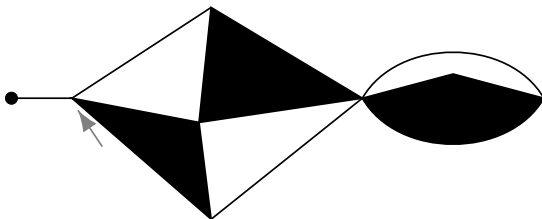


# Enumerating bicolored maps with an alternating boundary

Ariane Carrance (Laboratoire de Mathématiques d'Orsay)  
joint work with Jérémie Bouttier

Journées Combinatoires de Bordeaux - February 3<sup>rd</sup> 2021

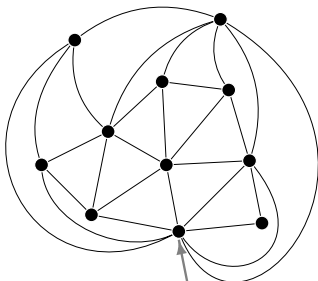


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## Initial problem

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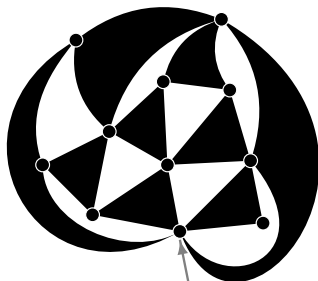
- (Rooted) maps whose faces are all of degree 3



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- (Rooted) maps whose faces are all of degree 3, and can be properly bicolored in black and white



Define **Boltzmann** Eulerian triangulations:

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- Random planar maps with a *boundary*, (picked in a certain family of maps) with fixed perimeter  $p$ , sampled according to *size*:

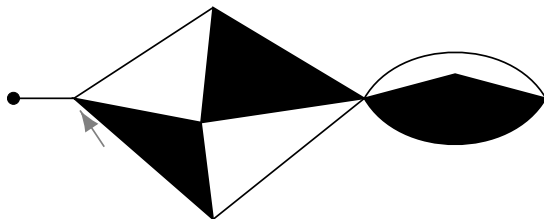
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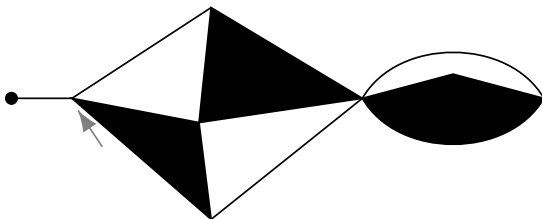


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- Natural boundary condition for Boltzmann triangulations: **alternating**
- Necessary in particular to show convergence of Eulerian triangulations to *Brownian map* in [C., 2019]



- For the probability law  $\mathbb{P}_p$  to be well-defined:

$$1 = \sum_n \sum_{|m|=n} \mathbb{P}_p(m) = c \sum_n C^{-n} A_{n,p}$$

$A_{n,p} = \#$ maps with a boundary of perimeter  $p$  and size  $n$  (in the family)



- For the probability law  $\mathbb{P}_\rho$  to be well-defined:

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$A_{n,p} = \# \text{maps with a boundary of perimeter } p \text{ and size } n \text{ (in the family)}$

- In fact: for all “nice” families of maps

$$A_{n,p} \underset{n \rightarrow \infty}{\sim} C(p) \rho^n n^{-5/2} \forall p$$

$$\implies Z(\rho) := \sum_n A_{n,p} \rho^{-n} < \infty, \quad \mathbb{P}_\rho(\mathbf{m}) = \frac{\rho^{-n}}{Z(\rho)}$$

## Theorem (Bouttier-C. 2020+)

Let  $A(t, w) = \sum_{n,p} A_{n,r} t^n w^r$  be the generating function for planar Eulerian triangulations with an alternating boundary, with  $t$  enumerating the vertices, and  $w$  the half-perimeter. Then:

$$[t^n]A(t, w) \underset{n \rightarrow \infty}{\sim} f(w)8^n n^{-5/2} \quad \forall w \in [0, \frac{1}{4}), \quad (1)$$

with  $f$  explicit. Consequently:

$$A_{n,r} \underset{n \rightarrow \infty}{\sim} C(r)8^n n^{-5/2} \quad \forall r, \quad (2)$$

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- Same as other “classical” families of planar maps with boundaries
- **Explicit**  $f$  in (1)  $\implies$  uniform bound by asymptotics (2):

$$c' C(r)8^n n^{-5/2} \leq A_{n,r} \leq c'' C(r)8^n n^{-5/2} \quad (4)$$

(crucial for establishing convergence to Brownian map in [C., 2019])

- **$m$ -constellations**: maps whose faces are properly colored in black and white, where all black faces have degree  $m$  and all white faces have degree some multiple of  $m$
- $A_r$ : GF of  $m$ -constellations with an alternating boundary of length  $2r$ , with a weight  $t$  per vertex and a weight  $x_i$  per white inner face of degree  $mi$  for every  $i \geq 1$ .

