Hey Series, How do you make Macdonald polynomials?

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We'll talk about

- Modules, symmetric functions, combinatorics
- Macdonald goes wild (1980's)
- Catalanimals, a series perspective

Example: Harmonics

Polynomials in n variables

$$\mathcal{M} = \{f(\mathrm{x}) : \left(\partial^a_{x_1} + \cdots + \partial^a_{x_n}\right) f(\mathrm{x}) = 0 \ \forall a > 0\}$$

= span of all partial derivatives of Vandermonde

$$\Delta = \det \begin{vmatrix} x_1^2 & x_1^1 & 1 \\ x_2^2 & x_2^1 & 1 \\ x_3^2 & x_3^1 & 1 \end{vmatrix} = x_1^2(x_2 - x_3) - x_2^2(x_1 - x_3) + x_3^2(x_1 - x_2)$$

S_n-module

$$\mathrm{sp}\Big\{\Delta,\ 2x_1(x_2-x_3)-x_2^2+x_3^2,\ \ 2x_2(x_3-x_1)-x_3^2+x_1^2,\ \ x_3-x_1,\ \ x_1-x_2,\ \ 1\Big\}$$

$1\mapsto 2 \ \ 2\mapsto 3 \ \ 3\mapsto 1$

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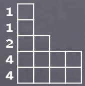
Harmonics Module

$\mathcal{M}$ = span of all partial derivatives of Vandermonde

$$\underbrace{\mathrm{sp}\{\Delta\}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}} \oplus \underbrace{\mathrm{sp}\{1\}}$$

Irreducible submodules indexed by partitions

$\lambda =$



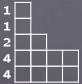
$= (4, 4, 2, 1, 1)$

Harmonics Module

$\mathcal{M}$ = span of all partial derivatives of Vandermonde

$$\underbrace{\mathrm{sp}\{\Delta\}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}} \oplus \underbrace{\mathrm{sp}\{1\}}$$

Irreducible submodules indexed by partitions

$\lambda =$  $= (4, 4, 2, 1, 1)$

how many copies of each irreducible?

Symmetric functions

Multivariate polynomials invariant under S_n-action:
$\sigma : z_i \mapsto z_{\sigma(i)}$

symmetric: $3z_1^2 + 3z_2^2 - 7z_1 z_2$

not symmetric: $z_1^5 z_2 \neq z_2^5 z_1$

Symmetric functions

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Bases are indexed by partitions

symmetrizing a monomial $\rightarrow$ symmetric function

$$z_1^5 z_2^1 \longrightarrow z_1^5 z_2^1 + z_2^5 z_1^1$$

indexed by $(5, 1) = \begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline \; \\ \cline{1-1}\end{array}$

Schur functions

Def. $s_\lambda(z)$ is the Weyl symmetrization of $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$

$$\sigma\big(f(z)\big) = \sum_{w \in \mathcal{S}_n} w\left(\frac{f(z)}{\prod_{i<j}(1 - z_j/z_i)}\right)$$

Schur functions

Def. $s_\lambda(z)$ is the Weyl symmetrization of $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$

Prop. $s_\lambda(z)$ is the sum over Young tableaux of shape λ

fillings which increase up columns
and weakly increase across rows

$$T = \begin{array}{|c|c|c|c|c|} \cline{1-1} 4 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 1 & 1 & 2 & 4 \\ \hline \end{array} \quad \longrightarrow \quad \mathbf{z}^T = \prod_{i=\text{label in } T} z_i = z_1 z_1 z_1 z_2 z_2 z_3 z_4 z_4$$

$$s_{(2,1)}(\mathbf{z}) = \begin{array}{|c|c|} \cline{1-1} 3 \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \cline{1-1} 2 \\ \hline 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|} \cline{1-1} 2 \\ \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \cline{1-1} 2 \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \cline{1-1} 3 \\ \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \cline{1-1} 3 \\ \hline 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|} \cline{1-1} 3 \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|} \cline{1-1} 3 \\ \hline 2 & 3 \\ \hline \end{array}$$

$$= 2z_1 z_2 z_3 \;+\; z_1^2 z_2 + z_1 z_2^2 + z_1^2 z_3 + z_1 z_3^2 + z_2^2 z_3 + z_2 z_3^2$$

