf{FQSym}$ through the map

$$F_\alpha \mapsto \sum_{\text{des}(\sigma)=\alpha} \mathbb{F}_\sigma.$$

This is nice, because otherwise, computing products in $QSym$ is hard:

$$F_\alpha \cdot F_\beta = K_{P_\alpha} \cdot K_{P_\beta} = K_{P_\alpha \sqcup P_\beta}.$$



P_{231}

Fair rooted trees

Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Recursive definition:

• $\in \mathcal{C}$,

•

Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Recursive definition:

$$\bullet \in \mathcal{C},$$

$$f \sqcup g \in \mathcal{C},$$

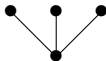
...

Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Recursive definition:

- $\in \mathcal{C}$,
- $f \sqcup g \in \mathcal{C}$,
- $\uparrow f \in \mathcal{C}$,

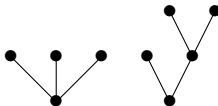


Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Recursive definition:

- $\in \mathcal{C}$,
- $f \sqcup g \in \mathcal{C}$,
- $\uparrow f \in \mathcal{C}$,

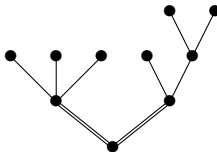


Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Recursive definition:

- $\in \mathcal{C}$,
- $f \sqcup g \in \mathcal{C}$,
- $\uparrow f \in \mathcal{C}$,
- $\uparrow\uparrow f \in \mathcal{C}$.

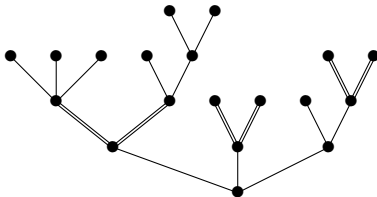


Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Recursive definition:

- $\in \mathcal{C}$,
- $f \sqcup g \in \mathcal{C}$,
- $\uparrow f \in \mathcal{C}$,
- $\uparrow\uparrow f \in \mathcal{C}$.

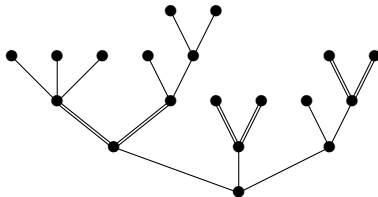


Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Recursive definition:

- $\in \mathcal{C}$,
- $f \sqcup g \in \mathcal{C}$,
- $\uparrow f \in \mathcal{C}$,
- $\uparrow\uparrow f \in \mathcal{C}$.



Lemma (Aval Djenabou & McNamara '23+)

The partition enumerator of a fair rooted tree is irreducible in QSym.

Technical. Has a counterpart in most proof of similar results.

Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Sketch of the proof of the Theorem:

Assume $K_f = K_g$. Use irreducibility and unique factorization in QSym to assume f and g are connected.

Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Sketch of the proof of the Theorem:

Assume $K_f = K_g$. Use irreducibility and unique factorization in QSym to assume f and g are connected.

If $f = \bullet \uparrow f'$, all linear extensions of f start with a descent:

$K_f = K_g = \sum_{\alpha} c_{\alpha} F_{1,\alpha}$ and so g can be written $\bullet \uparrow g'$.

Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Sketch of the proof of the Theorem:

Assume $K_f = K_g$. Use irreducibility and unique factorization in QSym to assume f and g are connected.

If $f = \bullet \uparrow f'$, all linear extensions of f start with a descent:

$K_f = K_g = \sum_{\alpha} c_{\alpha} F_{1,\alpha}$ and so g can be written $\bullet \uparrow g'$.

Furthermore, $K_{f'} = K_{g'} = \sum_{\alpha} c_{\alpha} F_{\alpha}$, and we iterate.

Theorem (Aval Djenabou McNamara '23+)

Partition enumerators distinguish fair rooted forests.

Sketch of the proof of the Theorem:

Assume $K_f = K_g$. Use irreducibility and unique factorization in QSym to assume f and g are connected.

