Discrete planes and substitutions
The Penrose rhombus tiling

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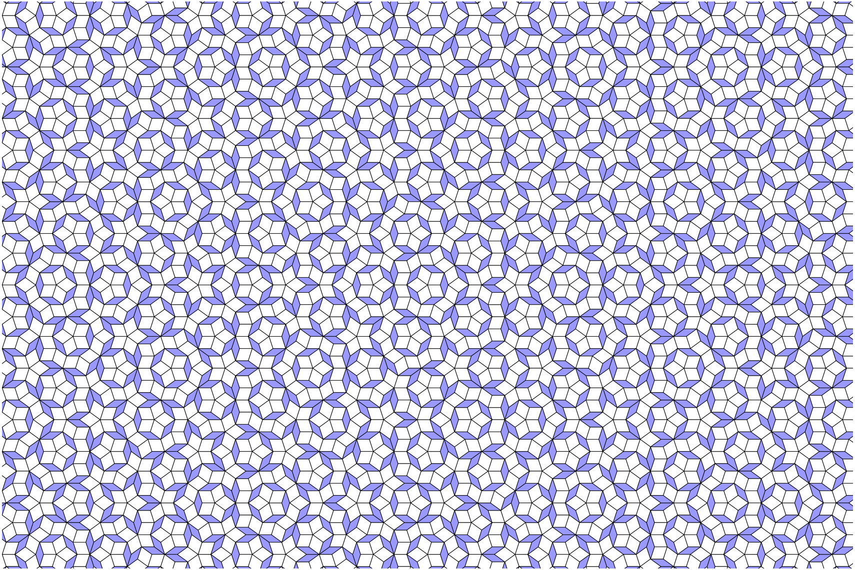
with Thomas Fernique

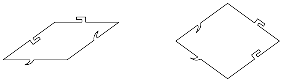
2021

https://lipn.univ-paris13.fr/~lutfalla/documents/
arXiv:2004.10128
arXiv:2010.01879

Penrose: tiling



Penrose: local rules



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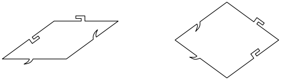
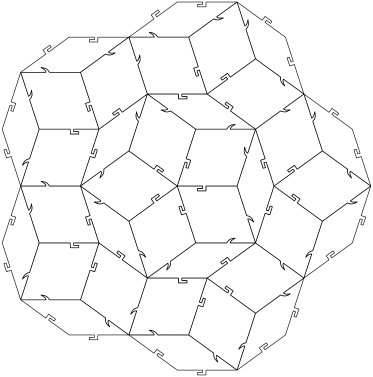
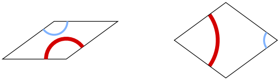
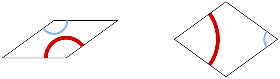


