Weakly-unambiguous Parikh automata and their link with holonomic series

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Generating series

Main tool for a combinatorial approach of formal languages

The generating series of a language $\mathcal{L}$ is the formal power series

$$L(x) = \sum_{n\in\mathbb{N}} \ell_n x^n,\ \ \ell_n := \text{ number of words of length } n \text{ in } \mathcal{L}$$

Example 1: all words on $\Sigma = \{a, b\}$, the generating series is

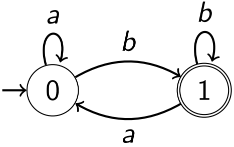
$$L(x) = \sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x}$$

Example 2: well-bracketed words, the generating series is

$$C(x) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{2n} = \frac{1-\sqrt{1-4x^2}}{2x^2}$$

Link between automata and combinatorics

regular languages $\subseteq$ unambiguous context-free languages



$$\begin{cases} S \to aSB \mid \varepsilon \\ B \to cB \mid bS \end{cases}$$

$abb \in \mathcal{L}(\mathcal{A})$

$S \to aSB \to aB \to abS \to ab \in \mathcal{L}(\mathcal{G})$

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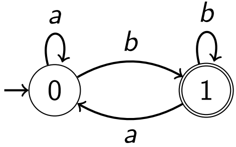

$$\begin{cases} S \to aSB \mid \varepsilon \\ B \to cB \mid bS \end{cases}$$

$$\begin{cases} q_0(x) = xq_0(x) + xq_1(x) \\ q_1(x) = 1 + xq_1(x) + xq_0(x) \end{cases}$$

$$\begin{cases} S(x) = xS(x)B(x) + 1 \\ B(x) = xB(x) + xS(x) \end{cases}$$

$$q_0(x) = \frac{x}{1-2x}$$

$$x^2S(x)^2 - (1-x)S(x) + 1 - x = 0$$

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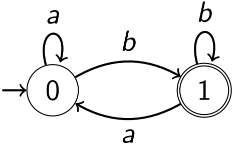

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rational series $L(x) = \dfrac{P(x)}{Q(x)}$ $\subseteq$ algebraic series $P(L(x), x) = 0$

Analytic criteria for inherent ambiguity

Flajolet's idea: if the series of a context-free language is not algebraic, then it is an inherently ambiguous context-free language.

Theorem [Product of palindromes, Crestin 72]:
$C = \{w_1 w_2 \,:\, w_1, w_2 \in \{a, b\}^* \text{ are palindromes}\}$ is inherently ambiguous.

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Analytic proof:

- Its generating series is [Kemp 80]

$$C(x) = 1 + 2\sum_{m\geq 1} \psi(m) \frac{x^m(1+x^m)(1+2x^m)}{(1-2x^{2m})^2}$$

- $C(x)$ has an infinite number of singularities: it is not algebraic [Flajolet 87]

Inclusion problem for unambiguous automata

Problem: Given $\mathcal{A}$ and $\mathcal{B}$ two unambiguous NFA, $L(\mathcal{A}) \subseteq L(\mathcal{B})$?

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- $C(x) := \sum c_n x^n = B(x) - A(x)$ is rational
- The coefficients of $C(x)$ satisfy a linear recurrence:

$$\forall n \geq r, c_n = \alpha_1 c_{n-1} + \cdots + \alpha_r c_{n-r}$$

- the order r is at most $|Q_{\mathcal{A}}| + |Q_{\mathcal{B}}|$

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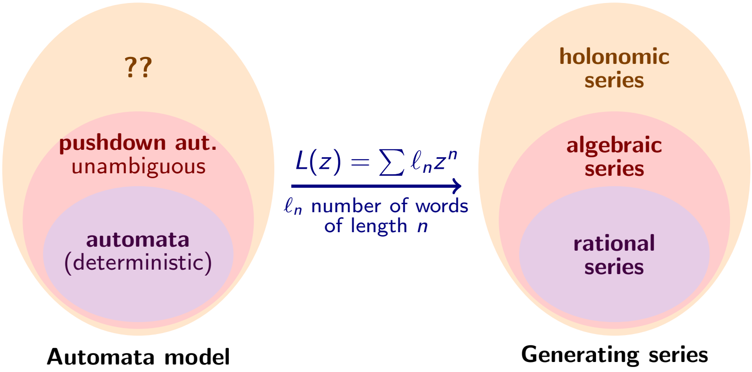
Theorem [Stearns and Hunt 85]: The inclusion problem for unambiguous NFA is polynomial.