If $f = \bullet \uparrow f'$, all linear extensions of f start with a descent:

$K_f = K_g = \sum_{\alpha} c_{\alpha} F_{1,\alpha}$ and so g can be written $\bullet \uparrow g'$.

Furthermore, $K_{f'} = K_{g'} = \sum_{\alpha} c_{\alpha} F_{\alpha}$, and we iterate.

The case $f = \bullet \uparrow f'$ is similar.



Fair series-parallel posets

Recursive definition:

Recursive definition:

$$\bullet \in \mathcal{C},$$



• • •

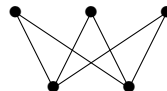
Recursive definition:

• $\in \mathcal{C}$,

$P \sqcup Q \in \mathcal{C}$,

Recursive definition:

- $\bullet \in \mathcal{C},$
- $P \sqcup Q \in \mathcal{C},$
- $P \uparrow Q \in \mathcal{C},$

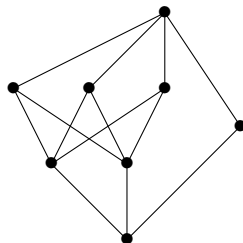


Recursive definition:

$\bullet \in \mathcal{C},$

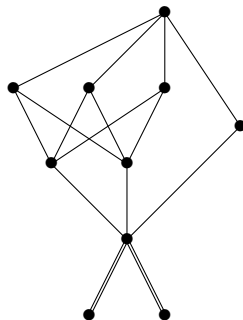
$P \sqcup Q \in \mathcal{C},$

$P \uparrow Q \in \mathcal{C},$



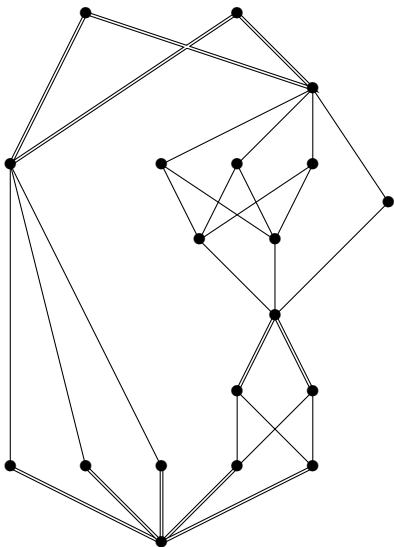
Recursive definition:

- $\bullet \in \mathcal{C},$
- $P \sqcup Q \in \mathcal{C},$
- $P \uparrow Q \in \mathcal{C},$
- $P \uparrow\uparrow Q \in \mathcal{C}.$



Recursive definition:

- $\in \mathcal{C}$,
- $P \sqcup Q \in \mathcal{C}$,
- $P \uparrow Q \in \mathcal{C}$,
- $P \uparrow\uparrow Q \in \mathcal{C}$.



Theorem (A. Aval & Mlodecki '23+)

Partition enumerators distinguish fair series-parallel posets.

Theorem (A. Aval & Mlodecki '23+)

Partition enumerators distinguish fair series-parallel posets.

Lemma (A. Aval & Mlodecki '23+)

The partition enumerator of a connected fair series-parallel poset is irreducible in QSym.

Theorem (A. Aval & Mlodecki '23+)

Partition enumerators distinguish fair series-parallel posets.

Lemma (A. Aval & Mlodecki '23+)

The partition enumerator of a connected fair series-parallel poset is irreducible in QSym.

The latter being a consequence of:

Lemma (A. Aval & Mlodecki '23+)

Let f be a homogeneous quasisymmetric function, such that all elements of its support in the fundamental basis have a common descent. Then f is irreducible in QSym.