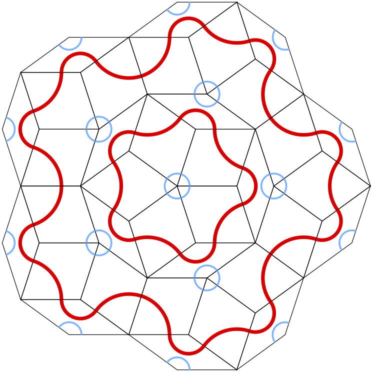


Penrose: local rules 2

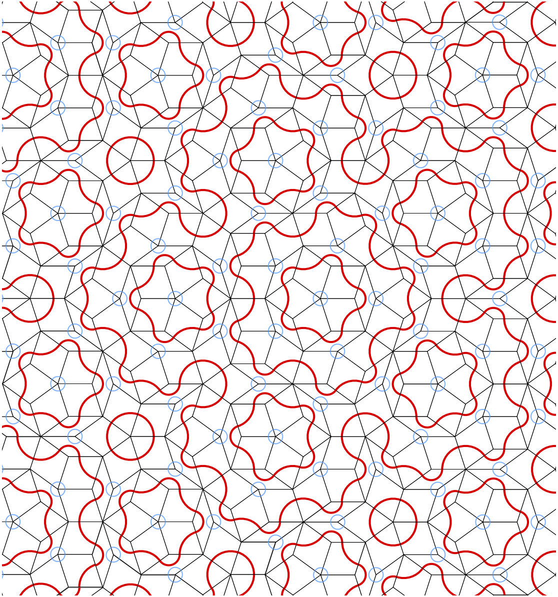


Penrose: local rules 2





Penrose: local rules 3



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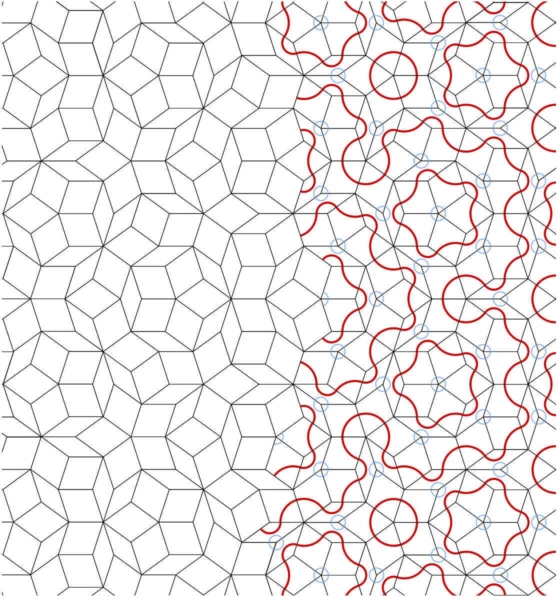

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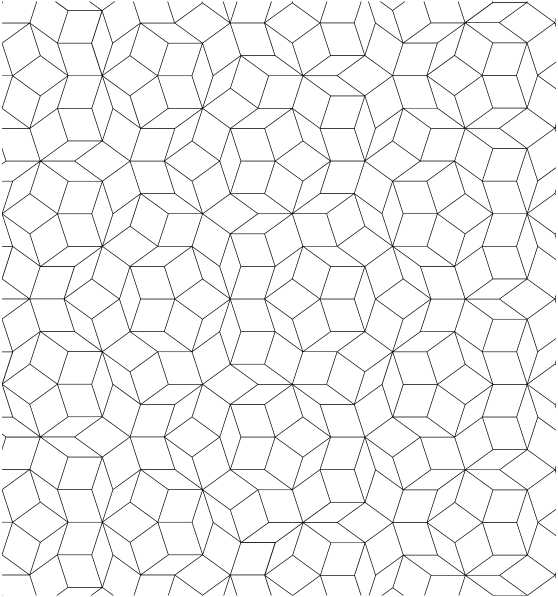
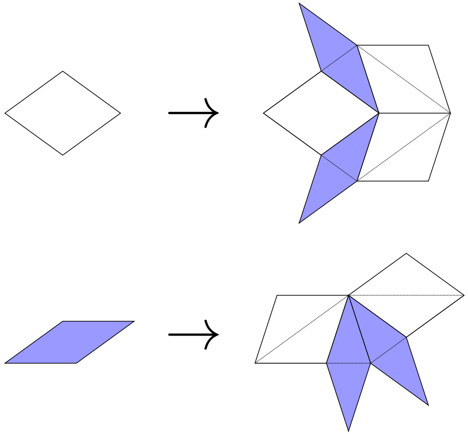
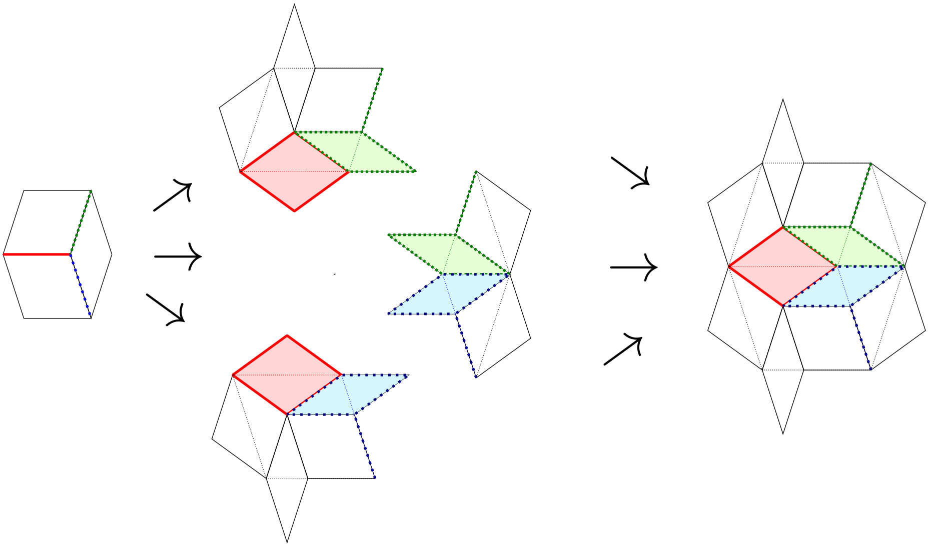


