Surreal numbers with exponential function and omega-exponentiation.

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joint work with A. Berarducci, S. Kuhlmann and V. Mantova

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On games...

How to win a

Combinatorial partial games?

Example

Blue–Red Hackenbush

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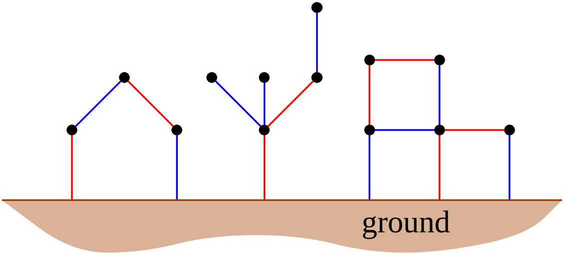
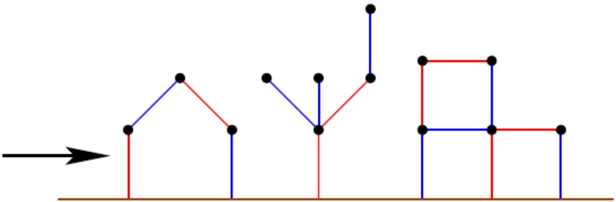
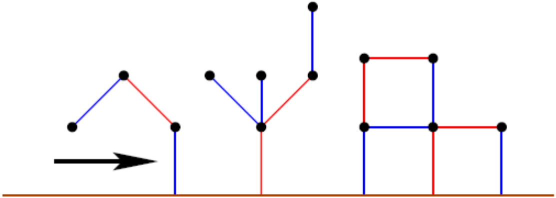


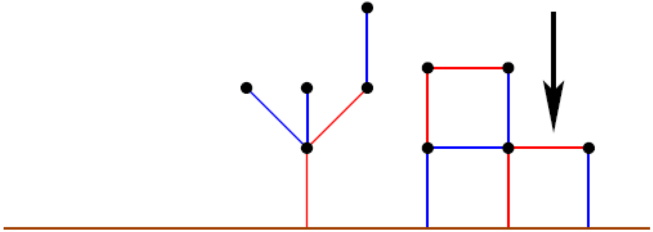
Red starts



Then Blue plays



Then Red



ETC...

On games... and numbers

J.H. Conway (1976):

Any GAME has a NUMBER (not a nimber!):

- $n(G) < 0 \Leftrightarrow$ Blue has a winning way;
- $n(G) > 0 \Leftrightarrow$ Red has a winning way;
- $n(G) = 0 \Leftrightarrow$ the first to play loses.

Examples.

$n(|) = 1$ $\qquad$ $n(|) = -1$,

$n(\text{empty game}) = 0 = n(||)$, $\qquad$ $n\left(\begin{array}{c}|\\|\end{array}\right) = 1/2$

Any NUMBER is an EQUIVALENCE CLASS OF GAMES

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Examples.

$n(\textcolor{red}{|}) = 1$ $\qquad n(\textcolor{blue}{|}) = -1,$

$n(\text{empty game}) = 0 = n(\textcolor{red}{|}\textcolor{blue}{|}),$ $\qquad n\left(\begin{smallmatrix}\textcolor{blue}{|}\\ \textcolor{red}{|}\end{smallmatrix}\right) = 1/2$

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Let there be... numbers!

RULE 1

$$x := \{X_G \mid X_D\}$$

where $x^D \not\leq x^G$ for any $x^G \in X_G$ and $x^D \in X_D$.

Examples:

$$0 := \{\emptyset \mid \emptyset\}, \quad 1 := \{0 \mid \emptyset\}, \quad 1/2 := \{0 \mid 1\}, \quad \text{etc...}$$

RULE 2 Given $x = \{X_G \mid X_D\}$ and $y = \{Y_G \mid Y_D\}$,

$$x \leq y \quad :\Longleftrightarrow \quad y \not\leq x^G \text{ and } y^D \not\leq x$$

for any $x^G \in X_G$ and $y^D \in Y_D$.

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Old and young numbers.

Day 0 $\{\emptyset \mid \emptyset\}$

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Day 0	0

Old and young numbers.

Day 0	0
Day 1	$\{\emptyset \mid 0\} < \{\emptyset \mid \emptyset\} < \{0 \mid \emptyset\}$

Old and young numbers.

Day 0	0
Day 1	$-1 < 0 < 1$

Old and young numbers.