- $L(\mathcal{A}) \not\subseteq L(\mathcal{B}) \Leftrightarrow L(\mathcal{A}) \cap L(\mathcal{B}) \subsetneq L(\mathcal{A})$
- Compute coefficients up to $|Q_{\mathcal{A}}||Q_{\mathcal{B}}| + |Q_{\mathcal{A}}|$ (dynamic prog.)

Extension to holonomic series



Goals of the talk

- suitable class of languages and unambiguous automata models
- proofs of inherent ambiguity
- algorithmic consequences

Holonomic series in one variable [Stanley 80]

A series $f(x) = \sum_n a_n x^n$ is holonomic (or D-finite) if it satisfies a differential equation of the form:

$$P_k(x) f^{(k)}(x) + \ldots + P_0(x) f(x) = 0 \quad \text{with } P_i(x) \in \mathbb{Q}[x]$$

where $f^{(1)}(x) = f'(x) = \sum_{n\geq 1} n a_n x^{n-1}$ is the formal derivative

Equivalently the coefficients a_n satisfy a linear recurrence

$$p_0(n) a_n + \ldots + p_r(n) a_{n-r} = 0 \quad \text{with } p_i(n) \in \mathbb{Q}[n]$$

Example: $F(x) = e^x := \sum \frac{x^n}{n!}$ is holonomic but not algebraic

- differential equation: $F' - F = 0$
- recurrence relation: $(n+1) f_{n+1} - f_n = 0$

Holonomic series in several variables [Lipshitz 89]

A series $f(x_1, \ldots, x_n)$ is holonomic (or D-finite) if it satisfies a system of partial derivative equations of the form:

$$\begin{cases} A_{1,r_1}(\vec{x})\ \partial_{x_1}^{r_1} f(\vec{x}) + \ldots + A_{1,1}(\vec{x})\ \partial_{x_1} f(\vec{x}) + A_{1,0}(\vec{x})\ f(\vec{x}) = 0 \\ \vdots \\ A_{n,r_n}(\vec{x})\ \partial_{x_n}^{r_n} f(\vec{x}) + \ldots + A_{n,1}(\vec{x})\ \partial_{x_n} f(\vec{x}) + A_{n,0}(\vec{x})\ f(\vec{x}) = 0 \end{cases}$$

with $A_{i,j}(\vec{x}) \in \mathbb{Q}[\vec{x}],$ and $\vec{x} = (x_1, \ldots, x_n)$ and

$$\partial_{x_1} f(\vec{x}) := \sum_{m_1,\ldots,m_n} m_1 a_{m_1,\ldots,m_n}\ {x_1}^{m_1-1} x_2^{m_2} \cdots x_n^{m_n}$$

In this talk, we mainly rely on closure properties and not on this formal definition

Closure properties of holonomic series

Holonomic series are closed under [Lipshitz 88,89]:

- arithmetic operations $+, \times, -$
- specialization to 1, when it is well-defined: if $f(x_1, \ldots, x_n)$ is holonomic, then $f(x, 1, \ldots, 1)$ is holonomic too
- Hadamard's product $\odot$ (component-wise product)

$$f(\vec{x}) = \sum_{\vec{\imath} \in \mathbb{N}^n} a(\vec{\imath})\vec{x}^{\vec{\imath}}, \quad g(\vec{x}) = \sum_{\vec{\imath} \in \mathbb{N}^n} b(\vec{\imath})\vec{x}^{\vec{\imath}}$$

$$f \odot g(\vec{x}) = \sum_{\vec{\imath} \in \mathbb{N}^n} a(\vec{\imath}) b(\vec{\imath})\vec{x}^{\vec{\imath}},$$

where $\vec{x} = (x_1, \ldots, x_n)$, $\vec{\imath} = (i_1, \ldots i_n)$ and $\vec{x}^{\vec{\imath}} = x_1^{i_1} \cdots x_n^{i_n}$