## Theorem (Bouttier-C. 2020+)

Let  $d$  be a positive integer and assume that  $x_i = 0$  for  $i > d$ . Then,  $A(w) := \sum_{r \geq 0} A_r w^{r+1}$  is algebraic and admits a rational parametrization: for  $s$  a formal variable we have

$$A(w(s)) = 1 - \frac{(1 + \sum_{k=1}^d \alpha_k s^k)^{m-1}}{1 + Vs}, \quad w(s) = s \frac{(1 + \sum_{k=1}^d \alpha_k s^k)^{m-2}}{(1 + Vs)^2} \quad (5)$$

$$\text{where } V = t + \sum_{i=1}^d \binom{mi-1}{i} x_i V^{(m-1)i} \quad (6)$$

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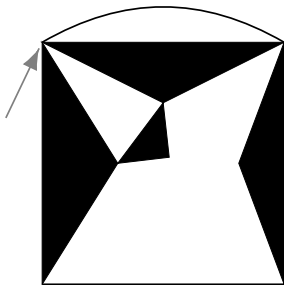
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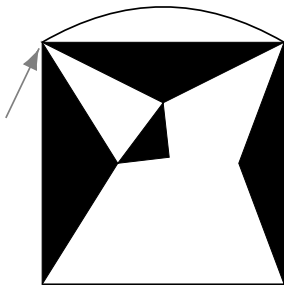
- Eulerian triangulations:  $m = 3$  and  $x_i = 0$  for  $i > 1$
- Asymptotic results obtained by application of the **transfer theorem** [Flajolet-Odlyzko 1990]



- Families  $\mathcal{F}$  of planar bicolored maps with an alternating boundary:



- Families  $\mathcal{F}$  of planar bicolored maps with an alternating boundary: characterized by possible degrees for black and white faces, enumerated respectively by variables  $\{t_i\}$  and  $\{\tilde{t}_i\}$
- Generating function  $A(t, \{t_i\}, \{\tilde{t}_i\}; w) := \sum_{r \geq 0} A_r w^{r+1}$

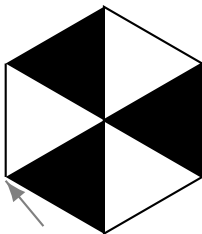


## Looking for a recurrence equation

- Strategy: **peel** the root edge, to get recurrence equation [Tutte 60's,...]

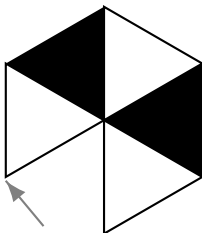
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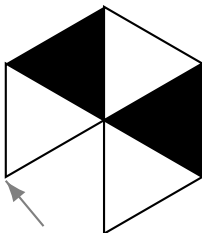
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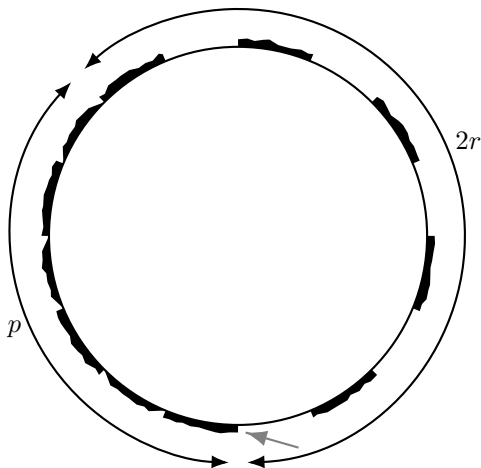


## Looking for a recurrence equation

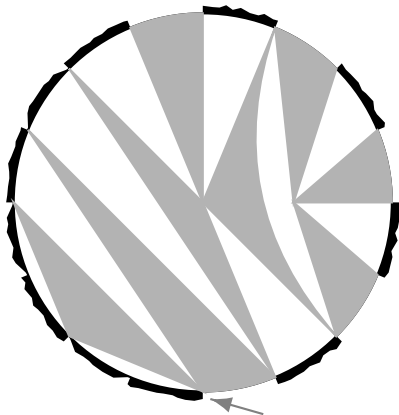
- Strategy: **peel** the root edge, to get recurrence equation [Tutte 60's,...]
- Problem: alternating boundary is **not** stable under peeling!
- Solution: start from more general boundary condition