Harmonics Module

$\mathcal{M}$ = span of all partial derivatives of Vandermonde

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

Frobenius

irreducible $\chi^\lambda \mapsto s_\lambda$

$$????? = s_{\ydiagram{1,1,1}} + s_{\ydiagram{1,2}} + s_{\ydiagram{1,2}} + s_{\ydiagram{3}}$$

coefficient of $s_{\ydiagram{1,2}}$ in ???? = # of irreducibles $\ydiagram{1,2}$

Harmonics Module

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Frobenius

irreducible $\chi^\lambda \mapsto s_\lambda$

$$????? = s_{\ydiagram{1,1,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{2,1}} + s_{\ydiagram{3}}$$

$$\mathcal{F}(\mathcal{M}) = (z_1+z_2+z_3)^3$$

Schur expansions

$$(z_1 + z_2 + z_3)^3 = s_{(1,1,1)} + s_{(2,1)} + s_{(2,1)} + s_{(3)}$$

Expand $(z_1 + z_2 + z_3)^3$ into Schur functions

```
sage: pol = Sym.from_polynomial((z1+z2+z3)^3)
sage: s(pol)

s[1,1,1] + 2*s[2,1] + s[3]
```

List tableaux with one each of letters 1,2,3

```
sage: Tableaux.options.convention="french"
sage: SYT=StandardTableaux(3)
sage: [taby.pp() for taby in SYT]
3
2       2       3
1       13      12      1 2 3
```

Modules, symmetric functions, combinatorics

- S_n-Module
  $\mathcal{M}$ = Harmonic polynomials
- Identify symmetric function = Frobenius image

  $\mathcal{F}(\mathcal{M}) = (z_1+z_2+z_3)^3$
- Explore combinatorics
  Young tableaux

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

$$s_{\ytableaushort{3,2,1}} + s_{\ytableaushort{3,12}} + s_{\ytableaushort{2,13}} + s_{\ytableaushort{123}}$$

$$(z_1+z_2+z_3)^3$$

The quantum craze

- Combinatorics: basic hypergeometric series, q-counting

$$\to \quad (1 + q + q^2 + q^3 + q^4)$$

- Geometry: q-deformation of cohomology
- Physics: conformal and topological quantum field theory

Modules/symmetric functions/combinatorics?

Harmonics Module

$\mathcal{M}$ = span of all partial derivatives of Vandermonde

$$\underbrace{\mathrm{sp}\{\Delta\}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\text{degree 2 polynomials}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\text{degree 1 polynomials}} \oplus \underbrace{\mathrm{sp}\{1\}}$$

what are the polynomial degrees in a given irreducible?

Harmonics Module

$\mathcal{M}$ = span of all partial derivatives of Vandermonde

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

degree 2 polynomials

degree 1 polynomials

Frobenius

degree d irreducible $\mapsto q^d s_\lambda$

$$????? = q^3 s_{\ydiagram{1,1,1}} + q^2 s_{\ydiagram{1,2}} + q\, s_{\ydiagram{1,2}} + s_{\ydiagram{3}}$$

Identify the symmetric function $\mathcal{F}(\mathcal{M})$

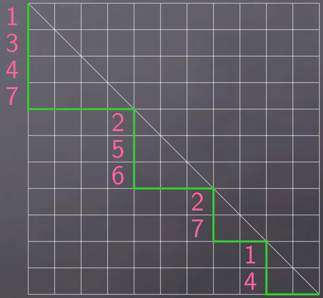
Hall-Littlewood Polynomials

For partition $\mu = (\mu_1, \ldots, \mu_\ell)$,

$$\tilde{H}_\mu(\mathbf{z}; q) = \sum_{\substack{\text{label } L \text{ of} \\ \text{path in } \mu-\text{blocks}}} q^{\mathrm{dinv}(L)} \, \mathbf{z}^L$$

Dyck path of partition $\mu = (4, 3, 2, 2)$
has blocks of sizes $4, 3, 2, 2$

label L: increase down vertical runs



Hall-Littlewood Polynomials

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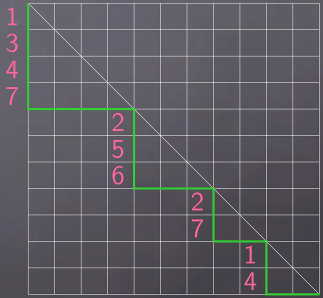
Dyck path of partition $\mu = (4, 3, 2, 2)$ has blocks of sizes $4, 3, 2, 2$

label L: increase down vertical runs

$$\mathbf{z}^L = \prod_{\text{label } \ell} z_\ell = z_1^2 z_2^2 z_3 z_4^2 z_5 z_6 z_7^2$$

dinv(L) counts inverted labels



Modules, symmetric functions, combinatorics

- Graded S_n-Module
  $\mathcal{M}$ = Harmonic polynomials

- Identify (Schur positive) function = q-Frobenius image
  $\mathcal{F}(\mathcal{M})$ = Hall-Littlewood polynomial

