Dimers, double-dimers, and the PT/DT correspondence

Helen Jenne

joint work with Gautam Webb and Ben Young

CNRS, Institut Denis Poisson, Université de Tours and Université d'Orléans

Journées de combinatoire de Bordeaux

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Outline

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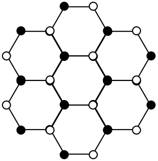
Dimer configurations

- Today $G = (V_1, V_2, E)$ is a finite, bipartite, planar graph.
- We will also assume G has a fixed embedding in the plane. This embedding divides the plane into *faces*, one of which is unbounded.



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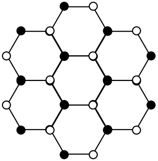
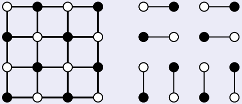


Definition (Dimer configuration/Perfect matching)



A collection of edges that covers each vertex exactly once

Throughout this talk, we'll assume $|V_1| = |V_2|$.

Weighted graphs and the partition function

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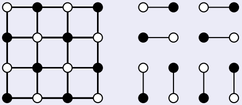

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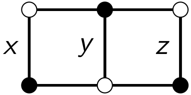
- Given a graph, we can assign a weight $w(e)$ to each edge.



- If M is a perfect matching (dimer configuration), $w(M) = \prod_{e \in M} w(e)$

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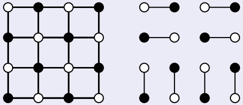

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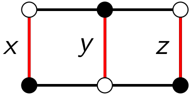

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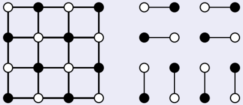

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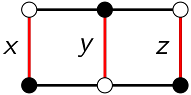

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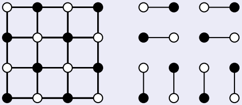

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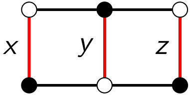

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$$Z^D(G) = xyz + x + z$$

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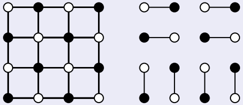

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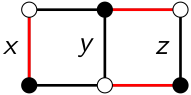

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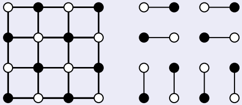

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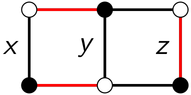

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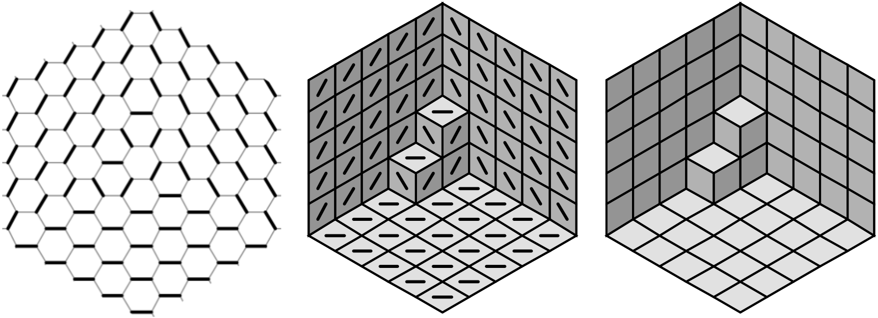
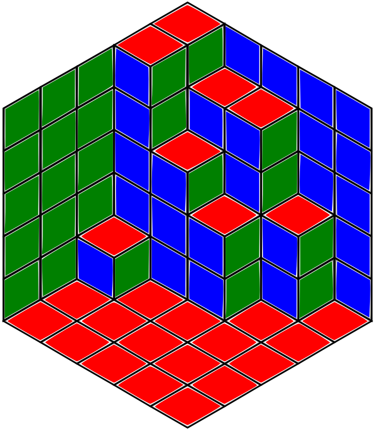
Plane partitions

A well-studied example of the dimer model are *plane partitions.*





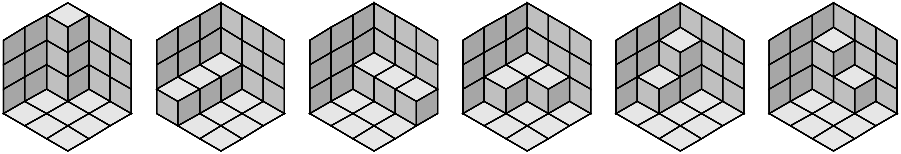
$$\begin{matrix} 5 & 4 & 4 & 2 \\ 5 & 3 & 2 & \\ 1 & & & \end{matrix}$$

Definition (Plane partition)

A plane partition is a finite array π of positive integers that is nonincreasing in rows and columns.