- **Mixed** boundary condition: monochromatic + alternating



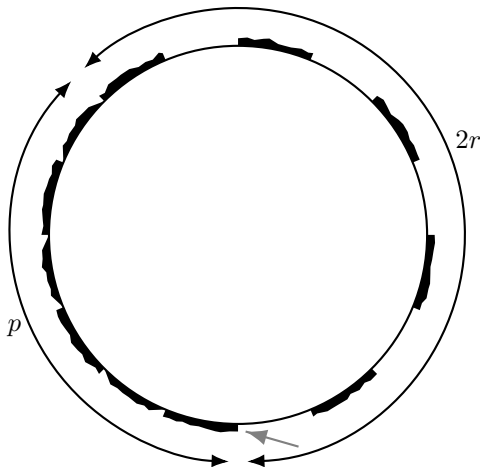
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## New boundary condition

- **Mixed** boundary condition: monochromatic + alternating
- Monochromatic boundary: well known [Bousquet-Mélou-Schaeffer 2000, Albenque-Bouttier 2012]



- $F_p$ : GF of maps in  $\mathcal{F}$  with a monochromatic border of length  $p$

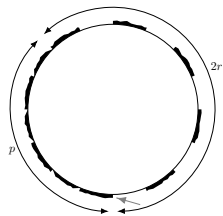
- $F_p$ : GF of maps in  $\mathcal{F}$  with a monochromatic border of length  $p$
- $M_{p,r}$ : GF of maps in  $\mathcal{F}$  with a border made of a monochromatic part of length  $p$ , and an alternating part of length  $2r$

$$W(x) = \sum_{p \geq 0} \frac{F_p}{x^{p+1}} \quad (8)$$

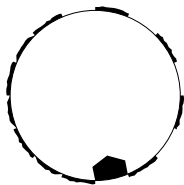
$$A(w) = \sum_{r \geq 0} A_r w^{r+1} \quad (9)$$

$$M(x, w) = \sum_{r \geq 1, p \geq 0} \frac{M_{p,r}}{x^{p+1}} w^r. \quad (10)$$

## Recursion for mixed boundary

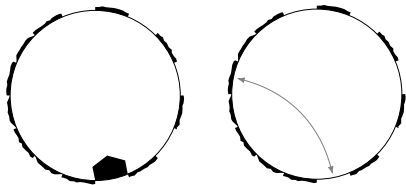


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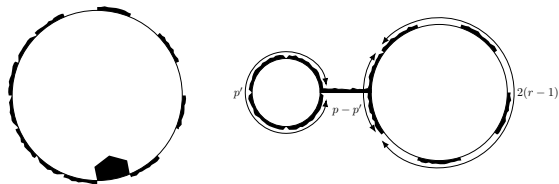
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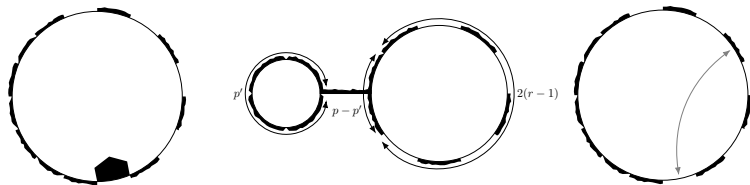
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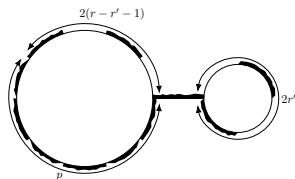
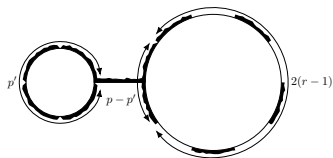
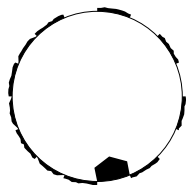
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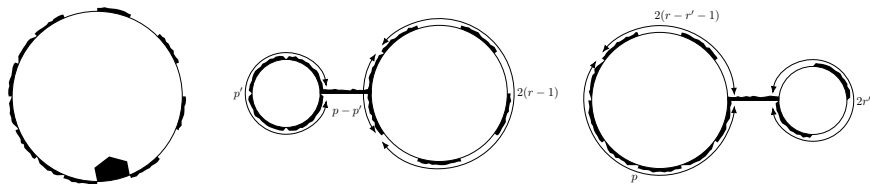


