Enumeration of 3-connected bipartite planar maps

joint work with Marc Noy and Juanjo Rué

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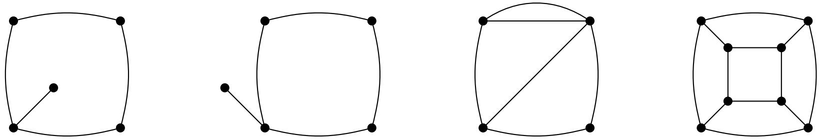
Journées Combinatoires de Bordeaux
10/06/2022

Bipartite planar maps

Planar map: fixed embedding of a connected planar multigraph on the sphere.



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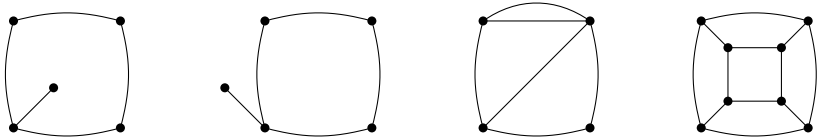

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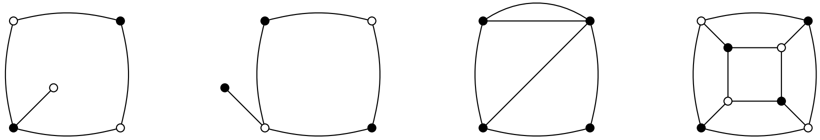


k-**connnected map**: needs to remove at least k vertices to disconnect the underlaying graph.

Bipartite map: admits a partition of its verticres into two independent sets

$\Leftrightarrow$ admits a proper 2-colouring of its vertices

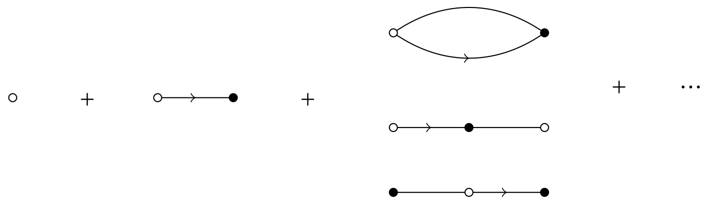
$\Leftrightarrow$ every face has even degree (number of corners)

$\Leftrightarrow$ dual map is eulerian.

Counting bipartite maps

The generating function of bipartite (planar) maps counted by edges is:

$$M(z) = 1 + z + 3z^2 + 12z^3 + 56z^4 + 288z^5 + 1584z^6 + 9152z^7 + 54912z^8 + \dots$$



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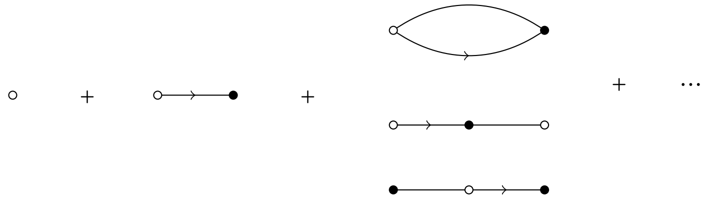

Tutte (1962): The number of bipartite maps with n edges is

$$\frac{3}{(n+1)(n+2)} 2^{n-1} \binom{2n}{n} \sim m \cdot n^{-5/2} \cdot 8^n, \qquad \text{as } n \to \infty \text{ and with } m > 0.$$

Counting bipartite maps by connectivity

Liskovets & Walsh (2004): The number of 2-connected bipartite maps with n edges is

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$$t_n \sim t \cdot n^{-5/2} \cdot \tau^n, \qquad \text{as } n \to \infty \text{ and with } \tau \approx 2.40958 \text{ and } t > 0.$$

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Class of maps	Arbitrary	2-connected	3-connected
All	12	$27/4$	4
Bipartite cubic	2	2	$8/5$
Bipartite	8	$128/25$	$\tau \approx 2.41$

Other motivations

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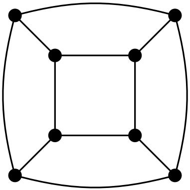
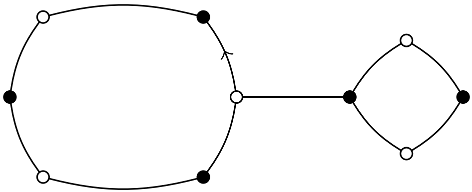
Steinitz (1922): The skeletons of convex polyhedra are exactly the 3-connected planar graphs.