Sketch of the irreducibility proof:

Assume that $K_P = f_1 f_2$.

$$\left(\sum_{\alpha \models n} c_\alpha F_\alpha \right) := K_P = f_1 f_2 =: \left(\sum_{\beta \models n_1} d_\beta F_\beta \right) \cdot \left(\sum_{\gamma \models n_2} e_\gamma F_\gamma \right).$$

Sketch of the irreducibility proof:

Assume that $K_P = f_1 f_2$.

$$\left(\sum_{\alpha \models n} c_\alpha F_\alpha \right) := K_P = f_1 f_2 =: \left(\sum_{\beta \models n_1} d_\beta F_\beta \right) \cdot \left(\sum_{\gamma \models n_2} e_\gamma F_\gamma \right).$$

$$\sum_{\sigma \in \mathfrak{S}_n} c_{\text{des}(\sigma)} \mathbb{F}_\sigma = \sum_{\sigma \in \mathfrak{S}_n} \left(\sum_{\sigma \in \tau \sqcup \nu} d_{\text{des}(\tau)} \cdot e_{\text{des}(\nu)} \right) \mathbb{F}_\sigma.$$

Sketch of the irreducibility proof:

Assume that $K_P = f_1 f_2$.

$$\left(\sum_{\alpha \models n} c_\alpha F_\alpha \right) := K_P = f_1 f_2 =: \left(\sum_{\beta \models n_1} d_\beta F_\beta \right) \cdot \left(\sum_{\gamma \models n_2} e_\gamma F_\gamma \right).$$

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n} c_{\text{des}(\sigma)} \mathbb{F}_\sigma &= \sum_{\sigma \in \mathfrak{S}_n} \left(\sum_{\sigma \in \tau \sqcup \nu} d_{\text{des}(\tau)} \cdot e_{\text{des}(\nu)} \right) \mathbb{F}_\sigma. \\ &= \sum_{\sigma \in \mathfrak{S}_n} d_{\text{des}(\sigma|^{[1, n_1]})} \cdot e_{\text{des}(\sigma|^{[n_1+1, n]})} \mathbb{F}_\sigma. \end{aligned}$$

$$\sum_{\sigma \in \mathfrak{S}_n} c_{\text{des}(\sigma)} \mathbb{F}_\sigma = \sum_{\sigma \in \mathfrak{S}_n} d_{\text{des}(\sigma|_{[1, n_1]})} \cdot e_{\text{des}(\sigma|_{[n_1+1, n]})} \mathbb{F}_\sigma.$$

Assume $P = Q \uparrow R$. Then all linear extensions of P have a descent in position $q := \#Q$.

$$\sum_{\sigma \in \mathfrak{S}_n} c_{\text{des}(\sigma)} \mathbb{F}_\sigma = \sum_{\sigma \in \mathfrak{S}_n} d_{\text{des}(\sigma|_{[1, n_1]})} \cdot e_{\text{des}(\sigma|_{[n_1+1, n]})} \mathbb{F}_\sigma.$$

Assume $P = Q \uparrow R$. Then all linear extensions of P have a descent in position $q := \#Q$.

$$\forall \sigma \in \mathfrak{S}_n, q \notin \text{des}(\sigma) \Rightarrow c_{\text{des}(\sigma)} = 0.$$

$$\sum_{\sigma \in \mathfrak{S}_n} c_{\text{des}(\sigma)} \mathbb{F}_\sigma = \sum_{\sigma \in \mathfrak{S}_n} d_{\text{des}(\sigma|^{[1, n_1]})} \cdot e_{\text{des}(\sigma|^{[n_1+1, n]})} \mathbb{F}_\sigma.$$

Assume $P = Q \uparrow R$. Then all linear extensions of P have a descent in position $q := \#Q$.

$$\forall \sigma \in \mathfrak{S}_n, q \notin \text{des}(\sigma) \Rightarrow c_{\text{des}(\sigma)} = 0.$$

Let $\beta \models n_1$ and $\gamma \models n_2$ such that $d_\beta \neq 0$. It is easy to build $\sigma_{\beta, \gamma}$ such that :

$$\begin{cases} q \notin \text{des}(\sigma_{\beta, \gamma}), \\ \text{des}(\sigma|^{[1, n_1]}) = \beta, \\ \text{des}(\sigma|^{[n_1+1, n]}) = \gamma. \end{cases}$$

$$\sum_{\sigma \in \mathfrak{S}_n} c_{\text{des}(\sigma)} \mathbb{F}_\sigma = \sum_{\sigma \in \mathfrak{S}_n} d_{\text{des}(\sigma|^{[1, n_1]})} \cdot e_{\text{des}(\sigma|^{[n_1+1, n]})} \mathbb{F}_\sigma.$$

