Frobenius conjecture: Christoffel and Sturmian words

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Journée combinatoire de Bordeaux
February 2020

Outline

We propose a generalization to the

Frobenius conjecture

using combinatorics on words.

We want to

- link the Frobenius conjecture with combinatorics of words
- propose a generalization of the conjecture
- prove it for the language of a Christoffel word and a Sturmian word

Frobenius conjecture

Markoff numbers

- Markoff triple : a solution of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz$$

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$$1, 2, 5, 13, 29, 34, 89, 169, 194, \ldots$$

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- $\mathcal{M}$: set of Markoff numbers

Markoff numbers

- Approximation theorem for irrational real numbers :

Theorem (Markoff, 1879-1880)

The Lagrange spectrum below 3 *is given by*

$$\mathcal{L}_{<3} = \left\{ \frac{\sqrt{9m^2 - 4}}{m} : m \in \mathcal{M} \right\}.$$

- Infimum of real indefinite binary quadratic form

Theorem (Markoff, 1879-1880)

Let $f(x, y)$ be an indefinite binary quadratic form. Assume that its infimum $L(f)$ and its discriminant $d(f)$ satisfy the inequality $\sqrt{d(f)} < 3L(f)$. Then the infimum of f are attained and

$$\frac{\sqrt{d(f)}}{L(f)} = \sqrt{9 - 4\sqrt{m^2}}$$

Markoff tree

- From a proper Markoff triple, we can construct two others distinct Markoff triples :

$$\begin{tikzcd}[column sep=small] & \{x, z, y\} \arrow[dl, no head] \arrow[dr, no head] & \\ \{x, 3xz - y, z\} & & \{z, 3zy - x, y\} \end{tikzcd}$$

Markoff tree

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$$\{x, z, y\}$$

$$\{x, 3xz - y, z\} \quad \{z, 3zy - x, y\}$$

$$\{1, 5, 2\}$$

$$\{1, 13, 5\} \qquad \{5, 29, 2\}$$

$$(1, 34, 13) \quad \{13, 194, 5\} \quad \{5, 433, 29\} \quad \{29, 169, 2\}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Markoff tree

Theorem

All proper Markoff triples appear exactly one in the Markoff tree.

$$\{1, 5, 2\}$$

$$\{1, 13, 5\} \qquad \{5, 29, 2\}$$

$$(1, 34, 13) \qquad \{13, 194, 5\} \qquad \{5, 433, 29\} \qquad \{29, 169, 2\}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Frobenius conjecture

Conjecture (Frobenius,1913)

Every Markoff number appears exactly once as the maximum of a Markoff triple in the Markoff tree.

$$
\begin{tikzcd}[column sep=tiny]
 & & & \{1,5,2\} \arrow[dll, no head] \arrow[drr, no head] & & & \\
 & \{1,13,5\} \arrow[dl, no head] \arrow[dr, no head] & & & & \{5,29,2\} \arrow[dl, no head] \arrow[dr, no head] & \\
(1,34,13) & & \{13,194,5\} & & \{5,433,29\} & & \{29,169,2\} \\
\vdots & & \vdots & & \vdots & & \vdots
\end{tikzcd}
$$

Frobenius conjecture

- A lot of equivalent formulation of the conjecture are known (Aigner, 2013).
- Today, we will study this conjecture using combinatorics on words.

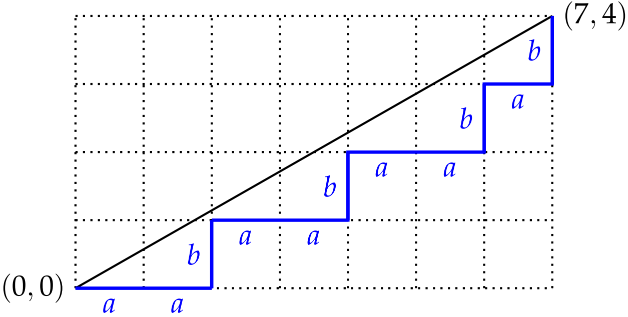
Christoffel words

Combinatorics on words

- Word : a sequence of element in a set $A = \{a, b\}$
- Factor : u is a factor of w if there exist two words x, y such that $w = xuy$.
- Conjugate : u and v are conjugates if there exist two words xy such that $u = xy$ and $v = yx$.
- Circular factor : u is a circular factor of w if it is a factor of a conjugate of w.
- Palindrome : A word equal to its reversal.

Christoffel words

Examples : The Christoffel words of slope 7/4



Christoffel words in Markoff works

The continued fraction expansion of the Markoff constant

$$\sqrt{9 - \frac{4}{m^2}}, m \in \mathcal{M},$$

as an infinite words, have the form p^∞, where p is a Christoffel word written on the alphabet $\{11, 22\}$.

Markoff numbers and Christoffel words

- $\mu : \{a,b\}^* \to \mathrm{SL}_2(\mathbb{N})$

$$\mu(a) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

- $M : \{a,b\}^* \to \mathbb{N}$:

$$M(w) = \mu(w)_{12}$$

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- $M : \{a, b\}^* \to \mathbb{N}$:

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Theorem (Bombieri, 2007 ; Reutenauer, 2009)

The function f from the set of Christoffel words to the set of Markoff triples define as

$$f(w) = \{M(u), M(w), M(v)\},$$

where uv is the standard factorization of w, is a bijection.

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Conjecture (Frobenius)

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Conjecture (Frobenius)

The function M restricted to the set of Christoffel words is injective.