Counting bipartite maps: recursion

Just like the recursion for Cayley trees, we count rooted maps:

- distinguish an edge,
- orient it.



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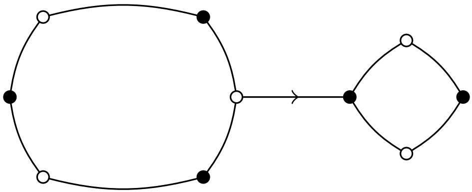

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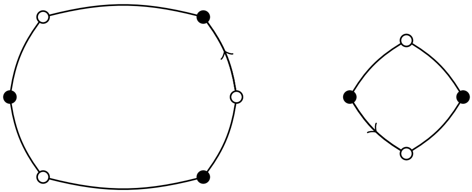

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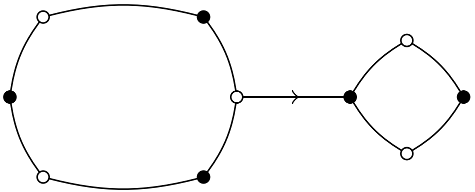

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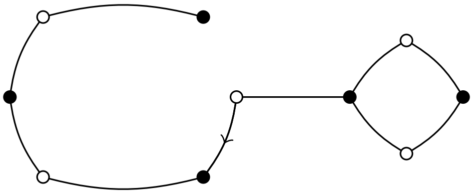

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- Consider the bivariate generating function $M(z, y) = \sum_{d \geq 0} M_d(z) y^d$.

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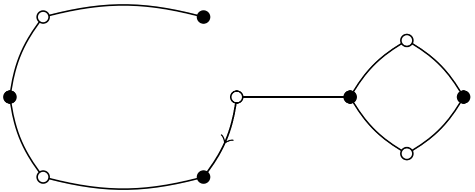

$$M(z,y) = 1 + zyM(z,y)^2 + z\sum_{d\geq 0} M_d(z)(y + y^2 + \ldots + y^d)$$

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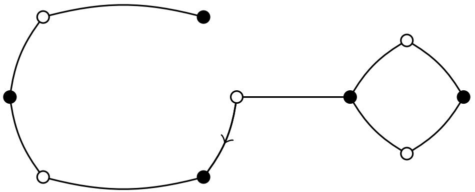

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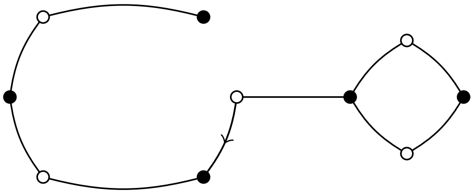

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- Encode the parameter d = root-face degree/2 into the catalytic variable y.
- Consider the bivariate generating function $M(z, y) = \sum_{d \geq 0} M_d(z) y^d$.
- Cannot obtain univariate $M(z) = M(z, 1)$ directly by setting $y = 1$