Assume $P = Q \uparrow R$. Then all linear extensions of P have a descent in position $q := \#Q$.

$$\forall \sigma \in \mathfrak{S}_n, q \notin \text{des}(\sigma) \Rightarrow c_{\text{des}(\sigma)} = 0.$$

Let $\beta \models n_1$ and $\gamma \models n_2$ such that $d_\beta \neq 0$. It is easy to build $\sigma_{\beta, \gamma}$ such that :

$$\begin{cases} q \notin \text{des}(\sigma_{\beta, \gamma}), \\ \text{des}(\sigma|^{[1, n_1]}) = \beta, \\ \text{des}(\sigma|^{[n_1+1, n]}) = \gamma. \end{cases}$$

So that any product $d_\beta e_\gamma$ with $d_\beta \neq 0$ is null. Which is absurd. \square

Theorem (A. Aval & Mlodecki '23+)

Partition enumerators distinguish fair series-parallel posets.

Sketch of the Theorem's proof:

Suppose $K_P = K_{P'}$. As before, use irreducibility and unique factorization to assume P and P' are connected.

Theorem (A. Aval & Mlodecki '23+)

Partition enumerators distinguish fair series-parallel posets.

Sketch of the Theorem's proof:

Suppose $K_P = K_{P'}$. As before, use irreducibility and unique factorization to assume P and P' are connected.

Assume $P = Q \uparrow\uparrow R$, then K_P has a global descent in q , and so does $K_{P'}$.

This implies that P' is of the form $Q' \uparrow\uparrow R'$, with $\#Q' = q$.

Theorem (A. Aval & Mlodecki '23+)

Partition enumerators distinguish fair series-parallel posets.

Sketch of the Theorem's proof:

Suppose $K_P = K_{P'}$. As before, use irreducibility and unique factorization to assume P and P' are connected.

Assume $P = Q \uparrow\uparrow R$, then K_P has a global descent in q , and so does $K_{P'}$.

This implies that P' is of the form $Q' \uparrow\uparrow R'$, with $\#Q' = q$.

Indeed, all linear extensions of P' have the same first q values.

By contradiction, let $\sigma, \nu \in \mathcal{L}(P')$, $i \in \sigma_{|[q]} \setminus \nu_{|[q]}$ and $j \in \nu_{|[q]} \setminus \sigma_{|[q]}$.

Pick i and j to minimize their distance to position q in σ .

By contradiction, let $\sigma, \nu \in \mathcal{L}(P')$, $i \in \sigma_{|[q]} \setminus \nu_{|[q]}$ and $j \in \nu_{|[q]} \setminus \sigma_{|[q]}$.

Pick i and j to minimize their distance to position q in σ .

$$\begin{array}{rcl} \sigma & = & ik \mid j \\ \nu & = & j \mid i \end{array}$$

By contradiction, let $\sigma, \nu \in \mathcal{L}(P')$, $i \in \sigma_{|[q]} \setminus \nu_{|[q]}$ and $j \in \nu_{|[q]} \setminus \sigma_{|[q]}$.

Pick i and j to minimize their distance to position q in σ .

$$\begin{array}{rcl} \sigma & = & ik \mid j \\ \nu & = & j \mid i \end{array}$$

$$k \not\leq_{P'} i$$

By contradiction, let $\sigma, \nu \in \mathcal{L}(P')$, $i \in \sigma_{|[q]} \setminus \nu_{|[q]}$ and $j \in \nu_{|[q]} \setminus \sigma_{|[q]}$.

Pick i and j to minimize their distance to position q in σ .

$$\begin{array}{rcl} \sigma & = & ik \mid j \\ \nu & = & j \mid i \end{array}$$

$$k \not\prec_{P'} i \text{ and } k \not\prec_{P'} i$$

By contradiction, let $\sigma, \nu \in \mathcal{L}(P')$, $i \in \sigma_{|[q]} \setminus \nu_{|[q]}$ and $j \in \nu_{|[q]} \setminus \sigma_{|[q]}$.