- Explore combinatorics
  labelled Dyck paths, dinv-statistic, Young tableaux

$$\underbrace{\mathrm{sp}\{\Delta\}}_{\ydiagram{1,1,1}} \oplus \underbrace{\mathrm{sp}\{2x_1(x_2-x_3)-x_2^2+x_3^2,\, x_1^2-2x_2(x_1-x_3)-x_3^2\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{x_3-x_1,\, x_2-x_3\}}_{\ydiagram{1,2}} \oplus \underbrace{\mathrm{sp}\{1\}}_{\ydiagram{3}}$$

$$q^3\, s_{\begin{ytableau}3\\2\\1\end{ytableau}} + q^2\, s_{\begin{ytableau}3\\1&2\end{ytableau}} + q\, s_{\begin{ytableau}2\\1&3\end{ytableau}} + s_{\begin{ytableau}1&2&3\end{ytableau}}$$

$$\sum_{\substack{P=\text{labelling of}\\ \text{path } \mu}} q^{\mathrm{dinv}(P)}\, z^P$$

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Macdonald initiates q, t–theory

Generalization of Selberg's integral led to q, t symmetric functions

$$\frac{-4q}{t-1}x_1^2x_2 + \frac{-4q}{t-1}x_2^2x_3 + \frac{-4q}{t-1}x_1^2x_3 + (t^2 - 7q)x_1x_2x_3$$

Macdonald prove the existence of a basis

– defined by orthogonality (use Gram-Schmidt)
– specializing to Hall-Littlewoods
– conjectured to have certain positive expansion

(Garsia-modified) conjecture: q, t-sum of Schur functions

$$\tilde{H}_{2,2} = t^2 s_{4} + (qt^2 + qt + t)\, s_{31} + (q^2t^2 + 1)\, s_{22} + (q^2t + qt + q)\, s_{211} + q^2 s_{1111}$$

Garsia-Haiman modules

Wanted: for each partition,

S_n-module in $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ under $\sigma : x_i y_j \mapsto x_{\sigma(i)} y_{\sigma(j)}$

degree in x degree in y

$$\tilde{H}_{2,1} = t^0\, q^0\, s_{\ydiagram{3}} + t^0\, q^1\, s_{\ydiagram{2,1}} + t^1\, q^0\, s_{\ydiagram{2,1}} + t^1\, q^1\, s_{\ydiagram{1,1,1}}$$

modified Macdonald polynomial

Garsia-Haiman modules

Found

$\mathcal{M}_\mu$ – span of partial derivatives of Δ_μ

$x_r^{\text{row-1}} y_r^{\text{col-1}}$

$$\Delta_{\begin{smallmatrix}(2,1)&\\(1,1)&(1,2)\end{smallmatrix}} = \det\begin{vmatrix}1&y_1&x_1\\1&y_2&x_2\\1&y_3&x_3\end{vmatrix} = x_3y_2 - y_3x_2 - y_1x_3 + y_1x_2 + y_3x_1 - y_2x_1$$

$$\mathcal{M}_{2,1} = \underbrace{\mathrm{sp}\{1\}}_{(0,0)} \oplus \underbrace{\mathrm{sp}\{x_3 - x_1, x_1 - x_2\}} \oplus \underbrace{\mathrm{sp}\{y_3 - y_1, y_1 - y_2\}}_{(1,0)} \oplus \underbrace{\mathrm{sp}\{\Delta_{(2,1)}\}}_{(1,1)}$$

degree 0 in y degree 1 in x

$$\tilde{H}_{2,1} = t^0 q^0 s + t^0 q^1 s + t^1 q^0 s + t^1 q^1 s$$

modified Macdonald polynomial

q,t-Example

2 Bi-Graded Module (Garsia/Haiman'92)

$$\mathcal{M}_\mu = \text{linear span of partial derivatives of } \Delta_\mu$$

1 Symmetric function = Frobenius image (Haiman'02)

$$\mathcal{F}(\mathcal{M}_\mu) = \text{modified Macdonald polynomial}$$

3 Combinatorics...

Another Bigraded example

Harmonics = polynomials in n variables

$\mathcal{M} = \{f(\mathrm{x}) : \left(\partial^a_{x_1} + \cdots + \partial^a_{x_n}\right) f(\mathrm{x}) = 0 \ \forall a > 0\}$