Enumerating plane partitions

Example. There are six plane partitions of 3:



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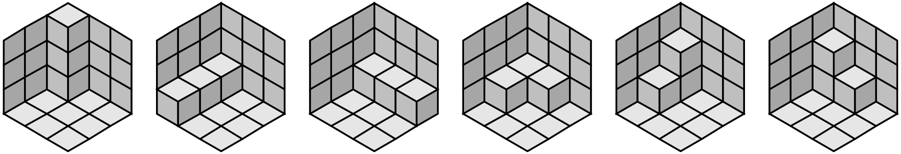

Theorem (MacMahon, 1916)

$$\sum_{\pi} q^{|\pi|} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$

where the sum is over all plane partitions π *and* $|\pi| = \sum \pi_{i,j}$.

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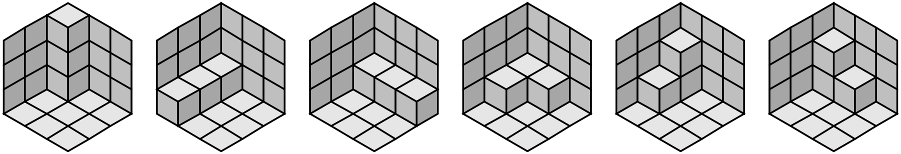

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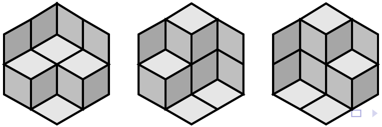
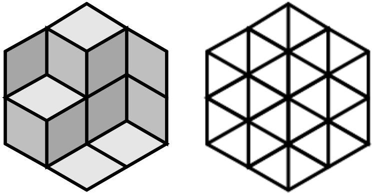
Let $\mathcal{B}(r,s,t)$ denote a "room" with dimensions $r \times s \times t$.
There are three plane partitions π of 3 such that $\pi \subset \mathcal{B}(2,2,2)$:



Connection to the dimer model

$\pi \in \mathcal{B}(r, s, t)$ is a rhombus tiling of a hexagonal region of triangles



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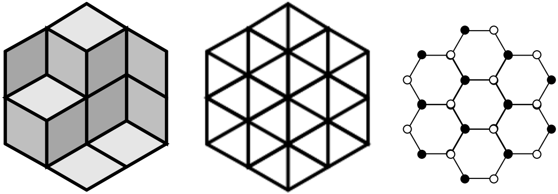
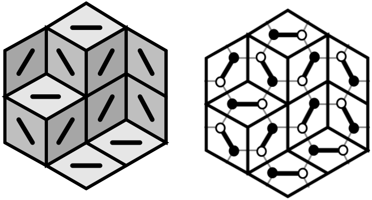
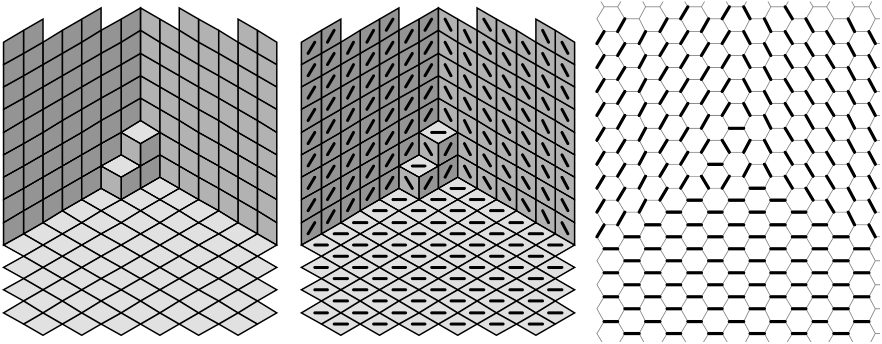


This is equivalent to a dimer configuration of the dual graph.



Connection to the dimer model

By making the room larger, we can extend the dimer configuration to be a dimer configuration of the infinite honeycomb graph.



