q-deformations of numbers and matrices

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based on joint work
with Valentin Ovsienko arXiv:1908.04365 , arXiv:1812.00170
with Ludivine Leclere arXiv:2101.02953

q-integers

$n>0 \quad [n]_q = 1+q+q^2+\dots+q^n = \dfrac{1-q^n}{1-q}$

$n<0 \quad [n]_q = q^{-1}+q^{-2}+\dots+q^n$

q-rationals ? [MG-O]

$\mathbb{R}\cup\{\infty\} \qquad \mathbb{R}\cup\{\infty\}$

$R : x \longmapsto x+1$

$R^n(0) = n$

$R_q^n(0) = [n]_q$

$S : x \longmapsto -\dfrac{1}{x}$

$\mathbb{Q} \cup\{\frac{1}{0}\}$ = orbit of 0 under S and R

$$0 \xrightarrow{R} 1 \xrightarrow{R} 2 \xrightarrow{S} -\frac{1}{2} \xrightarrow{R} \frac{1}{2} \xrightarrow{S} -2$$

$$0 \to 1 \to 1+q \to -\frac{1}{q+q^2} \to \frac{q}{1+q} \to -q^{-1}+q^{-2}$$

q-deformation

$R_q : X \longmapsto qX+1$

$S_q : X \longmapsto -\dfrac{1}{qX}$

$\mathbb{R}(q) \to \mathbb{R}(q)$

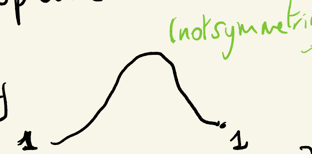
Defn: $q\text{-}\mathbb{Q} :=$ orbit of 0 under S_q and R_q

$\Rightarrow$ well defined map $\mathbb{Q} \longrightarrow \mathbb{Q}(q)$, $\dfrac{a}{b} \mapsto \left[\dfrac{a}{b}\right]_q$

Properties $\left[\dfrac{a}{b}\right]_q = \pm q^{\pm N} \dfrac{A(q)}{B(q)}$ $\quad A, B \in \mathbb{Z}[q]$ coprime

- positive coefficients
- monic and unitary

Conjecture: unimodality



[proved in particular cases by McConville Sagan Smyth]

Thm "stabilization" [MGO]

$x \notin \mathbb{Q} \quad x_n \in \mathbb{Q} \quad x_n \longrightarrow x$

$[x_n]_q = \sum_k \alpha_{k,n} q^k$

1) $\forall k \quad (\alpha_{k,n})_n$ converges

2) The limit $\alpha_k := \lim \alpha_{k,n}$ does not depend on (x_n) only on x !

$\Longrightarrow$ Defn: $[x]_q = \sum_k \alpha_k q^k$ $\quad$ q-reals !

Matrix approach:

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

generators of $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\pm Id$

(relation $S^2 = -Id$, $(RS)^3 = -Id$)

Möbius transf.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$$

q-deformation [MGO]-[LMG]

$$R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \quad S_q = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}$$

$$\in PGL_2(\mathbb{Z}[q,q^{-1}])$$

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, AD - BC = \pm q^{\pm N} \right\} / \pm q^{\pm N} Id$$

Well defined map

$$[\ ]_q : PSL_2(\mathbb{Z}) \longrightarrow PSLq(2,\mathbb{Z})$$

$$M \qquad [M]_q$$

$$(R, S \longmapsto R_q, S_q)$$

$$\langle R_q, S_q \rangle_{subgp} \simeq PSL_2(\mathbb{Z})$$

Thm [LMG]

The $PSL_2(\mathbb{Z})$-actions commute:

$$[M \cdot x]_q = [M]_q \cdot [x]_q \quad \forall M \in PSL_2(\mathbb{Z}) \quad \forall x \in \mathbb{R}.$$

Cor: q-quadratic irrationals are quadratic!

$$\frac{a+\sqrt{b}}{c} \overset{q}{\rightsquigarrow} \frac{A+\sqrt{B}}{C} \qquad A, B, C \in \mathbb{Z}(q)$$

Thm [LMG]