Day 0 $\quad 0$

Day 1 $\quad -1 < 0 < 1$

Day 2 $\{\emptyset \mid -1\} = \{\emptyset \mid -1, 0\} = \{\emptyset \mid -1, 0, 1\} = \{\emptyset \mid -1, 1\} <$

$$\{\emptyset \mid 0\} = \{\emptyset \mid 0, 1\} < \{-1 \mid 0\} = \{-1 \mid 0, 1\} <$$

$$\{\emptyset \mid \emptyset\} = \{-1 \mid 1\} = \{-1 \mid \emptyset\} = \{\emptyset \mid 1\} <$$

$$\{0 \mid 1\} = \{-1, 0 \mid 1\} < \cdots$$

$$\ldots < \{1 \mid \emptyset\}$$

Old and young numbers.

Day 0	0
Day 1	$-1 < 0 < 1$
Day 2	$-2 < -1 < -\frac{1}{2} < 0 < \frac{1}{2} < 1 < 2$

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$\vdots$	$\vdots$
Day $\alpha \in$ **On**	A surreal number of birthdate α
$\vdots$	$\vdots$

Simple and complicated numbers.

H. Gonshor's alternative definition of surreal numbers (1986):

$$\begin{array}{rcl} a \in \mathbf{No} & :\Leftrightarrow & a : \alpha \to \{\ominus, \oplus\} \text{ pour un certain } \alpha \in \mathbf{On} \\ & \Leftrightarrow & a = \underbrace{\oplus \oplus \ominus \oplus \ominus \ominus \cdots}_{\textbf{length } \alpha} \end{array}$$

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Examples:
0= empty sequence, $1 = \oplus$, $-1 = \ominus$, $1/2 = \oplus\ominus$,...

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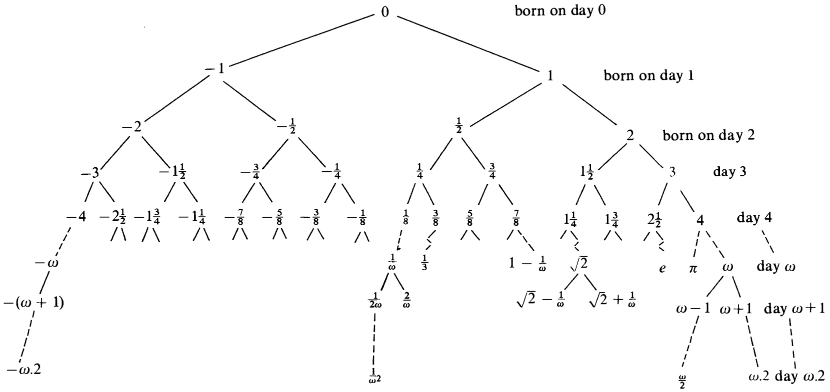
$\rightarrow$ **Total ordering**: lexicographical based on

$$\ominus \;<\; \text{no symbol} \;<\; \oplus$$

$\rightarrow$ **Partial ordering *simplicity***:

$$a \leq_s b :\Leftrightarrow a \text{ is an initial subsequence of } b.$$

Full rooted binary tree representation.



Algebraic structure.

- **Addition:**

$$a + b := \langle a^L + b,\ a + b^L \mid a^R + b,\ a + b^R \rangle$$

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inverse element:

$$-a := \langle -a^R \mid -a^L \rangle$$

neutral element:

$$0 = \langle \emptyset \mid \emptyset \rangle$$

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- **Multiplication:**

$$\begin{aligned} a \cdot b :=& \\ &\langle a^L \cdot b + a \cdot b^L - a^L \cdot b^L, \quad a^R \cdot b + a \cdot b^R - a^R \cdot b^R \mid \\ &\qquad a^L \cdot b + a \cdot b^R - a^L \cdot b^R, \qquad a^R \cdot b + a \cdot b^L - a^R \cdot b^L \rangle \end{aligned}$$

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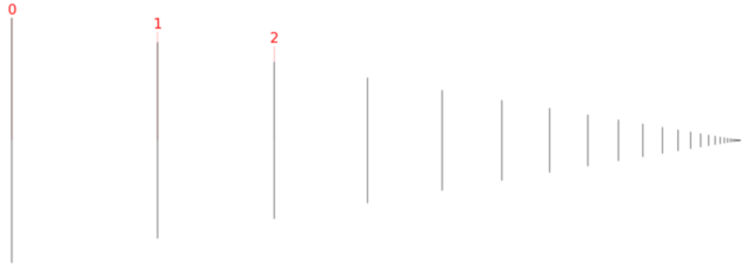
Inverse element..., neutral element = 1

From real to surreal.