Counting bipartite maps: solution

Transform the previous into a (quadratic) polynomial equation:

$$1 - y - zyM(z) + (1 + y(z-1))M(z,y) + zy(y-1)M(z,y)^2 = 0$$

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Obtain $M(z) = M(z,1)$ via the quadratic method (Tutte (1962); Brown (1965); Bousquet-Mélou & Jehanne (2006)):

$$z^2 + 11z - 1 - (8z^2 + 12z - 1)M(z) + 16z^2M(z)^2 = 0$$

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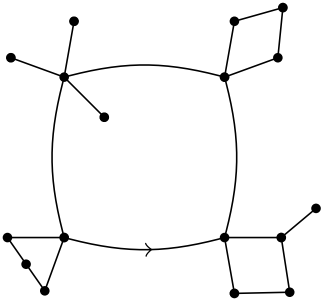
Solving for $M(z)$ gives

$$M(z) = \frac{8z^2 + 12z - 1 + (1-8z)^{3/2}}{32z^2} = \sum_{n \geq 0} \frac{3}{(n+1)(n+2)} 2^{n-1} \binom{2n}{n} z^n$$

Counting non-separable maps

Folklore + Tutte(1962): maps can be uniquely decomposed into maximal 2-connected components (blocks).



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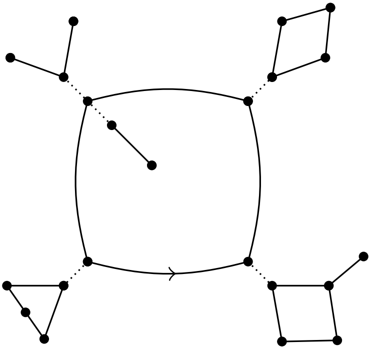

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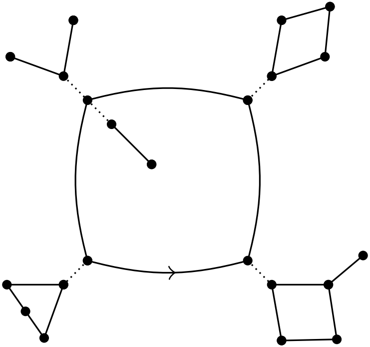

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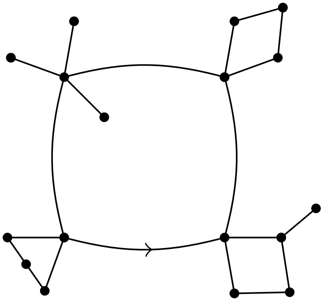

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- The generating function $N(z)$ can be obtained by inversion from $M(z)$:

$$M(z) = 1 + N\left(zM(z)^2\right)$$

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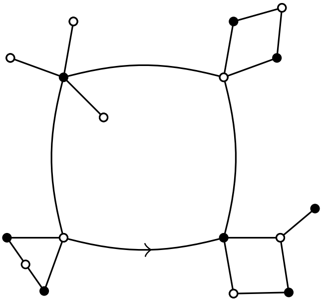

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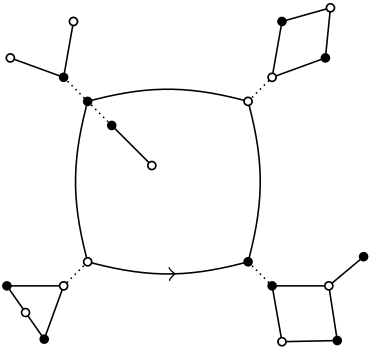
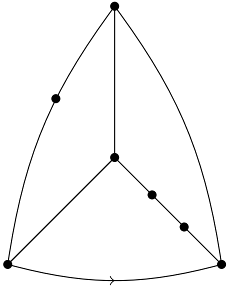


3-connected graphs and maps

Trakhtenbrot (1958); Tutte (1962): a 2-connected graph admits a unique decomposition into components that are series, parallel and polyhedral.