$=$ span of all partial derivatives of Vandermonde

$\downarrow$

extended to Δ_μ

Diagonal Harmonics Module

Polynomials in 2 sets of variables

$$\mathcal{DH}_n = \left\{ f(x, y) : \sum_i \partial_{x_i}^a \partial_{y_i}^b f(x, y) = 0 \ \forall a + b > 0 \right\}$$

$$\mathcal{DH}_2 = \mathrm{sp}\{x_1 - x_2, y_1 - y_2, 1\}$$

$$\swarrow \qquad\quad \downarrow \qquad\quad \searrow$$

$$??? = q^1 \, s_{\substack{\square \\ \square}} \; + \; t^1 \, s_{\substack{\square \\ \square}} \; + \; s_{\square\square}$$

What symmetric function is the Frobenius image of $\mathcal{DH}_n$?

The Frobenius image of $\mathcal{F}(\mathcal{DH}_n)$

Example. $\mathcal{F}(\mathcal{DH}_2) = (q+t)\, s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} + s_{\square\square}$

Macdonald expand $\mathcal{F}(\mathcal{DH}_2)$

```
sage: HMact( (q+t)*s([1,1])+s([2]) )
```

$$\mathcal{F}(\mathcal{DH}_2) = \frac{t}{t-q} H_{1,1} - \frac{q}{t-q} H_2$$

Macdonald expand $s_{1^2} = x_1 x_2$

```
sage: HMact( s([1,1]) )
```

$$s_{1^3} = \frac{1}{t-q} H_{1,1} - \frac{1}{t-q} H_2$$

The Frobenius image of $\mathcal{F}(\mathcal{DH}_n)$

Example. $\mathcal{F}(\mathcal{DH}_2) = (q+t)\, s_{\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}} + s_{\tiny\begin{array}{|c|c|}\hline & \\ \hline\end{array}}$

Macdonald expand $\mathcal{F}(\mathcal{DH}_2)$

```
sage: HMact( (q+t)*s([1,1])+s([2]) )
```

$$\mathcal{F}(\mathcal{DH}_2) = \frac{t}{t-q} H_{1,1} - \frac{q}{t-q} H_2$$

Def. ∇ is a Macdonald eigenoperator

Macdonald expand $s_{1^2} = x_1 x_2$

```
sage: HMact( s([1,1]) )
```

$$s_{1^3} = \frac{1}{t-q} H_{1,1} - \frac{1}{t-q} H_2$$

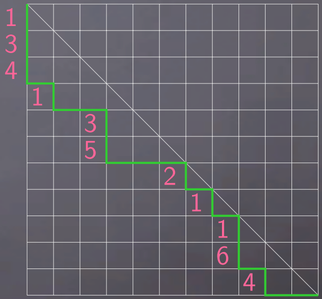
[Haiman'02] $\mathcal{F}(\mathcal{DH}_n) = \nabla s_{1^n}$

The Shuffle Conjecture [HHLRU'05]

$$\nabla s_{1^n} = \sum_{\pi} t^{\mathrm{area}(\pi)} \sum_{P \text{ labels } \pi} q^{dinv(P)} z^P$$

Dyck Paths π:

lattice path from $(0, n)$ to $(n, 0)$ below diagonal



LLT-polynomial

Labellings P

increase down vertical runs

dinv(P) counts inverted labels

The Shuffle Conjecture [HHLRU'05]

$$\nabla s_{1^n} = \sum_{\pi} t^{\mathrm{area}(\pi)} \sum_{P \text{ labels } \pi} q^{dinv(P)} z^P$$

Dyck Paths π:

lattice path from $(0, n)$ to $(n, 0)$ below diagonal

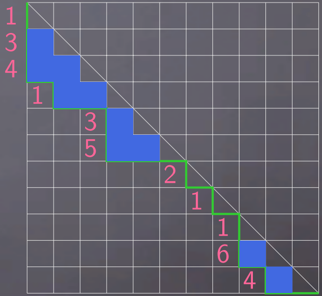
area counts squares above π

LLT-polynomial

Labellings P

increase down vertical runs

dinv(P) counts inverted labels



Garsia-Haiman Modules...combinatorics

2 Bi-graded Module (Garsia/Haiman'92)

$$\mathcal{M}_\mu = \text{linear span of partial derivatives of } \Delta_\mu$$

1 Symmetric function (Haiman'02)

$$\mathcal{F}(\mathcal{M}_\mu) = \text{modified Macdonald polynomial } \tilde{H}_\mu$$