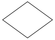
Penrose: substitution

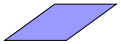


Penrose: substitution on patches



Penrose: substitution metatiles





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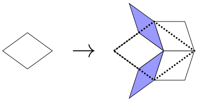

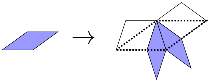

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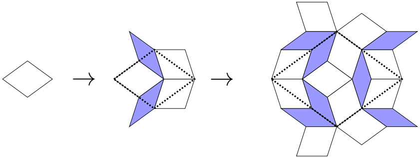

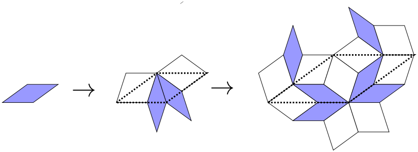

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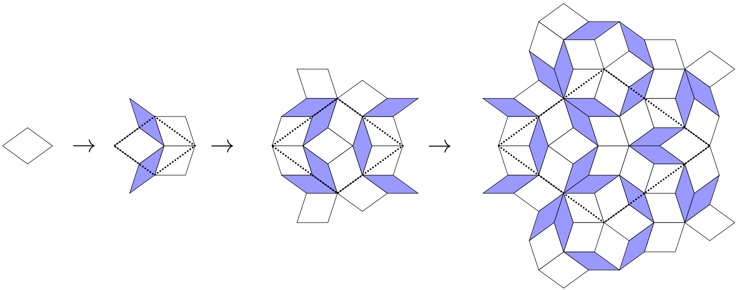
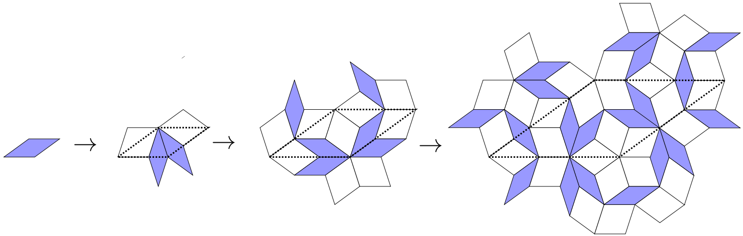
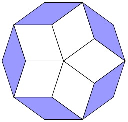




Penrose: substitution seed



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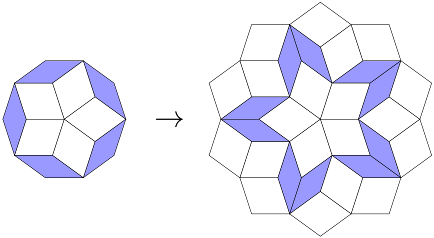