On **DAY** ω, the first infinite ordinal:

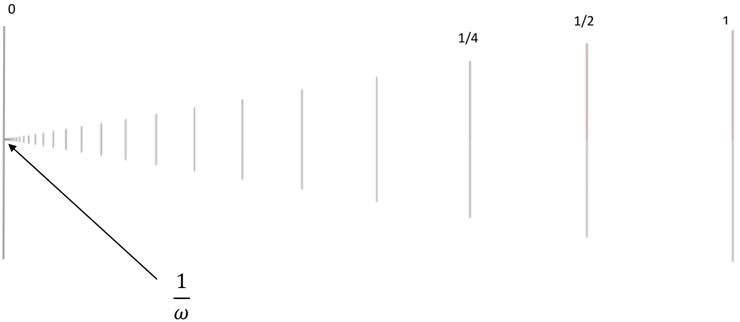
$$\omega := \{1, 2, 3, \ldots \mid \emptyset\}$$



From real to surreal.

has now an *infinitesimal inverse*:

$$\frac{1}{\omega} := \left\{ 0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \right\}$$



A fundamental result.

Theorem (Conway-Kruskal)

No *is a real closed Field which canonically contains the field* $\mathbb{R}$ *and the Semiring* **On**.

Conway's ω-map.

- The **natural valuation** on **No**:

$$\begin{array}{rccl} \mathrm{val}: & (\mathrm{NO}, \cdot, \leq) & \to & (\mathrm{NO}_{/\sim_+} \cup \{\infty\}, +, \leq) \\ & a & \mapsto & [a]_+ \end{array}$$

via the Archimedean equivalence relation $\sim_+$:

$$a \sim_+ b \Leftrightarrow \exists n,\ \frac{1}{n}|a| \leq |b| \leq n|a|$$

Conway's ω-map.

- **Conway's ω-map** Ω = *canonical* section of the natural valuation:

$$\begin{array}{cccc} \Omega & (\mathrm{NO},+,\leq) & \hookrightarrow & (\mathrm{NO}_{>0},\cdot,\leq) \\ & a & \mapsto & \omega^a := \langle 0,\ n\omega^{a^L} \mid \omega^{a^R}/2^n \rangle \end{array}$$

such that $\mathrm{val} \circ \Omega = \mathrm{Id}$.

$\omega^{\mathbf{No}}$ is the group of **monomials** and $\mathbb{R} \simeq$ residue field

Conway's ω-map.

Examples:

- $\omega^0 = \{0 \mid \emptyset\} = 1$
- $\omega^1 = \{0, 2^n \mid \emptyset\} = \omega$
- $\omega^{-1} = \{0 \mid 1/2^n\} = \dfrac{1}{\omega}$

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Theorem (Conway)

The ω-map is an exponentiation which extends ordinal exponentiation. It is a canonical section of val, so in particular:

$$\mathbf{No} \simeq \mathrm{val}\left(\mathbf{No} \setminus \{0\}\right).$$

Consequence of Kaplansky's embedding theorem.

Surreal numbers as **generalized power series**:

$\forall a \in \mathbf{No}$,

$$a = \sum_{i<\lambda} r_i\, \omega^{a_i}$$

where $\lambda \in \mathbf{On}$, $(a_i)_{i<\lambda}$ is a decreasing sequence in **No** and $(r_i)_{i<\lambda}$ is a sequence in $\mathbb{R} \setminus \{0\}$.

Results of Alling and Kruskal - Gonshor.

Alling (1987): **No** carries **restricted analytic functions**: for any $a = a_0 + \varepsilon$,

$$f(a) = f(a_0 + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{f^{(n)}(a_0)}{n!} \varepsilon^n$$

Results of Alling and Kruskal - Gonshor.

Alling (1987): In particular, exp for *bounded* surreal numbers:

$$\exp(a) = \exp(a_0 + \varepsilon) = \sum_{n \in \mathbb{N}} \frac{e^{a_0}}{n!} \varepsilon^n$$

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Gonshor (1986): **No** carries an **exponential** and a **logarithm** $\log = \exp^{-1}$:

$$\exp : (\mathbf{NO}, +, \leq) \to (\mathbf{NO}_{>0}, ., \leq)$$

Gonshor's exp.

- Global definition: $\forall a \in \mathbf{No},\ \exp(a) :=$

$$\left\langle 0,\ \exp(a^L)E_n(a-a^L),\ \exp(a^R)E_{2n+1}(a-a^R) \mid \frac{\exp(a^R)}{E_n(a^R-a)},\ \frac{\exp(a^L)}{E_{2n+1}(a^L-a)} \right\rangle$$

- Examples:

$\exp(0) = 1$, $\exp(1) = e$ and $\exp(r) = e^r$

$$\exp(a_0+\varepsilon) = \sum_{n\in\mathbb{N}} \frac{e^{a_0}}{n!}\varepsilon^n$$

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Exponential and ω-map.