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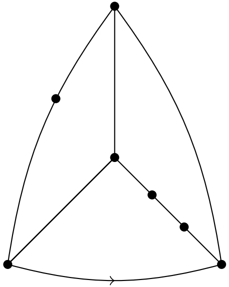

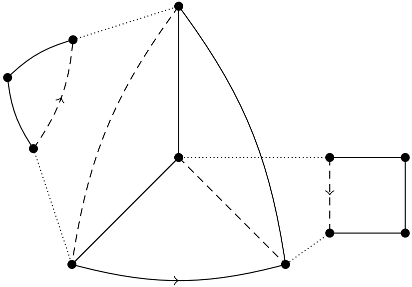

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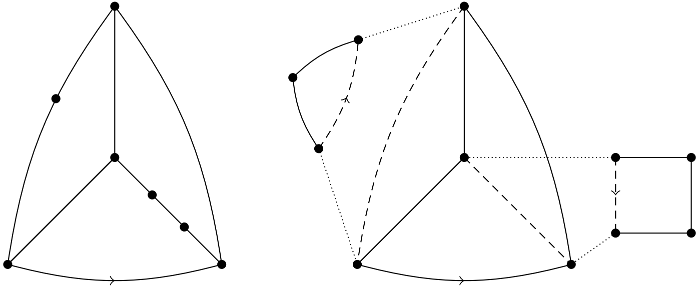
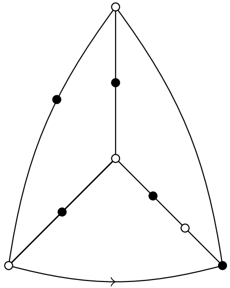


Tutte (1962): the GF of 3-connected maps can be obtained as some algebraic inverse of the GF of polyhedral components:

$$N(z) = z + S(z) + P(z) + \frac{T(N(z))}{N(z)}, \qquad P(z) = S(z) = N(z)(N(z) - S(z)).$$

3-connected bipartite planar maps

Problem: 3-connected components of bipartite maps are not bipartite!



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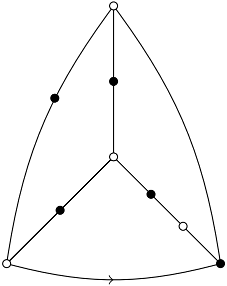

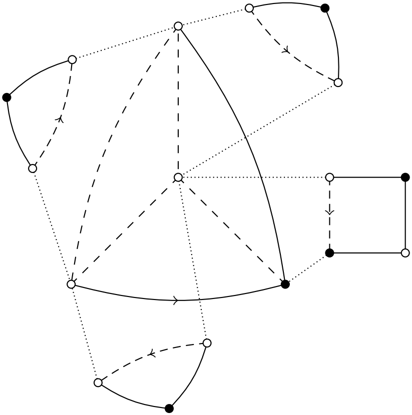

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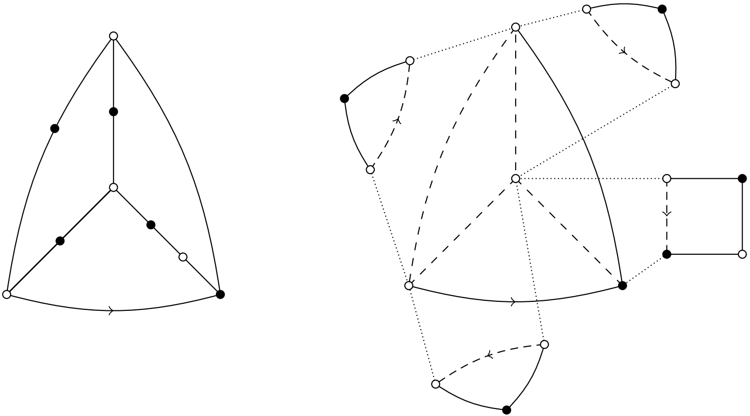