Other functions

- Thue-Morse morphism : $\theta(a) = ab, \theta(b) = ba$
- $w^{(+)} = xp\widetilde{x}$, with p is the longest palindrome suffix of w
- Iterated palindromization : $Pal(\varepsilon) = \varepsilon$,

$$Pal(w_1 \ldots w_n) = (Pal(w_1 \ldots w_{n-1})w_n)^{(+)}$$

Theorem (Reutenauer and Vuillon, 2017)

For each word $v \in \{a, b\}^*$, the number

$$|Pal \circ \theta \circ Pal(av)| + 2$$

is a Markoff number. The mapping is surjective and the injection is equivalent to Frobenius conjecture.

Generalized Frobenius conjecture

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Conjecture

The function M restricted to the set of circular factors of Christoffel words is injective.

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Conjecture

The function M restricted to the set of circular factors of Christoffel words is injective.

- We don't have a proof of this conjecture.
- The generalized Frobenius conjecture implies the Frobenius conjecture.
- We can prove that the function M is injective on two subsets of the circular factors of Christoffel words.

Language of a Christoffel word

Proposition

Let w be a Christoffel word and H_w the set of its circular factors. The function M restricted to H_w is injective.

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Example : $H_{aaabaab}$

a	b	aa	ab	ba	aaa	aab	aba	baa	$\ldots$
1	2	3	5	7	8	13	17	19	$\ldots$

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Let w be a Christoffel word and H_w the set of its circular factors. The function M restricted to H_w is injective.

Example : $H_{aaabaab}$

$$\begin{array}{cccccccccc} a & b & aa & ab & ba & aaa & aab & aba & baa & \dots \\ 1 & 2 & 3 & 5 & 7 & 8 & 13 & 17 & 19 & \dots \end{array}$$

Remark : In fact, we show that M is strictly increasing according to the radix order on H_w.

Language of a Christoffel word

$$\begin{array}{cccccccccccccccc}
w_1 & a & a & a & b & a & a & b & & 1 & 3 & 8 & 34 & 115 & 311 & 1325 \\
w_2 & a & a & b & a & a & a & b & & 1 & 3 & 13 & 44 & 119 & 313 & 1327 \\
w_3 & a & a & b & a & a & b & a & & 1 & 3 & 13 & 44 & 119 & 507 & 1715 \\
w_4 & a & b & a & a & a & b & a & \mapsto & 1 & 5 & 17 & 46 & 121 & 513 & 1735 \\
w_5 & a & b & a & a & b & a & a & & 1 & 5 & 17 & 46 & 196 & 663 & 1793 \\
w_6 & b & a & a & a & b & a & a & & 2 & 7 & 19 & 50 & 212 & 717 & 1939 \\
w_7 & b & a & a & b & a & a & a & & 2 & 7 & 19 & 81 & 274 & 741 & 1949
\end{array}$$

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1. Next line Lemma

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1. Next line Lemma
2. Next column Lemma

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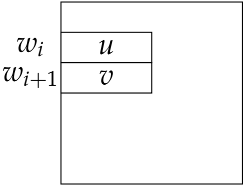
Lemma (Next line)

If u is a prefix of w_i and v is a prefix of w_{i+1} such that $|u| = |v|$ and $u \neq v$, then $M(u) < M(v)$.



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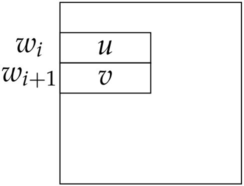

Sketch of proof :

- $w_i = pabs$ and $w_{i+1} = pbas$
- Notice that sp is a palindrome

$$
\begin{array}{ccccccc}
a & a & \underline{a} & \underline{b} & a & a & b \\
a & a & b & a & a & \underline{a} & \underline{b} \\
a & \underline{a} & \underline{b} & a & a & b & a \\
a & b & a & a & \underline{a} & \underline{b} & a \\
\underline{a} & \underline{b} & a & a & b & a & a \\
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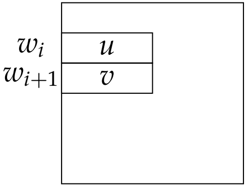

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- $w_i = pabs$ and $w_{i+1} = pbas$
- Two cases :
  - $u = pa, v = pb$
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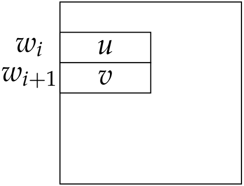

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- Main ideas :
  - $u = pab\widetilde{p}t, v = pba\widetilde{p}t$
  - $\mu(p) = \mu(\widetilde{p})^t$

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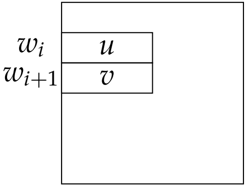

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- Calculation of $M(v) - M(u) > 0$

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Lemma (Next column)

If u is a prefix of length k of w_n and v is a prefix of length $k+1$ of w_1, then $M(u) < M(v)$.

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Sketch of proof :

- Since w is a Christoffel word, we have that $w_1 = amb$ and $w_n = bma$, where m is a palindrome.
- $u = bp$ and $v = apx$, where p is a prefix of m and x is a letter.
- Let $\mu(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in the monoid $\mu(A^*)$.
- $M(v) - M(u) = (\mu(apa) - \mu(bp))_{12} = 2a - 3b + c - d > 0$

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Proposition

Let w be a Christoffel word and H_w the set of its circular factors. The function M restricted to H_w is injective.

Language of a Sturmian word

A Sturmian word is an infinite aperiodic word s such that $P(s, n) = n + 1$.

Corollary

Let S be a Sturmian words and $L(S)$ the set of all its factors. The function M restricted to $L(S)$ is injective.

Sketch of proof :

- Let u and v two factors of $L(S)$.
- There exists a prefix p of S such that u and v are both factors of p and p is a Christoffel word.

Remark

- Values of M are not increasing for the radix order in general

a	b	aa	ab	ba	bb	aaa	aab	aba	abb	baa
1	2	3	5	7	12	8	13	17	29	19

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