- For any surreal number a:

$$\log\left(\omega^{\omega^{a}}\right)=\omega^{h(a)}$$

where

$$\begin{array}{rccl} h: & \mathbf{No} & \simeq & \mathbf{No}_{>0} \\ & a & \mapsto & h(a):=\left\langle 0,\ h(a^L)\mid h(a^R),\ \omega^a/2^n\right\rangle \end{array}$$

Universality!

Theorem (Conway 76, Ehrlich 2001)

No *is "the" universal domain for:*

- *linear orderings;*
- *ordered Abelian groups;*
- *real fields;*

$$\mathbf{No} \succcurlyeq \mathbb{R}(\textit{real closed fields}) \quad \text{and} \quad \mathbf{No} \succcurlyeq \mathbf{On}\ (\textit{semirings})$$

Universality!

Theorem (Alling 87, Gonshor 86, van den Dries-Ehrlich 2001)

No *is "the" universal domain for* ***real analytic and real exponential fields***

$$\mathbf{No} \succcurlyeq \mathbb{R}_{\mathrm{an,exp}}$$

Surreal derivation and transseries.

Surreal derivation: a derivation d such that

- $\ker d = \mathbb{R} \simeq$ residue field of the natural valuation;
- $a > \mathbb{R} \Rightarrow a' > 0$
- $\Rightarrow$ strong l'Hospital rule;
- rule for the logarithmic derivative;
- strong linearity;
- strong Leibniz rule;
- compatibility : $d(\exp(a)) = \exp(a) \cdot d(a)$

Transseries field:

Surreal derivation and transseries.

Surreal derivation:

Transseries field: field $(\mathbb{T} = \mathbb{R}((\mathfrak{M})), \log)$ such that:

(T1) the domain of $\log$ consists of the positive series;

(T2) $\log(\mathfrak{M}) \subseteq \mathbb{R}((\mathfrak{M}_{\succ 1}))$;

(T3) $\log(1+\varepsilon) = \sum_{n\geq 1}(-1)^{n+1}\frac{\varepsilon^n}{n}$ for any $\varepsilon \in \mathbb{R}((\mathfrak{M}_{\preccurlyeq 1}))$;

(T4) let $(\mathfrak{m}_n)_{n\in\mathbb{N}}$ be a sequence of monomials with $\mathfrak{m}_{n+1}$ in the support of $\log(\mathfrak{m}_n)$ for all n. There is an integer n_0 such that for any $n \geq n_0$, for any $\mathfrak{m}$ in the support of $\log(\mathfrak{m}_n)$, one has that $\mathfrak{m} \succcurlyeq \mathfrak{m}_{n+1}$ and the coefficient of $\mathfrak{m}_{n+1}$ in $\log(\mathfrak{m}_n)$ is ± 1.

Results of Berarducci - Mantova.

Theorem (Berarducci - Mantova 2018)

No *carries a surreal derivation* d_{BM} *which is "the simplest".*

HOW? By proving that **No** is a **field of transseries**

(in the sense of J.van der Hoeven–M. Schmeling, and also identifying the so-called log-atomic elements based on S.Kuhlmann–M.M, etc...)

Universality continued!

Theorem (Aschenbrenner-van den Dries-van der Hoeven 2019)

No *equipped with the derivation* δ_{BM} *is "the" universal domain for* ***H-fields****.*

Examples:

Hardy fields: **No** $\succcurlyeq \mathcal{H}$.

Transseries, log-exp series, exp-log series: **No** $\succcurlyeq \mathbb{T}$.

BUT...

Theorem (Berarducci - Mantova 2019)

There is a (partial) composition on **No** *by the subfield of ω-series, in particular by classical transseries / LE-series / EL-series:*

$$\circ : \mathbb{R}\langle\langle\omega\rangle\rangle \times \mathbf{No}^{>\mathbb{R}} \to \mathbf{No}$$

BUT δ_{BM} is not compatible with it!

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BUT δ_{BM} is not compatible with it!

Omega-fields.

Problem: to find a compatible derivation on **No**.

$\rightarrow$ **Omega-field** (Berarducci-Kuhlmann-Mantova-M.): a real-closed field which is isomorphic to the value group of its natural valuation:

$$\mathbb{K} \simeq \mathrm{val}(\mathbb{K} \setminus \{0\})$$

Some new results...

Theorem (Berarducci - Kuhlmann - Mantova - M. 2020)

Every omega-field of the form $\mathbb{K} = \mathbb{R}((G))_\kappa$ admits an analytic log (and therefore an exp) determined by the omega-map and by any isomorphism of ordered sets:

$$h : \mathbb{K} \simeq \mathbb{K}_{>0}$$

Depending on the choice of h, $\mathbb{K}$ is either a model of $T_{\mathrm{an,exp}}$, or not even o-minimal.

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Thank you for your attention!