3 Combinatorics (Haiman, Haglund, Loehr'05)

$$\tilde{H}_\mu = \sum_{D \subset \mu} t^{maj(D)} \, LLT_D(z; q)$$

Diagonal Harmonics

- Bigraded Module (Garsia/Haiman'92)

$$\mathcal{DH}_n = \text{harmonics in } 2n \text{ variables}$$

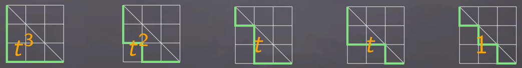
- Symmetric functions = Frobenius image (Haiman'02)

$$\mathcal{F}(\mathcal{DH}_n) = \nabla s_{1^n}$$

- Combinatorics (Carlsson/Mellit'15)

$$\nabla s_{1^n} = \sum_{\text{Dyck paths } \pi} t^{\text{area}(\pi)} \, LLT_\pi(\mathbf{z}; q)$$

attach an LLT to each path, t-weighted by its area



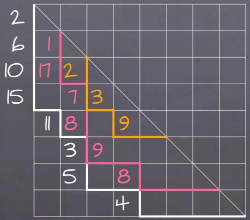
Other q, t-frameworks

Building on $\mathcal{F}(\mathcal{DH}_n) = \nabla s_{1^n}$

Conjecture. Loehr-Warrington (2008)

$$\nabla s_\lambda = \sum_{\substack{\pi \,=\, \text{nested paths} \\ \text{of lengths from } \lambda}} t^{\mathrm{area}(\pi)}\, LLT_\pi(\mathbf{z}; q)$$

attach an LLT to each nest of paths,
t-weighted by the sum of areas of each path in a nest

$P =$ 

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Back to the drawing board

Schur function = symmetrization of monomial

$s_\mu = \sigma(z^\mu)$ where

$$\sigma(f(z)) = \sum_{w \in \mathcal{S}_n} w\left(\frac{f(z)}{\prod_{i<j}(1 - z_j/z_i)}\right)$$

symmetric function theory develops from formulas like this!

until both q and t got involved.

- Macdonald polynomials
- ∇s_λ (q,t-labellings of nested Dyck paths)
  ∇s_{1^n} and friends

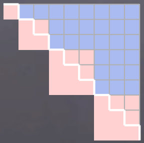
One parameter study

Hall-Littlewood polynomials (Shimozono-Weyman)

$$\tilde{H}_\mu(\boldsymbol{z};q) = \operatorname{pol}\left(\boldsymbol{\sigma}\left(\frac{z_1\cdots z_n}{\prod_{(i,j)\in R_\mu}(1-q\,z_i/z_j)}\right)\right)$$

Introduce q denominator terms

Def. R_μ = roots above Dyck path with $\mu_\ell,\ldots,\mu_1$-sized blocks

$R_{3321} =$ 

$R_{21} =$  $= \{(1,2),(1,3)\}$

Hall-Littlewood polynomial example

$$\tilde{H}_{(21)}(\boldsymbol{z};q) = \mathrm{pol}\left(\boldsymbol{\sigma}\left(\frac{z_1 z_2 z_3}{(1-qz_1/z_2)(1-qz_1/z_3)}\right)\right) \qquad R_{21} =$$ 

Series expand

$$\downarrow$$

$$z_1 z_2 z_3 + q\, z_1^2 z_3 + q^2\, z_1^3 z_3/z_2 + q\, z_1^2 z_2 + q^2\, z_1^3 + q^3\, z_1^5/z_2 z_3 + \cdots$$

Apply σ

$$\boldsymbol{\sigma}(\boldsymbol{z}^{111}) + q\boldsymbol{\sigma}(\boldsymbol{z}^{201}) + q^2\boldsymbol{\sigma}(\boldsymbol{z}^{3-11}) + q\boldsymbol{\sigma}(\boldsymbol{z}^{210}) + q^2\boldsymbol{\sigma}(\boldsymbol{z}^{300}) + q^3\boldsymbol{\sigma}(\boldsymbol{z}^{5-1-1}) + \cdots$$

s_{111} $\qquad\qquad$ $q\ s_{21}$

straightening

$\boldsymbol{\sigma}(x^\gamma) \to \pm\boldsymbol{\sigma}(x^\lambda)$ for weakly decreasing λ, or zero

Hall-Littlewood polynomial example

$$\tilde{H}_{(21)}(\mathbf{z};q) = \mathrm{pol}\left(\sigma\left(\frac{z_1 z_2 z_3}{(1-qz_1/z_2)(1-qz_1/z_3)}\right)\right) \qquad R_{21} =$$ 

Series expand

$$\downarrow$$

$$z_1 z_2 z_3 + q\, z_1^2 z_3 + q^2\, z_1^3 z_3/z_2 + q\, z_1^2 z_2 + q^2\, z_1^3 + q^3\, z_1^5/z_2 z_3 + \cdots$$