Crucial particular case: support series

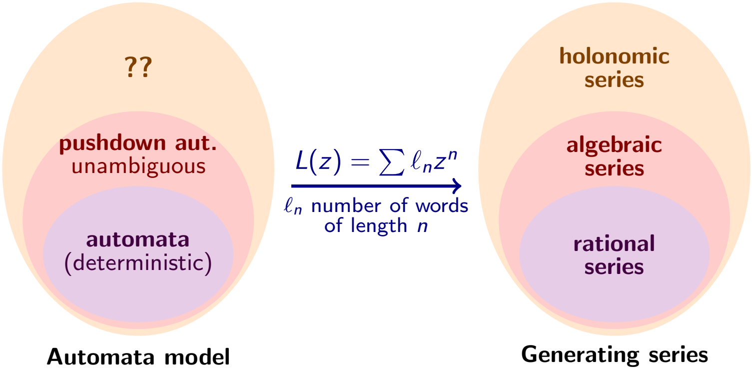
Let $\mathcal{S} \subseteq \mathbb{N}^n$. The support series of $\mathcal{S}$ is

$$g(x_1, \ldots, x_n) = \sum_{(i_1,\ldots,i_n)\in\mathcal{S}} x_1^{i_1} \ldots x_n^{i_n}$$

Let $f(x_1, \ldots, x_n) = \sum_{\vec{\imath}\in\mathbb{N}^n} a(\vec{\imath})\vec{x}^{\,\vec{\imath}}$. Then:

$$(f \odot g)(x_1, \ldots, x_n) = \sum_{\vec{\imath}\in\mathcal{S}} a(\vec{\imath})\vec{x}^{\,\vec{\imath}}$$

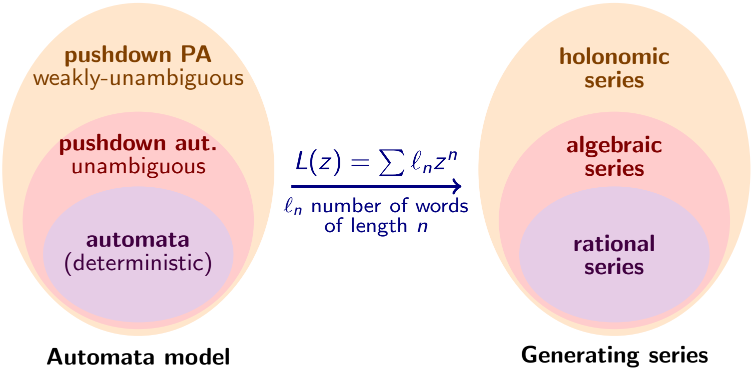
Completing the picture



Previous attempts at a link with formal languages

- semilinear constraints with Hadamard product [Lipshitz 88]
- ad-hoc family LCL [Massazza 93]
- RCM + conjecture link to RBCM [Castiglione, Massazza 2017]

Completing the picture

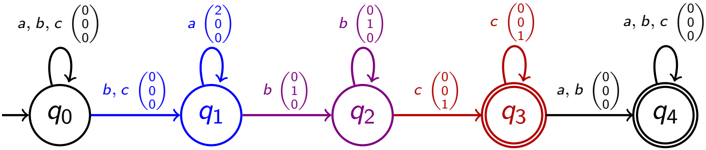


Our model: weakly unambiguous Pushdown Parikh automata

In this talk, we only focus on weakly-unambiguous PA.