Traces of $[M]_q$ are palindromic polynomials with positive coefficients! (up to $\pm q^{\pm N}$)

Stabilization

$\frac{12}{5} = 2.4$ $\qquad$ $\left[\frac{12}{5}\right] = \frac{1+2q+3q^2+3q^3+2q^4+q^5}{1+q+2q^2+q^3}$

$$= 1+q+q^4-2q^6+q^7+3q^8-3q^9-4q^{10}+7q^{11}+4q^{12}\dots$$

$\frac{70}{29} = 2.4137\dots$ $\qquad$ $\left[\frac{70}{29}\right] = \frac{1+3q+7q^2+11q^3+13q^4+13q^5+11q^6+7q^7+3q^8+q^9}{1+2q+5q^2+6q^3+6q^4+5q^5+3q^6+q^7}$

$$= 1+q+q^4-2q^6+q^7+4q^8-5q^9-7q^{10}+18q^{11}+6q^{12}\dots$$

$\frac{408}{169} = 2.41420\dots$ $\qquad$ $\left[\frac{408}{169}\right] = \frac{1+4q+12q^2+25q^3+41q^4+56q^5+65q^6+\dots+4q^{12}+q^{13}}{1+3q+9q^2+16q^3+24q^4+29q^5+29q^6+\dots+4q^{10}+q^{11}}$

$$= 1+q+q^4-2q^6+q^7+4q^8-5q^9-7q^{10}+18q^{11}+7q^{12}\dots$$

$1+\sqrt{2}$ $\qquad$ série limite

Stabilization

$$\frac{12}{5} = 2.4 \qquad \left[\frac{12}{5}\right]_q = \frac{1+2q+3q^2+3q^3+2q^4+q^5}{1+q+2q^2+q^3}$$

$$= 1 + q + q^4 - 2q^6 + q^7 + 3q^8 - 3q^9 - 4q^{10} + 7q^{11} + 4q^{12} \dots$$

$$\frac{70}{29} = 2.4137\dots \qquad \left[\frac{70}{29}\right]_q = \frac{1+3q+7q^2+11q^3+13q^4+13q^5+11q^6+7q^7+3q^8+q^9}{1+2q+5q^2+6q^3+6q^4+5q^5+3q^6+q^7}$$

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$$= 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} \dots$$

$\downarrow$ pt fixe $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ $\qquad\qquad \downarrow$

$$1+\sqrt{2} \qquad [1+\sqrt{2}]_q = \frac{q^3+2q-1+\sqrt{q^6+4q^4-2q^3+4q^2+1}}{2q}$$

$$=2.41421\dots \qquad = 1 + q + q^4 - 2q^6 + q^7 + 4q^8 - 5q^9 - 7q^{10} + 18q^{11} + 7q^{12} \dots$$

Examples of q-deformed matrices

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix} \qquad \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & -5 \\ 3 & -2 \end{pmatrix}_q = \begin{pmatrix} 1+2q+2q^2+q^3+q^4 & -q-2q^2-q^3-q^4) \\ 1+q+q^2 & -q-q^2 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 5 \\ 7 & 3 \end{pmatrix}_q = \begin{pmatrix} q+2q^2+3q^3+3q^4+2q^5+q^6 & 1+q+2q^2+q^3 \\ q+2q^2+2q^3+q^4+q^5 & 1+q+q^2 \end{pmatrix}$$

$$\begin{pmatrix} 31 & 13 \\ 19 & 8 \end{pmatrix}_q = \begin{pmatrix} q+3q^2+5q^3+6q^4+7q^5+5q^6+3q^7+q^8 & 1+2q+3q^2+3q^3+3q^4+q^5 \\ q+3q^2+4q^3+4q^4+4q^5+2q^6+q^7 & 1+2q+2q^2+2q^3+q^4 \end{pmatrix}$$

Examples of q-deformed matrices

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix}$$
$$\mathrm{Tr} = 1+q+q^2,$$