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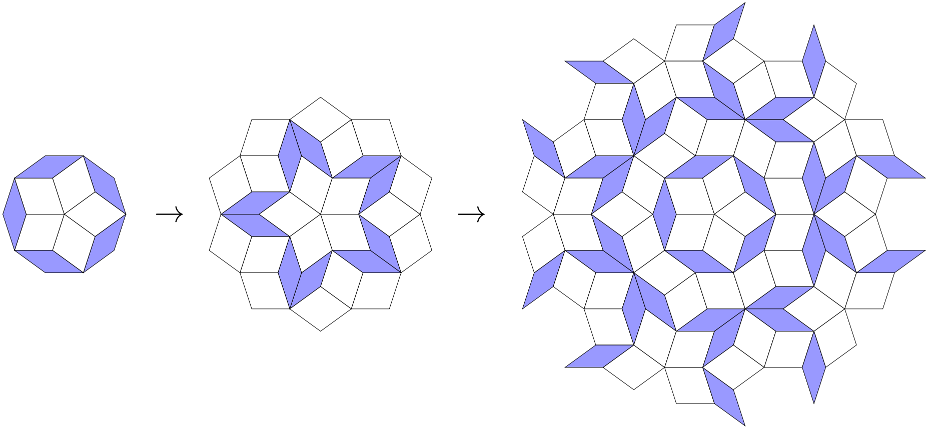
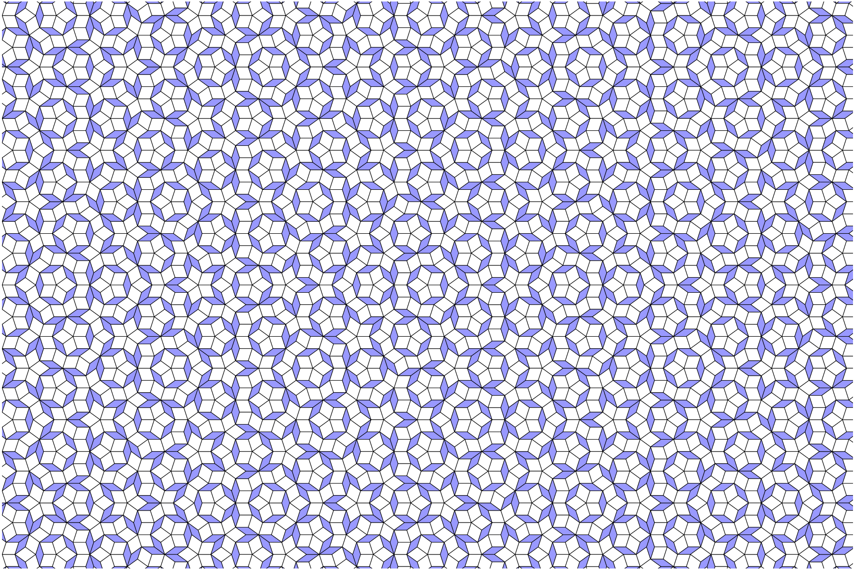
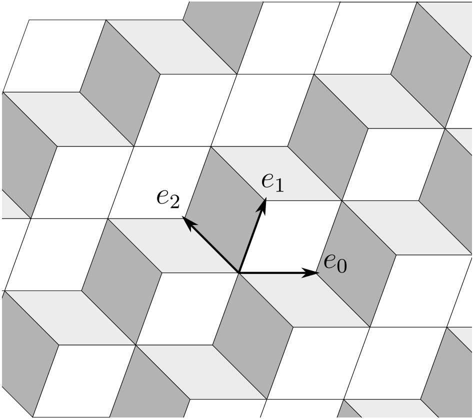
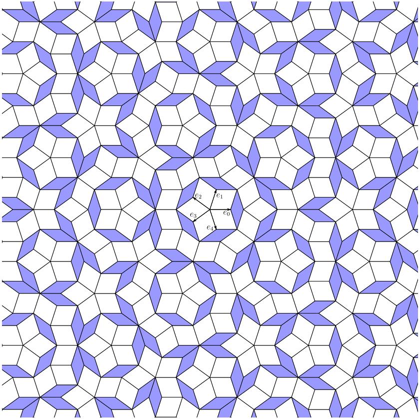