Main result

Recall the generating function for plane partitions:

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- Donaldson-Thomas theory and Pandharipande-Thomas theory are branches of enumerative geometry.
- Both have generating fns known as the *combinatorial Calabi-Yau topological vertices* which count plane-partition like objects.
- DT theory gen fn: $V(\mu_1, \mu_2, \mu_3)$
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$$V(\mu_1, \mu_2, \mu_3) = M(q)W(\mu_1, \mu_2, \mu_3)$$

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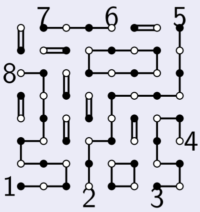
$\mathbf{N}$ is a set of special vertices called *nodes* on the outer face of G (with $|\mathbf{N}|$ even)

Definition (Double-dimer configuration on $(G, \mathbf{N})$)



- A "subgraph" (doubled edges allowed) of degree 1 at $v \in \mathbf{N}$ and degree 2 at all other vertices.
- Equivalently, a configuration of
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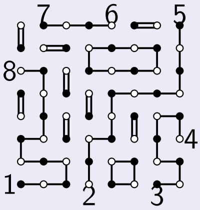

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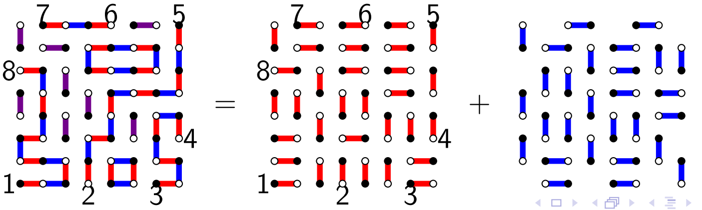
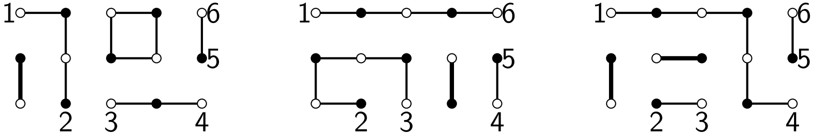
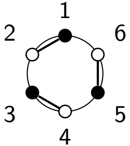
The weight of a DD configuration is the product of its edge weights $\times\ 2^{\ell}$



Double-dimer pairing probabilities

Kenyon and Wilson gave an explicit way to compute the probability that a random double dimer configuration has pairing σ.





$$\widehat{\Pr}\left(\begin{smallmatrix}1\\2\end{smallmatrix}\middle|\begin{smallmatrix}3\\4\end{smallmatrix}\middle|\begin{smallmatrix}5\\6\end{smallmatrix}\right) = X_{1,4}X_{2,5}X_{3,6} + X_{1,2}X_{3,4}X_{5,6}$$

$X_{i,j}$ is defined in terms of dimer configurations of G.

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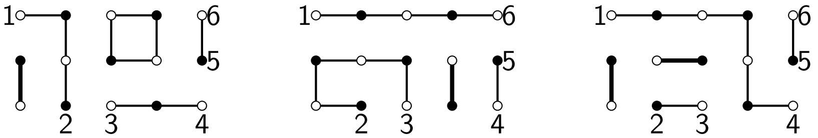

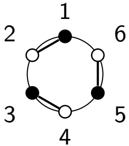

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$$\widehat{\Pr}\left(\begin{array}{c|c|c|c}1&3&5&7\\8&4&2&6\end{array}\right) = X_{1,8}X_{3,4}X_{5,2}X_{7,6} - X_{1,4}X_{3,8}X_{5,2}X_{7,6} + X_{1,6}X_{3,4}X_{5,8}X_{7,2}$$
$$-X_{1,8}X_{3,6}X_{5,2}X_{7,4} - X_{1,4}X_{3,6}X_{5,8}X_{7,2} + X_{1,6}X_{3,8}X_{5,2}X_{7,4}$$

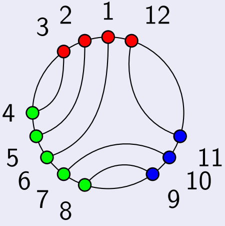
Theorem (KW11a, Theorem 1.3)

$\widehat{Pr}(\sigma)$ is an integer-coeff homogeneous polynomial in the quantities $X_{i,j}$

Tripartite pairings

Definition (Tripartite pairing)

A planar pairing σ of $\mathbf{N}$ is *tripartite* if the nodes can be divided into ≤ 3 sets of circularly consecutive nodes so that no node is paired with a node in the same set.