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}$$
$$\mathrm{Tr} = (1+q+q^2)(1+q^2)$$

$$\begin{pmatrix} 7 & -5 \\ 3 & -2 \end{pmatrix}_q = \begin{pmatrix} 1+2q+2q^2+q^3+q^4 & -q-2q^2-q^3-q^4) \\ 1+q+q^2 & -q-q^2 \end{pmatrix}$$
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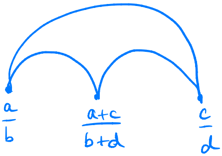
$$\begin{pmatrix} 12 & 5 \\ 7 & 3 \end{pmatrix}_q = \begin{pmatrix} q+2q^2+3q^3+3q^4+2q^5+q^6 & 1+q+2q^2+q^3 \\ q+2q^2+2q^3+q^4+q^5 & 1+q+q^2 \end{pmatrix}$$
$$\mathrm{Tr} = (1+q+q^2)(1+q+q^2+q^3+q^4)$$

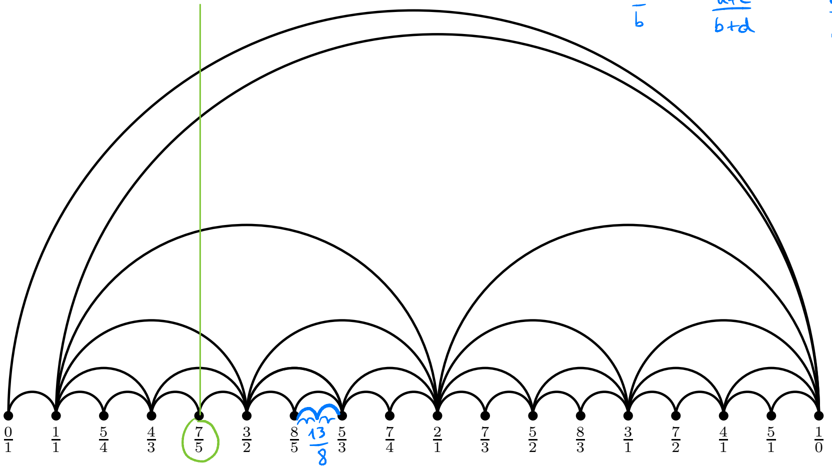
$$\begin{pmatrix} 31 & 13 \\ 19 & 8 \end{pmatrix}_q =$$
$$\begin{pmatrix} q+3q^2+5q^3+6q^4+7q^5+5q^6+3q^7+q^8 & 1+2q+3q^2+3q^3+3q^4+q^5 \\ q+3q^2+4q^3+4q^4+4q^5+2q^6+q^7 & 1+2q+2q^2+2q^3+q^4 \end{pmatrix}$$
$$\mathrm{Tr} = (1+q+q^2)(1+2q+2q^2+3q^3+2q^4+2q^5+q^6)$$

Farey graph

Vertices: $\mathbb{Q} \cup \{\frac{1}{0}\}$

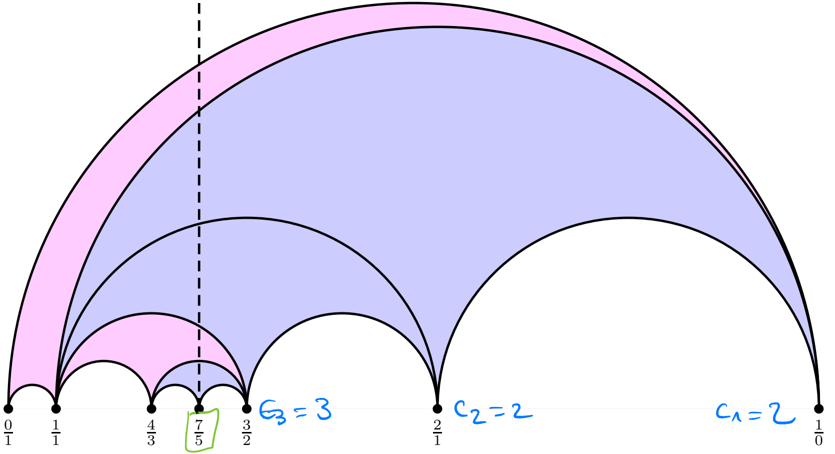
edges: $\frac{a}{b} - \frac{c}{d} \iff ad - bc = \pm 1$