Parikh automata (PA) [Klaedtke and Rueß 03]

Parikh automaton $=$ NFA over $\Sigma \times \mathbb{N}^d$ with semi-linear constraints



$$\mathcal{C} = \left\{ \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} : \ell_2 + \ell_3 \geq \ell_1 \text{ and } \ell_3 \geq 2 \right\}$$

$$q_0 \xrightarrow[b]{\begin{pmatrix}0\\0\\0\end{pmatrix}} q_0 \xrightarrow[a]{\begin{pmatrix}0\\0\\0\end{pmatrix}} q_0 \xrightarrow[b]{\begin{pmatrix}0\\0\\0\end{pmatrix}} q_0 \xrightarrow[c]{\begin{pmatrix}0\\0\\0\end{pmatrix}} q_1 \xrightarrow[a]{\begin{pmatrix}2\\0\\0\end{pmatrix}} q_1 \xrightarrow[a]{\begin{pmatrix}2\\0\\0\end{pmatrix}} q_1 \xrightarrow[b]{\begin{pmatrix}0\\1\\0\end{pmatrix}} q_2 \xrightarrow[b]{\begin{pmatrix}0\\1\\0\end{pmatrix}} q_2 \xrightarrow[b]{\begin{pmatrix}0\\1\\0\end{pmatrix}} q_2 \xrightarrow[c]{\begin{pmatrix}0\\0\\1\end{pmatrix}} q_3 \xrightarrow[c]{\begin{pmatrix}0\\0\\1\end{pmatrix}} q_3$$

This run accepts *babcaabbbcc* with the vector $\begin{pmatrix}4\\3\\2\end{pmatrix} \in \mathcal{C}$

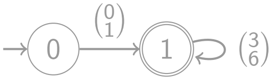
Semilinear sets in $\mathbb{N}^d$

$\bigwedge \bigvee$ of linear inequalities or equalities modulo constants

$$\{(3n, 6n+1) \,:\, n \in \mathbb{N}\} = \{(x_1, x_2) \,:\, x_1 \equiv 0[3] \,\wedge\, x_2 = 2x_1 + 1\}$$

Equivalent definitions

- Finite union of linear sets $\vec{c} + P^*$ where $P = \{p_1, \ldots, p_r\}$
  $(0,1) + \{(3,6)\}^*$
- Presburger arithmetic
  $\Phi(x_1, x_2) := \exists x, x_1 - 3x = 0 \;\wedge\; 1 + 2x_1 - x_2 = 0$
- (Unambiguous) rational subsets of $(\mathbb{N}^d, +)$



Semilinear sets in $\mathbb{N}^d$

Support series for $C \subseteq \mathbb{N}^d$

$$C(x_1, \ldots, x_d) := \sum_{(v_1,\ldots,v_d) \in C} x_1^{v_1} \ldots x_d^{v_d}$$

Theorem [Eilenberg and Schützenberger, Ito, 69]:
If C is semilinear, its support series $C(x_1, \ldots, x_d)$ is rational.

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If C is semilinear, its support series $C(x_1, \ldots, x_d)$ is rational.

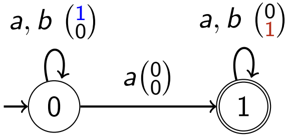
If $C = \bigcup_{i=1}^{m} \vec{c_i} + P_i^*$ is an unambiguous description of C:

$$C(x_1, \ldots, x_d) = \sum_{i=1}^{m} \frac{\vec{x}^{c_i}}{\prod_{p\in P_i}(1 - \vec{x}^{p})}$$

In the sequel we will deal with holonomic series of the form $f \odot C$, where C is the support series of a semilinear set.