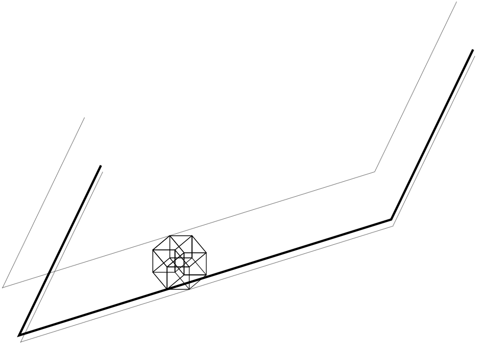


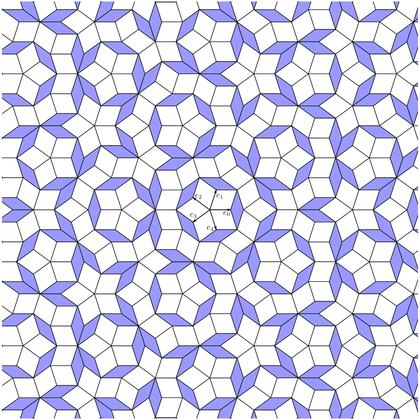
Penrose: lifting to $\mathbb{R}^5$





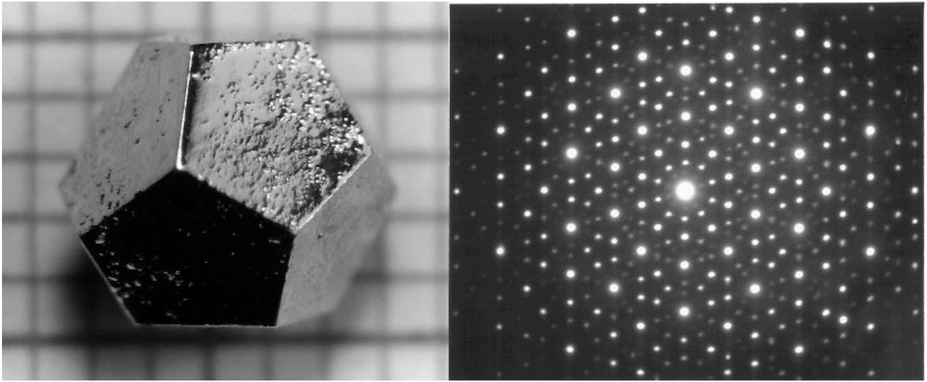
Penrose: planarity



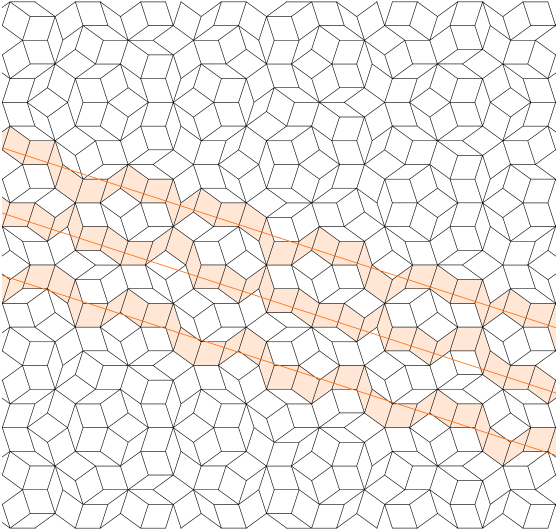


Slope of the Penrose tiling $\mathcal{E}_5 = \left\langle \begin{pmatrix} 1 \\ \cos\frac{2\pi}{5} \\ \cos\frac{4\pi}{5} \\ \cos\frac{6\pi}{5} \\ \cos\frac{8\pi}{5} \end{pmatrix}, \begin{pmatrix} 0 \\ \sin\frac{2\pi}{5} \\ \sin\frac{4\pi}{5} \\ \sin\frac{6\pi}{5} \\ \sin\frac{8\pi}{5} \end{pmatrix} \right\rangle = \left\langle \left(e^{\mathrm{i}\,\frac{2k\pi}{5}} \right)_{0 \leqslant k < 5} \right\rangle$

Planarity, long-range order and quasicrystals



Penrose: dual of a tiling



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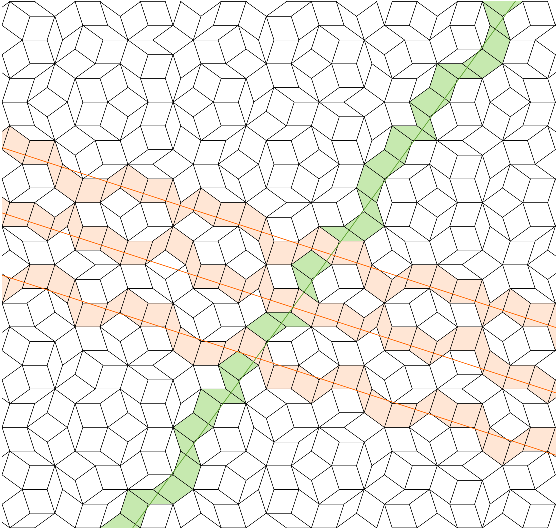
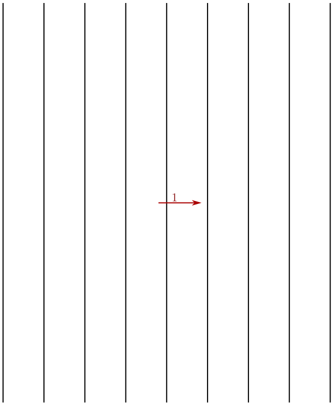


Penrose: multigrid 2



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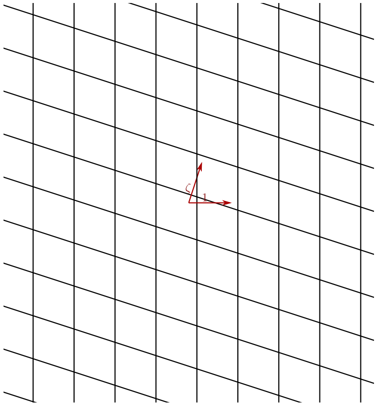