Farey graph

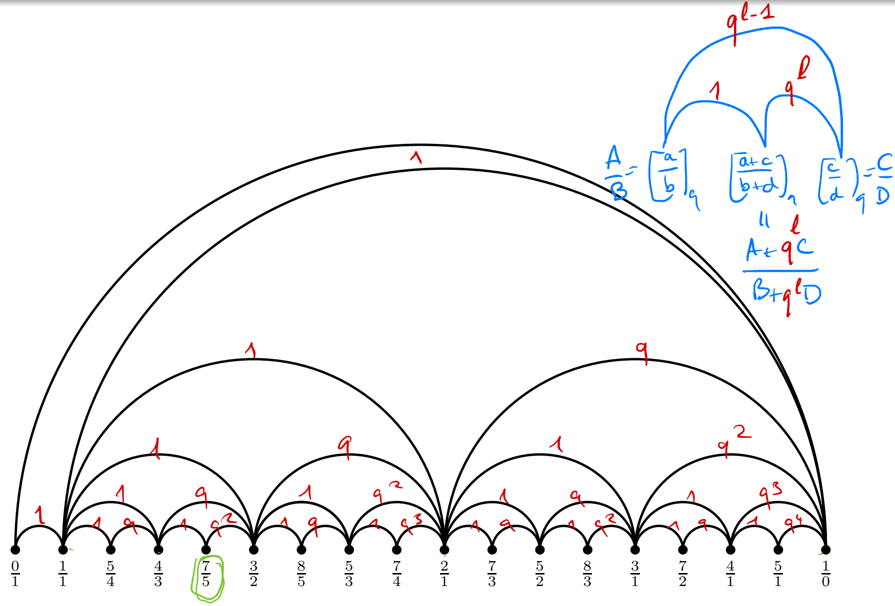
c_i = # triangles incident to vertex i



$$\frac{7}{5} = R^{c_1} S R^{c_2} S R^{c_3}(0)$$

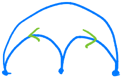
$$0 \xrightarrow{R^3} 3 \xrightarrow{S} -\frac{1}{3} \xrightarrow{R^2} \frac{5}{3} \xrightarrow{S} -\frac{3}{5} \xrightarrow{R^2} \frac{7}{5}$$

q-deformed Farey graph



q-deformed Farey graph

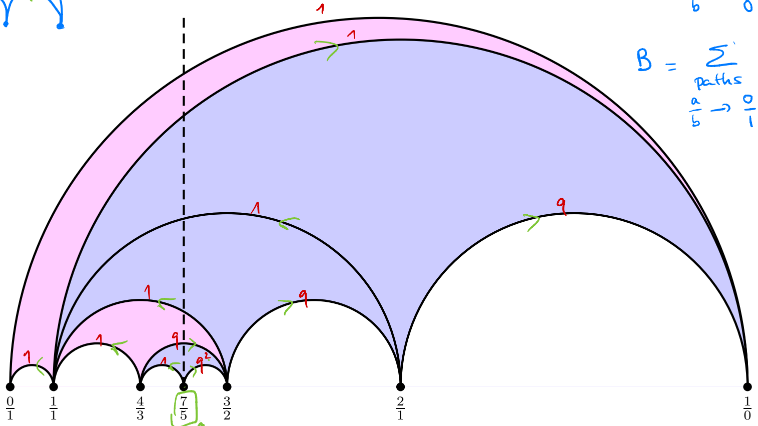
orientation



$$\left[\frac{a}{b}\right]_q = \frac{A}{B}$$

$$A = \sum_{\substack{\text{paths} \\ \frac{a}{b} \to \frac{1}{0}}} \mathrm{wt}(\text{path})$$

$$B = \sum_{\substack{\text{paths} \\ \frac{a}{b} \to \frac{0}{1}}} \mathrm{wt}(\text{path})$$



$$\boxed{\tfrac{7}{5}}_q = \frac{1+q+2q^2+2q^3+q^4}{1+q+2q^2+q^3}$$