Weakly-unambiguous Parikh automata

Weak-unambiguity: at most one accepting run (final state + semilinear constraint)



$C = \{(n, n) : n \in \mathbb{N}\}$

$L(\mathcal{A}) = \{$words with an a in the middle$\} = \{\ldots, abbabab, \ldots\}$

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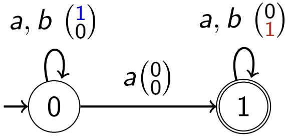

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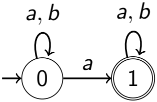
$\neq$ Unambiguous PA [Cadilhac, Finkel and McKenzie 13]:
the underlying NFA without the vectors is unambiguous



Weakly-unambiguous Parikh automata

Our first contribution is to show the robustness of this new class by relating it to already known classes

Weakly unambiguous PA are equivalent to:

- Regular languages with semilinear Constraints and an injective Morphism (RCM) [Castiglione Massazza 17]
- unambiguous two-way Reversal Bounded Counter Machines (RBCM) [Ibarra 78]

  $\Rightarrow$ stronger version of a conjecture of [Castiglione Massazza]

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Weakly unambiguous pushdown PA are equivalent to:

- LCL + an injective Morphism [Massazza 93]
- unambiguous one-way RBCM with a stack [Ibarra 78]

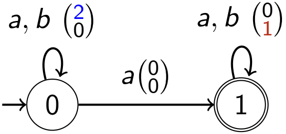
Holomicity of the generating series

Theorem: the series of a weakly-unambiguous PA is holonomic

Counting the number of runs by vectors

$$\overline{A}(x, y_1, \ldots, y_d) = \sum_{n, v_1, \ldots, v_d} a_{n, v_1, \ldots, v_d} x^n y_1^{v_1} \cdots y_d^{v_d} \quad \text{is rational}$$

where $a_{n,v_1,\ldots,v_d}$ counts the number of runs of length n, ending in a final state and computing the vector $(v_1, \ldots, v_d)$



$$\begin{cases} q_0(x, y_1, y_2) = 2x y_1^2 q_0(x, y_1, y_2) + x q_1(x, y_1, y_2) \\ q_1(x, y_1, y_2) = 2x y_2 + 1 \end{cases}$$

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Counting the number of runs by vectors

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Support series of the semilinear constraint C

$$C(y_1, \ldots, y_d) := \sum_{(v_1, \ldots, v_d) \in C} y_1^{v_1} \ldots y_d^{v_d} \quad \text{is rational}$$

Filtering non-accepting runs

$$\overline{A}(x, y_1, \ldots, y_d) \odot \frac{C(y_1, \ldots, y_d)}{1 - x}$$

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Filtering non-accepting runs

$$\overline{A}(x, y_1, \ldots, y_d) \odot \frac{C(y_1, \ldots, y_d)}{1 - x} \; [y_1 \to 1, \ldots, y_d \to 1]$$

weak-unambiguity $\Rightarrow$ counting accepting runs = counting words $\square$

Inherent ambiguity using non-holonomicity

Flajolet's idea: if the series of a context-free language is not algebraic, then it is an inherently ambiguous context-free language.

In our settings: if the series of a language accepted by a PA is not holonomic, then it is inherently weakly-ambiguous for PA.

Example: the following language is recognized by a PA
$\mathcal{D} = \{a^{n_1}b\ a^{n_2}b \ldots a^{n_k}b : k \in \mathbb{N}^*,\ n_1 = 1 \text{ and } \exists j < k, n_{j+1} \neq 2n_j\}$

Observe that:

- $w = aba^2ba^4ba^8b$ is a typical word that is not in $\mathcal{D}$
- $D(x) = \frac{x^2}{1-\frac{x}{1-x}} - \sum_{k\geq 1} x^{2^k-1+k}$
- $D(x)$ has infinitely many singularities: it is not holonomic

Hence $\mathcal{D}$ is inherently weakly-ambiguous for PA.

Limit of the method

Remark: Non-holonomicity is only a sufficient condition for inherent weakly-unambiguity for PA

Proposition: inherent weak-unambiguity is undecidable.

Proof: general theorem for undecidability [Greibach 68] using the undecidability of universality for PA

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Proposition: inherent weak-unambiguity is undecidable.