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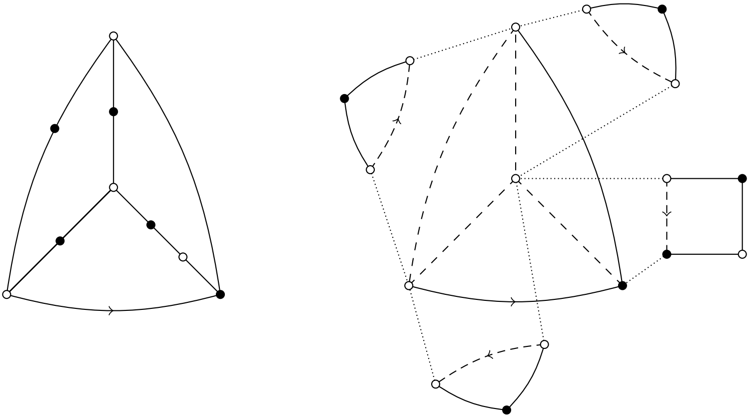

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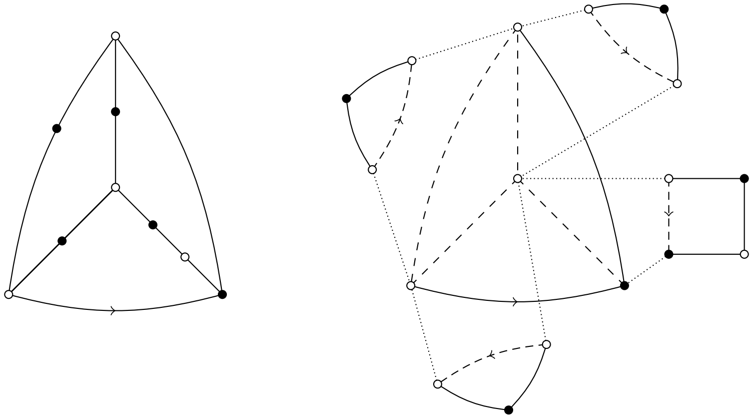

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- Set $\nu = 0$. Then $T(z, 0)$ is the GF of 3-connected bipartite maps.
- Need the GFs of maps rooted at a monochromatic/bichromatic edge.

Ising model on maps

Ising polynomial of a map M

- sum over all 2-colourings of the vertices of M, each monochromatic edge has weight ν:

$$P_M(2,\nu) = \sum_{c:V(M)\to\{0,1\}} \nu^{m(c)}$$

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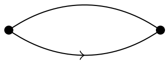
$M =$ 

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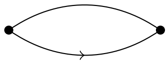
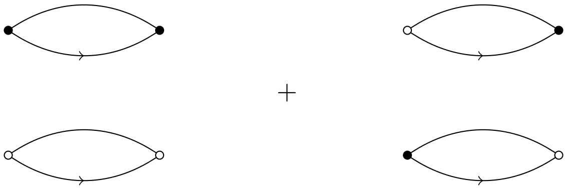
$M =$ 

$P_M(2,\nu) = 2\nu^2 + 2$:



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Recursive computation:

- If M has no edge

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- If e is an edge of M

$$P_M(q,\nu) = P_{M\setminus e}(q,\nu) + (\nu-1)P_{M/e}(q,\nu)$$

An equation for the Potts generating function of maps

Potts generating function of planar maps $\mathcal{M}$:

$$M = M(z, w, q, \nu) = \frac{1}{q} \sum_{M \in \mathcal{M}} z^{e(M)} w^{v(M)-1} P_M(q, \nu)$$

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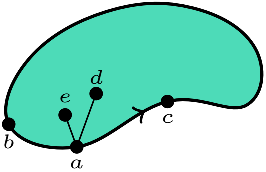

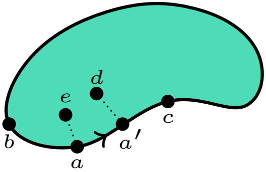

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- Includes $q = 1,2,3$. $j=2$: Beraha numbers.
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Rooting at a monochromatic/bichromatic edge (merci Mireille !)

Set $q = 2$. We have

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- Finally we do the same for $M_1(1,1)$.