Apply σ (straighten)

$$\sigma(\mathbf{z}^{111}) + q\sigma(\mathbf{z}^{201}) + q^2\sigma(\mathbf{z}^{3-11}) + q\sigma(\mathbf{z}^{210}) + q^2\sigma(\mathbf{z}^{300}) + q^3\sigma(\mathbf{z}^{5-1-1}) + \cdots$$

$$s_{111} + \quad 0 \quad -\underline{q^2\sigma(\mathbf{z}^{300})} + q\, s_{21} \quad + \underline{q^2\sigma(\mathbf{z}^{300})} + q^3\sigma(\mathbf{z}^{5-1-1}) + \cdots$$

$\mathrm{pol}(\sigma(x^\lambda)) = 0$ when $\lambda_n < 0$

$$s_{111} + q\, s_{21}$$

Why not roots above any Dyck path?

Catalan functions (Panyushev and Haiman)

For Dyck path π,

$$\tilde{H}_\pi(\boldsymbol{z}; q) = \mathrm{pol}\left(\boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^\lambda}{\prod_{(i,j)\in R_\pi}(1 - q\, z_i/z_j)}\right)\right)$$

Def. R_π = set of roots above any Dyck path

$R_\pi =$ 

Theorem. [Blasiak-M-Pun]

Catalan functions are Schur positive for any partition λ

It's time to get series!

Hall-Littlewood polynomials

$$\tilde{H}_\mu(\mathbf{z}; q) = \mathrm{pol}\left(\sigma\left(\frac{z_1 \cdots z_n}{\prod_{(i,j)\in R_\mu}(1 - q z_i/z_j)}\right)\right)$$

experiment with adding t

Example.

$$\mathcal{H} = \mathrm{pol}\left(\sigma\left(\frac{z_1 z_2}{(1-qz_1/z_2)(1-tz_1/z_2)}\right)\right)$$

sage: s($\mathcal{H}$)
sage: $(q+t)s_{\substack{\square\\\square}} + s_{\square\square}$

It's time to get series!

Hall-Littlewood polynomials

$$\tilde{H}_\mu(\mathbf{z};q) = \mathrm{pol}\left(\sigma\left(\frac{z_1\cdots z_n}{\prod_{(i,j)\in R_\mu}(1-q\,z_i/z_j)}\right)\right)$$

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sage: s($\mathcal{H}$)
sage: $(q+t)s_{\substack{\square\\\square}} + s_{\square\square}$

Example.

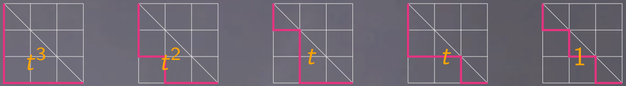
$$\mathcal{DH}_2 = \mathrm{sp}\{x_1 - x_2, y_1 - y_2, 1\}$$

$$\swarrow \qquad \downarrow \qquad \searrow$$

$$\nabla s_{1^2} = q^{1}\, s_{\substack{\square\\\square}} \;+\; t^{1}\, s_{\substack{\square\\\square}} \;+\; s_{\square\square}$$

Series for diagonal harmonics

$$\nabla s_{1^n} = \sum$$



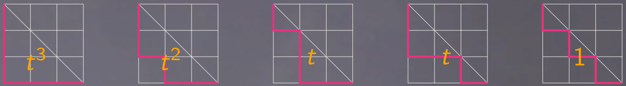
polynomial part of

$$\sigma\left(\frac{z_1\cdots z_n \prod_{i+1<j}(1-q\,t\,z_i/z_j)}{\prod_{i<j}(1-q\,z_i/z_j)(1-t\,z_i/z_j)}\right)$$

$$\sum_{\text{labellings } P \text{ of path}} q^{dinv(P)} x^P$$

Series for diagonal harmonics

$$\nabla s_{1^n} = \sum$$



$$\sum_{\text{labellings } P \text{ of path}} q^{dinv(P)} x^P$$

polynomial part of

$$\sigma\left(\frac{z_1 \cdots z_n \prod_{i+1<j}(1 - q\,t\,z_i/z_j)}{\prod_{i<j}(1 - q\,z_i/z_j)(1 - t\,z_i/z_j)}\right)$$