Proof: general theorem for undecidability [Greibach 68] using the undecidability of universality for PA

Proposition: the following language is recognized by a PA

$$\mathcal{L}_{even} = \{a^{n_1}b \dots a^{n_{2k}}b \ : \ k \in \mathbb{N}^*, \exists i \in [1,k], n_{2i-1} = n_{2i}\}$$

- its generating series is rational
- it is inherently weakly-ambiguous

Proof: involved pumping argument using Ramsey theorem
$\Rightarrow$ inherent ambiguity for a stronger model of PA
with arbitrary recursive constraint sets.

Inclusion problem for weakly-unambiguous PA

Problem: Given $\mathcal{A}$ and $\mathcal{B}$ weakly-unamb. PA, $L(\mathcal{A}) \subseteq L(\mathcal{B})$?

Known results

- deterministic PA $\rightarrow$ co-NEXP-complete [Filiot et al. 19]
- weakly-unambiguous $\rightarrow$ decidable without upper-bound
- non-deterministic PA $\rightarrow$ undecidable [Klaedtke-Rueß02]

Counting approach (similar to unambiguous NFAs)

- If $L(\mathcal{A}) \subsetneq L(\mathcal{B})$, bound the size of a witness by some N expressed in terms of $\|\mathcal{A}\|$ and $\|\mathcal{B}\|$
- Solve the inclusion problem using intersection and dynamic programming

Bound for non-inclusion $L(\mathcal{A}) \subsetneq L(\mathcal{B})$

$$C(x) := \sum c_n x^n = B(x) - A(x)$$

unambiguous NFAs	**weakly-unambiguous PAs**
$C(x)$ is rational	
c_n satisfies a linear rec. with constant coefficients	
◦ degree of $A(x)$ at most $\lvert Q_{\mathcal{A}} \rvert$ ◦ compute degree $B(x) - A(x)$ ◦ order of the recurrence for c_n	

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unambiguous NFAs

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- degree of $A(x)$ at most $|Q_{\mathcal{A}}|$
- compute degree $B(x) - A(x)$
- order of the recurrence for c_n

weakly-unambiguous PAs

$C(x)$ is holonomic

c_n satisfies a linear rec.
with polynomial coefficients

$C(x) = x^{100}$ satisfies $xC'(x) - 100C(x) = 0$ and $(n - 100)c_n = 0$

Bound for non-inclusion $L(\mathcal{A}) \subsetneq L(\mathcal{B})$

$$C(x) := \sum c_n x^n = B(x) - A(x)$$

unambiguous NFAs

$C(x)$ is rational

c_n satisfies a linear rec.
with constant coefficients

- degree of $A(x)$ at most $|Q_{\mathcal{A}}|$
- compute degree $B(x) - A(x)$
- order of the recurrence for c_n

weakly-unambiguous PAs

$C(x)$ is holonomic

c_n satisfies a linear rec.
with polynomial coefficients

- size sys. for $A(x)$ and $B(x)$
- size sys. for $B(x) - A(x)$
- recurrence order and largest root of leading coeff

$C(x) = x^{100}$ satisfies $xC'(x) - 100C(x) = 0$ and $(n - 100)c_n = 0$

Bounds for the series of a weakly-unambiguous PA

$\mathcal{A}$ weakly-unambiguous PA with a semilinear constraint C of dimension d (of the vectors)

$$A(x) = \underbrace{\bar{A}(x, y_1, \ldots, y_d)}_{\substack{\text{rat. series} \\ \text{obtained form } \mathcal{A}}} \odot \underbrace{C(x, y_1, \ldots, y_d)}_{\substack{\text{rat. series} \\ \text{obtained form } C}} \quad [y_1 \to 1, \ldots, y_d \to 1]$$

At each step, we bound the size of the system of differential equation satisfied by the series (order + coefficient)

- Main difficulty: the Hadamard product $\odot$
- Detailed analysis of the proof of closure from [Lipshitz 89]

Results for the inclusion problem

Assuming that the semilinear constraint is given in an unambiguous presentation and all numbers are encoded in unary.