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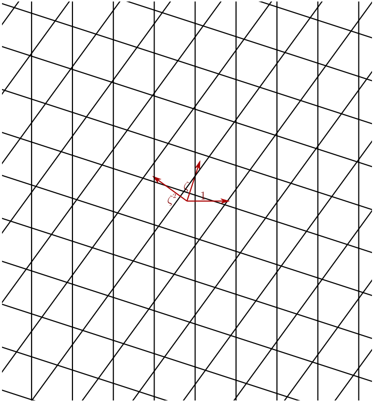

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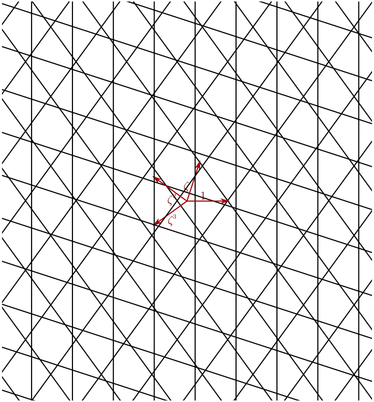

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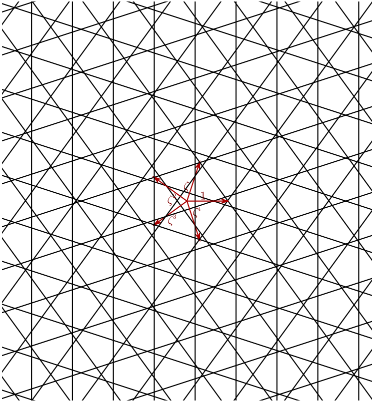

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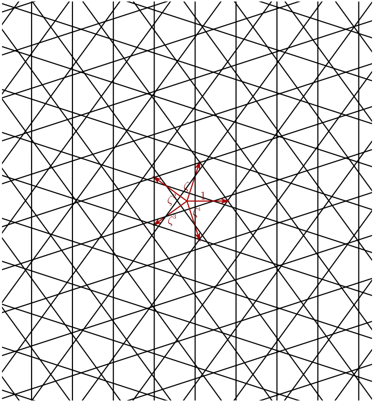
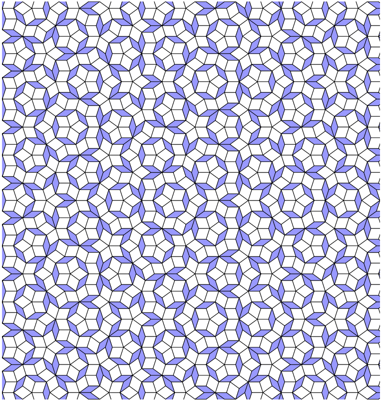
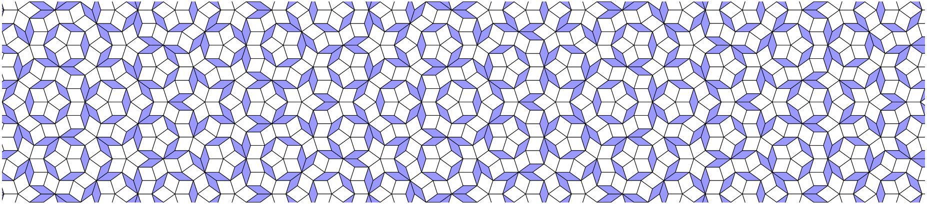




Penrose: overview



- Quasiperiodic with local 10-fold rotationnal symmetry
- Local rules
- Substitution tiling
- Planar with 5-fold slope $\mathcal{E}_5 = \left\langle \left(e^{\mathrm{i}\frac{2k\pi}{5}} \right)_{0 \leqslant k < 5} \right\rangle$
- Dual of a Pentagrid

Problem (Slope)

Let $\mathcal{E}$ be a 2D real plane of $\mathbb{R}^n$.
Does there exist some substitution discrete plane of slope $\mathcal{E}$.

n-fold planes are slopes of substitution discrete planes

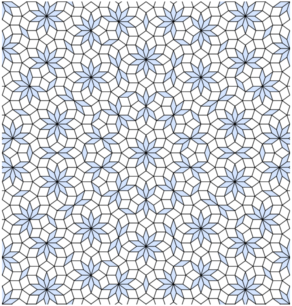
arXiv:2010.01879

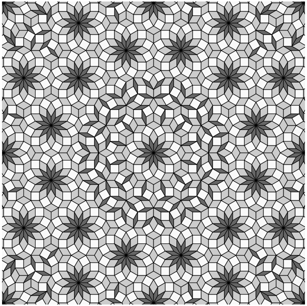
Let us define the *n*-fold plane as

$$\mathcal{E}_n := \left\langle \left(e^{\mathrm{i}\frac{2k\pi}{n}} \right)_{0 \leqslant k < n} \right\rangle = \left\langle \left(\cos \frac{2k\pi}{n} \right)_{0 \leqslant k < n}, \left(\sin \frac{2k\pi}{n} \right)_{0 \leqslant k < n} \right\rangle$$

Theorem (Kari,L.20)

$\mathcal{E}_n$ is the slope of a substitution discrete plane.