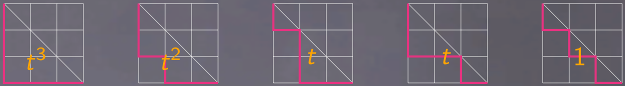
pol

LLT Series

$$\mathcal{L}_{\beta/\alpha} = \sigma\left(\frac{(w_0(F_\beta(z;q)\overline{E_\alpha(z;q)}))}{\prod_{i<j}(1 - qz_i/z_j)}\right)$$

for non-symmetric Hall-Littlewood polynomials $E_\alpha = (-q)^{\ell(w)} T_w z^{\alpha_+}$ and their duals

Series for diagonal harmonics

$\nabla s_{1^n} \;=\; \sum$ 

$$\sum_{\text{labellings } P \text{ of path}} q^{dinv(P)} x^P$$

~~polynomial part of~~

$$\cancel{ct}\left(\frac{z_1\cdots z_n\prod_{i+1<j}(1-q\,t\,z_i/z_j)}{\prod_{i<j}\cancel{(1-q\,z_i/z_j)}(1-t\,z_i/z_j)}\right)$$

~~pol~~

right hand side

$$\sum t^{\mathrm{area}(\beta/\alpha)}\cancel{ct}\left(\frac{(w_0(F_\beta(z;q)\overline{E_\alpha(z;q)})}{\cancel{\prod_{i<j}(1-q\,z_i/z_j)}}\right)$$

for non-symmetric Hall-Littlewood polynomials $E_\alpha = (-q)^{\ell(w)}T_w z^{\alpha_+}$ and their duals

Unstraightened Series Identity

Theorem

$$z_1 \cdots z_l \frac{\prod_{i+1<j}(1 - q\,t\,z_i/z_j)}{\prod_{i<j}(1 - t\,z_i/z_j)} = \sum_{a_1,\ldots,a_{l-1}\geq 0} t^{|a|} w_0(F_\beta(z;q)\overline{E_\alpha(z;q)})$$

where $\beta = (1, a_{l-1}+1, \ldots, a_1+1)$ and $\alpha = (a_1, \ldots, a_{l-1}, 0)$.

Proof: Cauchy Identity

$$\frac{\prod_{i<j}(1 - tqz_iy_j)}{\prod_{i\leq j}(1 - tz_iy_j)} = \sum_{a} t^{|a|} E_a(z;q^{-1}) F_a(y;q)$$

implies Shuffle Theorem by symmetrizing and taking the polynomial part

Catalanimals

For 3 sets of roots R_{qt}, R_q, R_t and an integer vector λ,

$$H(R_{qt}, R_q, R_t, \lambda) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^\lambda \prod_{(i,j)\in R_{qt}}(1 - q\,t\,z_i/z_j)}{\prod_{(i,j)\in R_q}(1 - q\,z_i/z_j)\prod_{(i,j)\in R_t}(1 - t\,z_i/z_j)}\right)$$

Example. ∇s_{1^n} is the polynomial part of

$$H(R_{qt}, R_q, R_t, 1^n) = \boldsymbol{\sigma}\left(\frac{z_1\cdots z_n \prod_{i+1<j}(1 - q\,t\,z_i/z_j)}{\prod_{i<j}(1 - q\,z_i/z_j)(1 - t\,z_i/z_j)}\right)$$

$R_q = R_t =$ all roots, $R_{qt} =$ all roots $\setminus \{(1,2),(2,3),\ldots,(n-1,n)\}$

is there a catalanimal H so $\mathrm{pol}(H) = \nabla s_\mu$?

what about Macdonald polynomials?

More sage experiments

Catalanimals for ∇s_μ

Def. the Schur catalanimal $H(R_{qt}, R_q, R_t; \lambda)$ has

- $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,
- $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,
- R_{qt} = all other pairs

1	3	6	
2	5	8	
4	7	9	10

$\mu = (433)$

Catalanimals for ∇s_μ

Def. the Schur catalanimal $H(R_{qt}, R_q, R_t; \lambda)$ has

- $R_+ \setminus R_q$ = pairs of boxes in the same diagonal,
- $R_t \setminus R_{qt}$ = pairs going between adjacent diagonals,
- R_{qt} = all other pairs

2	2	1	
2	1	0	
1	0	1	0

λ, as a filling of μ

the z^λ term: fill each diagonal D of μ with
$1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$.