Pick i and j to minimize their distance to position q in σ .

$$\begin{array}{rcl} \sigma' & = & ki \mid j \\ \nu & = & j \mid i \end{array}$$

$k \not\prec_{P'} i$ and $k \not\prec_{P'} i$, so $\sigma' \in \mathcal{L}(P')$.

By contradiction, let $\sigma, \nu \in \mathcal{L}(P')$, $i \in \sigma_{|[q]} \setminus \nu_{|[q]}$ and $j \in \nu_{|[q]} \setminus \sigma_{|[q]}$.

Pick i and j to minimize their distance to position q in σ .

$$\begin{array}{rcl} \sigma' & = & ki \mid j \\ \nu & = & j \mid i \end{array}$$

$k \not\prec_{P'} i$ and $k \not\prec_{P'} i$, so $\sigma' \in \mathcal{L}(P')$.

In the end, we have two linear extensions of the form :

$$\begin{array}{rcl} \tau_1 & = & i \mid j \\ \tau_2 & = & j \mid i \end{array}$$

and one of them has an ascent in q , which is absurd. □

Perspectives

Recall:

Theorem (Hasebe & Tsujie '17)

Strict partition enumerators distinguish rooted forests.

and

Theorem (Stanley '71)

$$K_{(P, \omega)} = \sum_{\sigma \in \mathcal{L}(P, \omega)} F_{\text{des}(\sigma)}.$$

Recall:

Theorem (Hasebe & Tsujie '17)

Strict partition enumerators distinguish rooted forests.

and

Theorem (Stanley '71)

$$K_{(P,\omega)} = \sum_{\sigma \in \mathcal{L}(P,\omega)} F_{\text{des}(\sigma)}.$$

Conjecture (A., Aval & Mlodecki 23+)

$\sum_{\sigma \in \mathcal{L}_1(P,\omega)} F_{\text{des}(\sigma)}$ *distinguishes naturally labeled rooted forests.*

We checked this for forests with up to 15 inner nodes

Conjecture

The partition enumerator of a connected binary tree is irreducible in QSym.

This was checked up to 11 inner nodes.

Conjecture

The partition enumerator of a connected binary tree is irreducible in QSym.

This was checked up to 11 inner nodes.

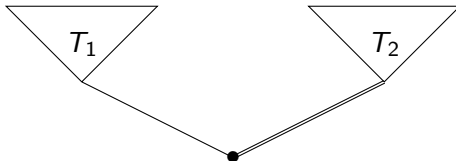
We can try to mimic the previous proofs. From K_T we recover $K_{T_1} \cdot K_{T_2}$, but if the root is unfair, we can't distinguish which tree goes where.

Conjecture

The partition enumerator of a connected binary tree is irreducible in QSym.

This was checked up to 11 inner nodes.

We can try to mimic the previous proofs. From K_T we recover $K_{T_1} \cdot K_{T_2}$, but if the root is unfair, we can't distinguish which tree goes where.

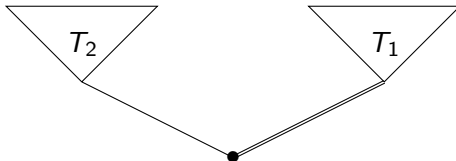


Conjecture

The partition enumerator of a connected binary tree is irreducible in QSym.

This was checked up to 11 inner nodes.

We can try to mimic the previous proofs. From K_T we recover $K_{T_1} \cdot K_{T_2}$, but if the root is unfair, we can't distinguish which tree goes where.



Other method: "Zhou-like".

In his '20 paper, Zhou proves injectivity of $T \mapsto \bar{K}_T$ by making explicit the reverse bijection.

Other method: "Zhou-like".

In his '20 paper, Zhou proves injectivity of $T \mapsto \bar{K}_T$ by making explicit the reverse bijection.

From the partition enumerator of a (mixed) binary tree, we can recover:

- the number of nodes,

Other method: "Zhou-like".

In his '20 paper, Zhou proves injectivity of $T \mapsto \bar{K}_T$ by making explicit the reverse bijection.