Motivation and strategy

Motivation: geometrical tilings and quasicrystals (long range order and rotational symmetry)

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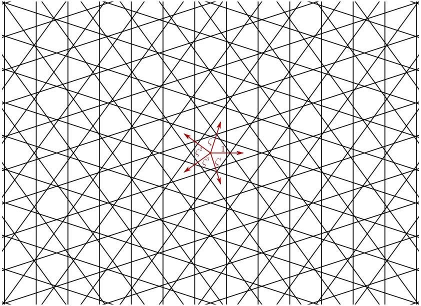
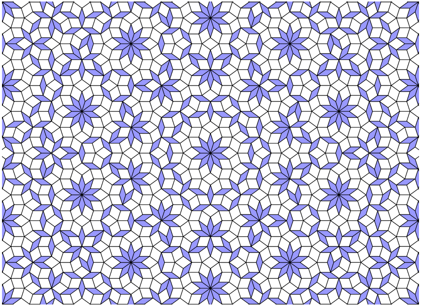
- Start from a discrete plane tiling with $\mathcal{E}_n$ slope and $2n$ fold rotational symmetry $P_n(\frac{1}{2})$ obtained by dualization of a De Bruijn n-grid with offset $\frac{1}{2}$ is suitable





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  We need a substitution that will keep the planarity and the slope of the lifted tiling.

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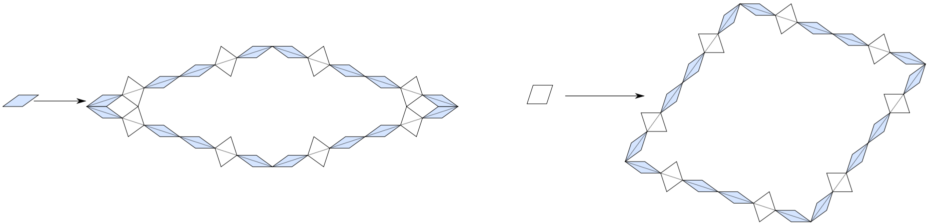
Difficulty:

- Planarity $\rightarrow$ shape of the substitution
- Shape of the substitution $\rightarrow$ well defined substitution



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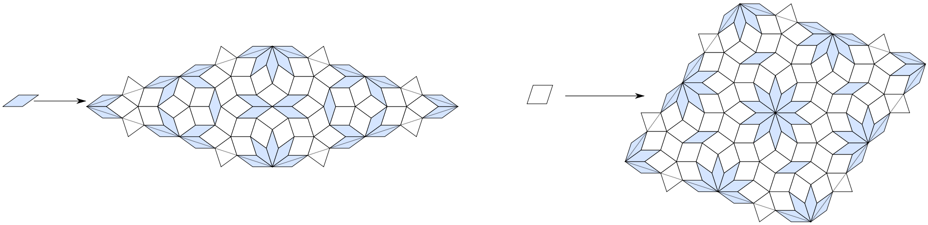

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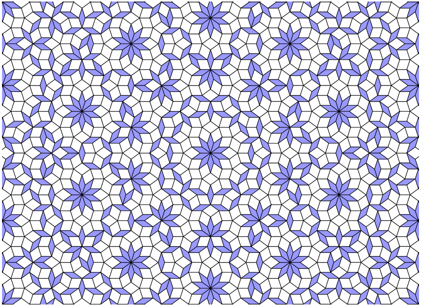

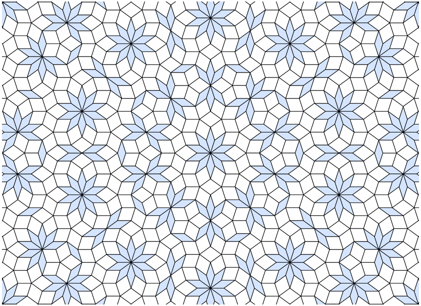

n-fold substitution discrete planes

Problem (Slope)

Let $\mathcal{E}$ be a 2D real plane of $\mathbb{R}^n$.
Does there exist some substitution discrete plane of slope $\mathcal{E}$.

$$\mathcal{E}_n := \left\langle \left(e^{\mathrm{i}\frac{2k\pi}{n}} \right)_{0 \leqslant k < n} \right\rangle = \left\langle \left(\cos \frac{2k\pi}{n} \right)_{0 \leqslant k < n}, \left(\sin \frac{2k\pi}{n} \right)_{0 \leqslant k < n} \right\rangle$$

Theorem (Kari,L.20)

$\forall n \in \mathbb{N}$, $\mathcal{E}_n$ is the slope of a substitution discrete plane.