Loehr-Warrington Conjecture

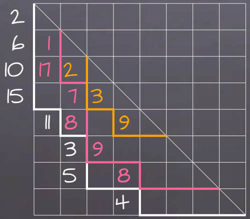
Schur catalanimal

$$\sigma\left(\frac{z^\lambda \prod_{(i,j)\in R_{qt}} (1 - q\,t\,z_i/z_j)}{\prod_{(i,j)\in R_q} (1 - q\,z_i/z_j) \prod_{(i,j)\in R_t} (1 - t\,z_i/z_j)} \right)$$

Theorem [Blasiak,Haiman,M,Pun,Seelinger]

$$\nabla s_\lambda = \sum_{\substack{\text{nested paths } \pi \\ \text{of shape } \lambda}} t^{\mathrm{area}(\lambda)} \sum_{\text{labelling } P \text{ of } \pi} q^{\mathrm{dinv}(P)} x^P$$

$P =$



LLT Series

$$\sigma\left(\frac{\left(w_0\left(F_\beta(z;q)\overline{E_\alpha(z;q)}\right)\right)}{\prod_{i<j}(1 - q z_i/z_j)} \right)$$

Playing with catalanimals

There are catalanimals for any LLT polynomial!

$$\mathrm{pol}(H) = \nabla(LLT_\nu(\boldsymbol{z}; q))$$

$$\tilde{H}_\mu(\boldsymbol{z}; q, t) = \sum_\nu t^{\mathrm{maj}(\nu)} LLT_\nu(\boldsymbol{z}; q)$$

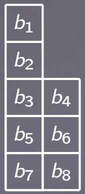
$$\Big\downarrow \text{ apply } \nabla$$

$$q^* t^* \tilde{H}_\mu(\boldsymbol{z}; q, t) = \sum_D t^{\mathrm{maj}(D)} \nabla(LLT_D(\boldsymbol{z}; q))$$

Theorem [Blasiak-Haiman-M.-Pun-Seelinger]

$$\tilde{H}_\mu = \mathrm{pol}\,\boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\prod_{(i,j)\in R_*} \left(1 - q^{\mathrm{arm}(b_i)+1}\, t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod_{(i,j)\in R_{qt}} \left(1 - qt\, z_i/z_j\right)}{\prod_{i<j} \left(1 - q\, z_i/z_j\right) \prod_{(i,j)\in R_t} \left(1 - t\, z_i/z_j\right)} \right).$$

Macanimals



row reading order

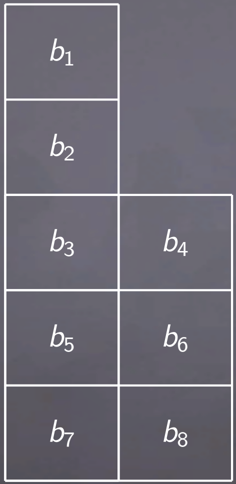
$R_t \setminus R_{qt}$ = top of a vertical domino in μ

R_{qt} = all pairs to right of

R_q = all roots

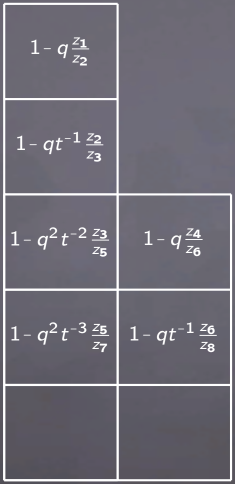
$$\tilde{H}_\mu = \mathrm{pol}\,\boldsymbol{\sigma}\left(z_1 \cdots z_n \frac{\prod\limits_{(i,j)\in R*} \left(1 - q^{\mathrm{arm}(b_i)+1}\, t^{-\mathrm{leg}(b_i)} z_i/z_j\right) \prod\limits_{(i,j)\in R_{qt}} \left(1 - qt\, z_i/z_j\right)}{\prod_{i<j} \left(1 - q\, z_i/z_j\right) \prod_{(i,j)\in R_t} \left(1 - t\, z_i/z_j\right)} \right).$$

Macanimals



partition $\mu = 22211$

Macanimals



numerator factors $1 - q^{\mathrm{arm}+1} t^{-\mathrm{leg}} z_i/z_j$

Now what?

For 3 sets of roots R_{qt}, R_q, R_t and an integer vector λ,

$$H(R_{qt}, R_q, R_t, \lambda) = \boldsymbol{\sigma}\left(\frac{\boldsymbol{z}^{\lambda}\prod_{(i,j)\in R_{qt}}(1 - q\,t\,z_i/z_j)}{\prod_{(i,j)\in R_q}(1 - q\,z_i/z_j)\prod_{(i,j)\in R_t}(1 - t\,z_i/z_j)}\right)$$

- revisit q,t-symmetric theory starting from catalanimal formulas
- explore new families of symmetric functions arising from catalanimals
- modules/combinatorics for new families
- develop theory around boosted Macanimals

thank you!