From the partition enumerator of a (mixed) binary tree, we can recover:

- the number of nodes,
- the nature of the root,

Other method: "Zhou-like".

In his '20 paper, Zhou proves injectivity of $T \mapsto \bar{K}_T$ by making explicit the reverse bijection.

From the partition enumerator of a (mixed) binary tree, we can recover:

- the number of nodes,
- the nature of the root,
- the enumerator of the child forest,

Other method: "Zhou-like".

In his '20 paper, Zhou proves injectivity of $T \mapsto \bar{K}_T$ by making explicit the reverse bijection.

From the partition enumerator of a (mixed) binary tree, we can recover:

- the number of nodes,
- the nature of the root,
- the enumerator of the child forest,
- the shape of the tree obtained by contracting all simple edges (provided irreducibility),

Other method: "Zhou-like".

In his '20 paper, Zhou proves injectivity of $T \mapsto \bar{K}_T$ by making explicit the reverse bijection.

From the partition enumerator of a (mixed) binary tree, we can recover:

- the number of nodes,
- the nature of the root,
- the enumerator of the child forest,
- the shape of the tree obtained by contracting all simple edges (provided irreducibility),
- the number of leaves and the number of double edges separating them from the root,

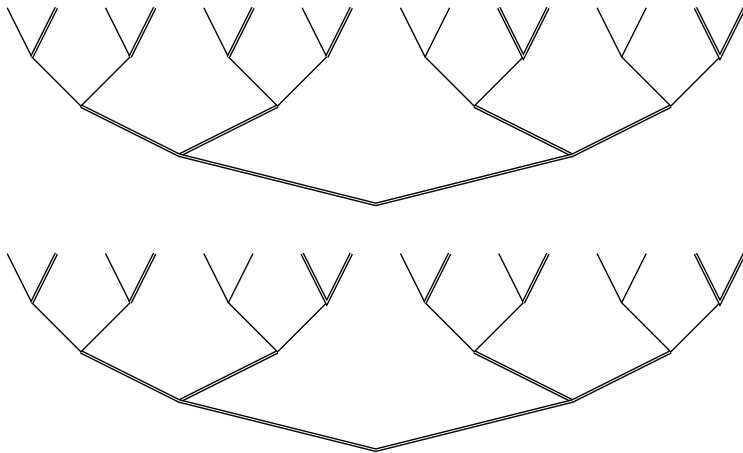
Other method: "Zhou-like".

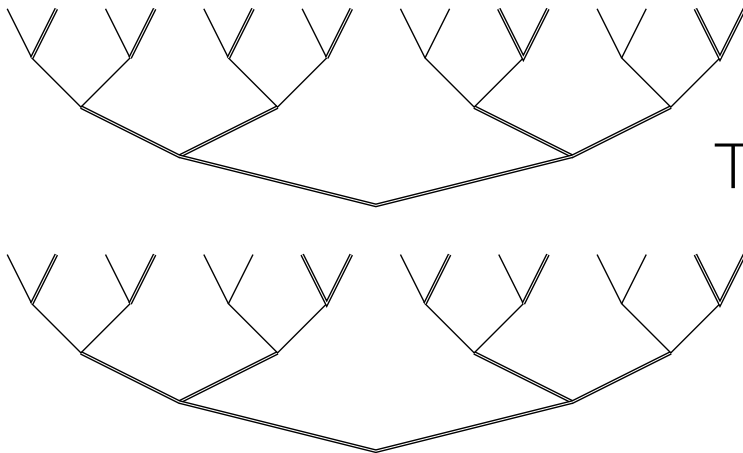
In his '20 paper, Zhou proves injectivity of $T \mapsto \bar{K}_T$ by making explicit the reverse bijection.

From the partition enumerator of a (mixed) binary tree, we can recover:

- the number of nodes,
- the nature of the root,
- the enumerator of the child forest,
- the shape of the tree obtained by contracting all simple edges (provided irreducibility),
- the number of leaves and the number of double edges separating them from the root,
- ...

but this is still not enough.





Thank
you!