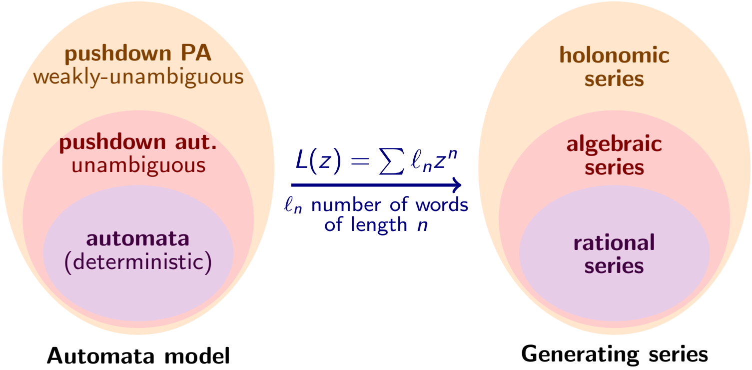
Proposition: If $L(\mathcal{A}) \subsetneq L(\mathcal{B})$, there exists a word $w \in L(\mathcal{A}) \setminus L(\mathcal{B})$ such that

$$|w| \leq 2^{2^{\mathcal{O}(d^2 \log(dM))}}$$

where $d = d_{\mathcal{A}} + d_{\mathcal{B}}$, $M = |\mathcal{A}|\,|\mathcal{B}|\,\|\mathcal{A}\|_\infty\,\|\mathcal{B}\|_\infty$.

Theorem: The inclusion problem for weakly-unambiguous PA is in $2-\mathrm{EXPTIME}$

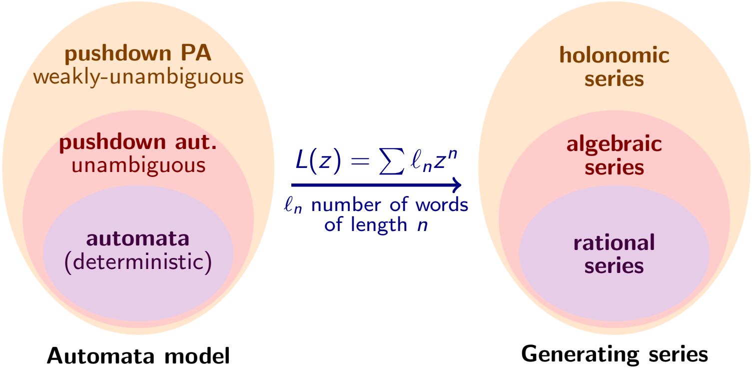
Completing the picture



Perspectives

- closure properties
- lower bounds
- universality with a stack
- better upper bound for inclusion

Completing the picture



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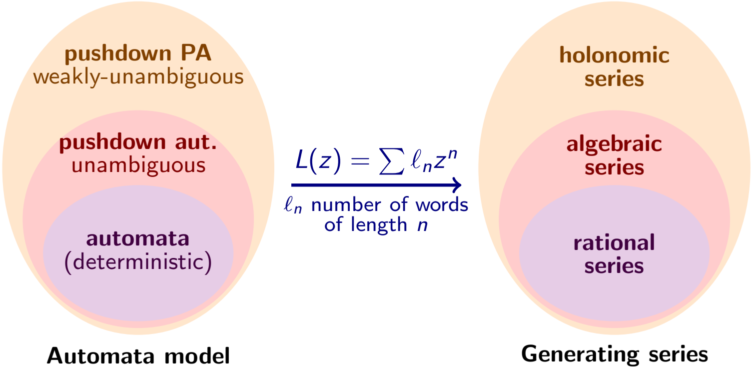
Extend the class of languages?

Proposition [Bell Chen 17]:
The support series of $C \subseteq \mathbb{N}^d$ is holonomic iff C is a semi-linear

Completing the picture

THANK YOU !



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