Generalized substitutions

Framework first introduced in [Arnoux and Ito, 2001] for the study of Rauzy fractals.

Let $\Sigma = \{0, 1, \dots n-1\}$ be the alphabet on n letters.
Let σ be a generalized substitution on alphabet Σ *i.e.* a substitution where we accept inverted letters denoted $\overline{a}$.
Example: $\sigma : 0 \to 1,\ 1 \to 2,\ 2 \to 3,\ 3 \to 3\overline{0}$
Remark: σ can be seen as a non-erasing morphism of the free group on n generators.

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Geometrical realizations of σ are:

- $E_0(\sigma)$ linear application on $\mathbb{R}^n$ obtained by abelianization of σ
- $E_1(\sigma)$ linear application on generalized discrete lines of $\mathbb{R}^n$ *i.e.* on formal sums of discrete segments $\mathbb{R}^n \times \Sigma$
- $E_2(\sigma)$ linear application on generalized discrete 2-surfaces of $\mathbb{R}^n$ *i.e.* on formal sums of discrete squares $\mathbb{R}^n \times \Sigma^2$
- $E_k(\sigma)$ linear application on generalized discrete k-surfaces of $\mathbb{R}^n$

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- $E_k(\sigma)$ linear application on generalized discrete k-surfaces of $\mathbb{R}^n$

We also define the geometrical realization of their dual applications denoted $E^k(\sigma)$.

In codimension 1

Theorem (Arnoux-Ito2001, Fernique2007, Jolivet2013)

Let $\mathcal{E}$ be a plane of $\mathbb{R}^3$. If there exists a unimodular integer matrix M such that

- $\mathcal{E}$ is stable under M
- M is expanding on $\mathcal{E}$

then $\mathcal{E}$ is the slope of a substitution discrete plane.

The proof uses a generalized Pisot substitution σ on three letters and its geometric realizations and their duals.

In codimension 2

There is the example studied in details in [Arnoux et al., 2011].

$$\sigma : \begin{matrix} 0 \to 1 \\ 1 \to 2 \\ 2 \to 3 \\ 3 \to 3\overline{0} \end{matrix}$$

$P(X) = X^4 - X^3 + 1$ with roots $\lambda, \overline{\lambda}, \mu, \overline{\mu}$.

Theorem (Arnoux-Furukado-Harriss-Ito2011)

The planes $\mathcal{E} = \left\langle (\lambda^k)_{0 \leqslant k < 4} \right\rangle$ and $\mathcal{C} = \left\langle (\mu^k)_{0 \leqslant k < 4} \right\rangle$ are the slopes of two different substitution discrete planes.

The key elements are:

- $E_0(\sigma) \in SL_4(\mathbb{Z})$
- $E_2(\sigma)$ and $E^2(\sigma)$ are geometric substitutions

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Theorem (Arnoux-Ito2001, Fernique2007, Jolivet2013)

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$\forall n \in \mathbb{N}$, $\mathcal{E}_n = \left\langle \left(e^{\mathrm{i}\frac{2k\pi}{n}} \right)_{0 \leqslant k < n} \right\rangle$ is the slope of a substitution discrete plane.

Theorem (Arnoux-Ito2001, Fernique2007, Jolivet2013)

Let $\mathcal{E}$ be a plane of $\mathbb{R}^3$. If there exists a unimodular integer matrix M such that $\mathcal{E}$ is a stable expanding subspace for M, then $\mathcal{E}$ is the slope of a substitution discrete plane.

Conjecture

Let $\mathcal{E}$ be a 2D real plane of $\mathbb{R}^n$.
There is a substitution discrete plane of slope $\mathcal{E}$ if and only if there exists an integer matrix M such that:

- $\mathcal{E}$ is an expanding stable subspace for M
- $\mathcal{E}^\perp$ is a contracting stable subspace for M

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