Ising generating function of non-separable maps

- Brown (1963): catalytic equation for the generating function.
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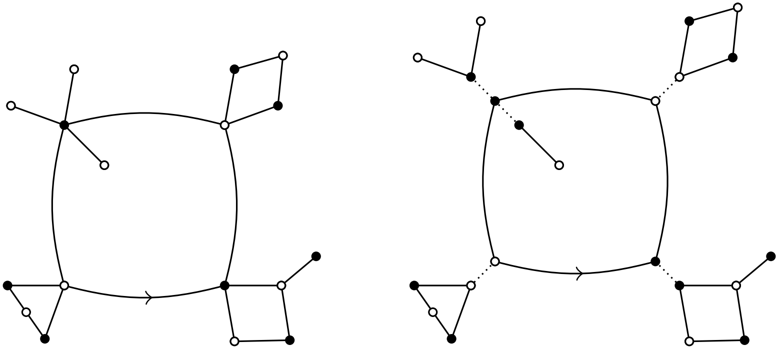
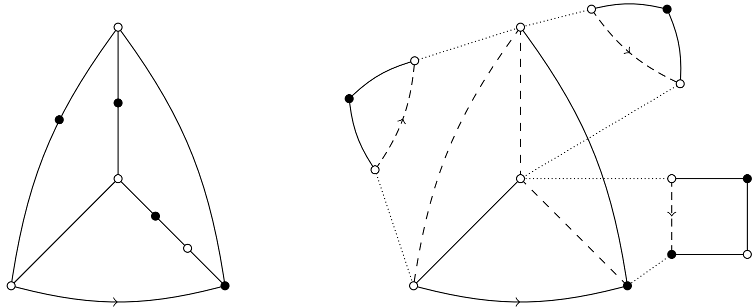


Bernardi & Bousquet-Mélou (2011): the Potts generating function of non-separable maps is algebraic over $\mathbb{Q}(q, \nu, z, w)$.

- The Ising generating function of non-separable maps has degree 15:

$$M_1(z,\nu,w) = N_1(zM^2, w, \nu) + (w+1)z\nu, \quad M_2(z,\nu,w) = N_2(zM^2, w, \nu) + zw.$$

Ising generating function of 3-connected maps



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$$P_1 = \frac{N_1^2}{1+N_1} \qquad P_2 = \frac{N_2^2}{1+N_2}$$

$$N_1 = z\nu + S_1 + P_1 + T_1(N_2,w,N_1/N_2) \qquad N_2 = z + S_2 + P_2 + T_2(N_2,w,N_1/N_2).$$

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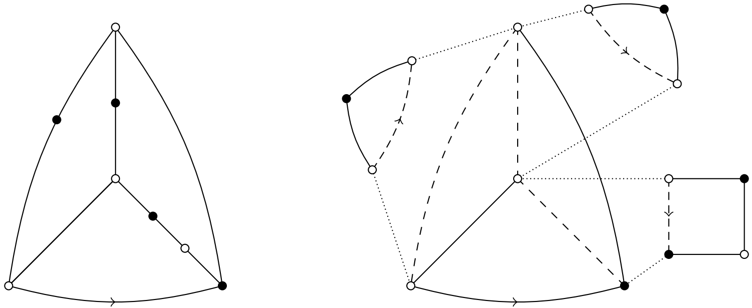

Noy, R. & Rué (2022+):

- The Ising generating function of 3-connected maps

$$T(z, w, \nu) = T_1(z, w, \nu) + T_2(z, w, \nu)$$

  is algebraic of degree 32 (same for T_1 and T_2).
- The generating function of bipartite 3-connected maps $T(z, 1, 0)$ is algebraic of degree 26.

Enumeration of bipartite planar graphs

Whitney's theorem:
Ising generating functions of 3-connected planar graphs is $T(z, w, \nu)/2$.

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Further question: Potts model over 3-connected maps?

- more colours seem to not behave well with respect to connectivity-decomposition
- Bodirsky, Gröpl, Johannsen & Kang (2005): bi-catalytic equation for 3-connected maps.