# Recursion for mixed boundary



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i.e.

$$M(x, w) = w \left( \sum_{i \geq 1} t_i \left[ x^i (M(x, w) + W(x)) \right]_{x < 0} + x W(x) (M(x, w) + W(x)) \right) + A(w) M(x, w) \quad (11)$$

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$$R(x, w) = wxW(x)Y(x) - w \sum_{i \geq 1} t_i \left[ x^i (M(x, w) + W(x)) \right]_{x \geq 0},$$

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- If we find parametrization  $\omega(x)$  such that  $K(x, \omega(x)) = 0$ , then  $R(x, \omega(x)) = 0$  as well  
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→ **system of two equations** for the two unknowns  $\omega(x), A(\omega(x))$
- In full generality: second term in  $R$  is hard to deal with, but we still have algebraicity of  $M(x, w)$  in  $w, x, W(x), t_i$  [[Bousquet-Mélou–Jehanne 2006](#)]



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## Parametrization for constellations

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By the kernel method: 
$$\begin{cases} w &= \frac{Y^2}{x^{m-2}} \\ A &= 1 - \frac{x^{m-1}}{Y}. \end{cases} \quad (12)$$
- Rational parametrization of the **spectral curve**  $(x, Y(x))$  (reformulation of a special case of the general results of [Eynard 2016]):

### Proposition (Bouttier-C., 2020+)

Let  $d$  be a positive integer and assume that  $x_i = 0$  for  $i > d$ , then, for some formal variable  $\zeta$ :

$$\begin{cases} x(\zeta) &= \zeta + \sum_{k \geq 1} \alpha_k \zeta^{km-1} \\ y(\zeta) &= \frac{V}{\zeta} + \zeta^{m-1}, \end{cases} \quad (13)$$

$$\text{where } V = t + \sum_{i=1}^d \binom{mi-1}{i} x_i V^{(m-1)i} \quad (14)$$

$$\alpha_k = \sum_{i=k}^d \binom{mi-1}{i-k} x_i V^{(m-1)i+k-1}, \quad k = 1, \dots, d. \quad (15)$$

- Combining system  $(w(x, Y), A(x, Y))$  and parametrization  $(x(\zeta), Y(\zeta))$  yields:

$$A(w(s)) = 1 - \frac{(1 + \sum_{k=1}^d \alpha_k s^k)^{m-1}}{1 + Vs}, \quad w(s) = s \frac{(1 + \sum_{k=1}^d \alpha_k s^k)^{m-2}}{(1 + Vs)^2}$$

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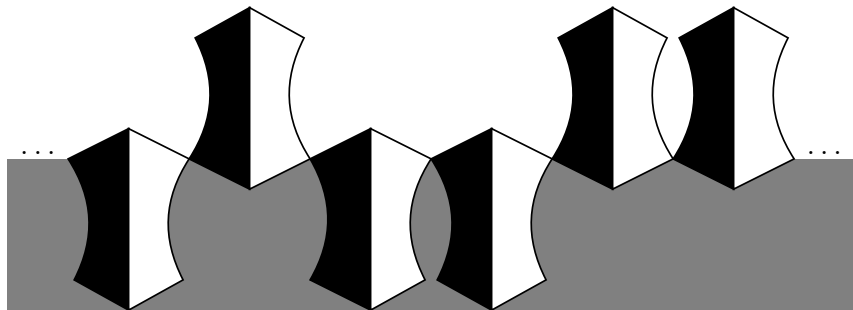
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- [Eynard 2016]: very general algebraic results for enumeration of bicolored maps with boundaries, **but** recurrence on the number of monochromatic parts of the boundary: alternating case is of maximal complexity, not really tractable



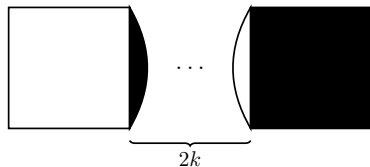
- More complex boundary conditions: eg. needed for **Boltzmann Eulerian quadrangulations**



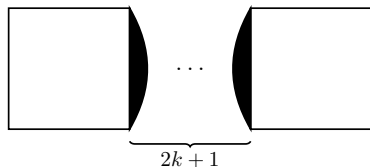


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- **Bijective** approach (*slices* [Bouttier-Guitter 2014, ...])

Thank you for your attention!