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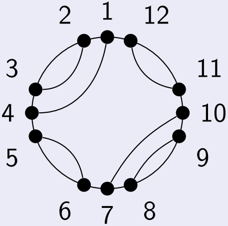

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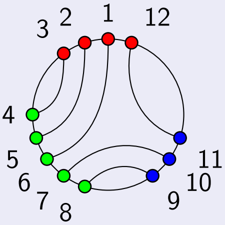

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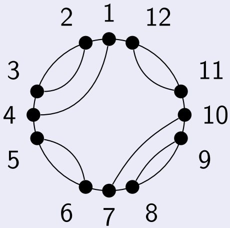

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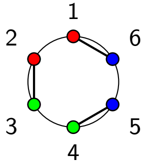
Dividing nodes into three sets R, G, and B defines a tripartite pairing.

Determinant formula

Theorem (KW09, Theorem 6.1)

When σ is a tripartite pairing,

$$\widehat{\Pr}(\sigma) = \det[1_{i,j \text{ RGB-colored differently}}\, X_{i,j}]_{j=\sigma(1),\sigma(3),\ldots,\sigma(2n-1)}^{i=1,3,\ldots,2n-1}.$$



$$\widehat{\Pr}\begin{pmatrix}1 & 3 & 5\\ 6 & 2 & 4\end{pmatrix} = \begin{vmatrix} X_{1,6} & 0 & X_{1,4} \\ X_{3,6} & X_{3,2} & 0 \\ 0 & X_{5,2} & X_{5,4} \end{vmatrix}$$

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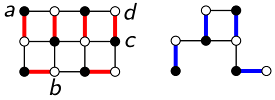
Kuo condensation

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\})=Z^D(G-\{a,b\})Z^D(G-\{c,d\})+Z^D(G-\{a,d\})Z^D(G-\{b,c\})$$



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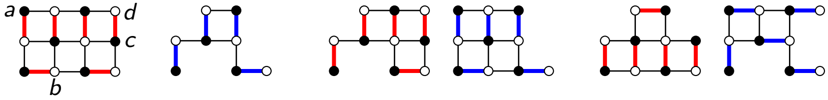

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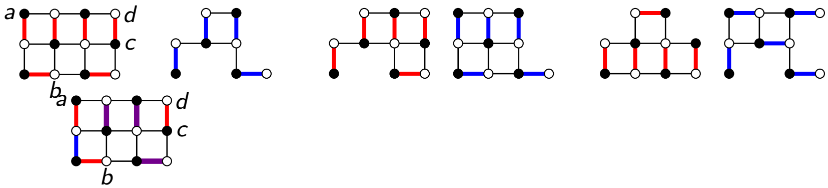

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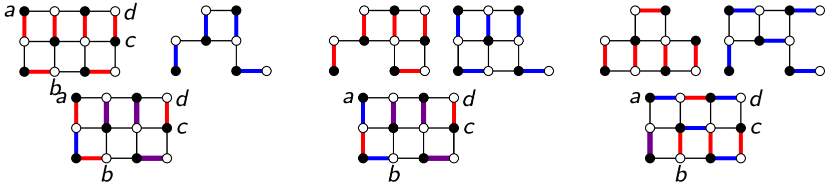

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Theorem (Desnanot-Jacobi identity/Dodgson condensation)

$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i)$$

M_i^j is the matrix M with the ith row and the jth column removed.

Double-dimer condensation

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Divide N *into sets* R, G, *and* B *and let* σ *be the corr. tripartite pairing. Let* $x, y, w, v \in \mathrm{N}$ *such that* $x < w \in V_1$ *and* $y < v \in V_2$. *If* $\{x, y, w, v\}$ *contains at least one node of each RGB color and* x, y, w, v *appear in cyclic order then*

$$Z^{DD}_{\sigma}(G, \mathrm{N}) Z^{DD}_{\sigma_{xywv}}(G, \mathrm{N} - \{x, y, w, v\}) =$$

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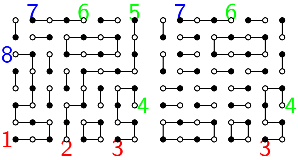
Example.

$$Z^{DD}_{\sigma}(\mathrm{N}) Z^{DD}_{\sigma_{1258}}(\mathrm{N} - 1, 2, 5, 8) = Z^{DD}_{\sigma_{12}}(\mathrm{N} - 1, 2) Z^{DD}_{\sigma_{58}}(\mathrm{N} - 5, 8) + Z^{DD}_{\sigma_{18}}(\mathrm{N} - 1, 8) Z^{DD}_{\sigma_{25}}(\mathrm{N} - 2, 5)$$



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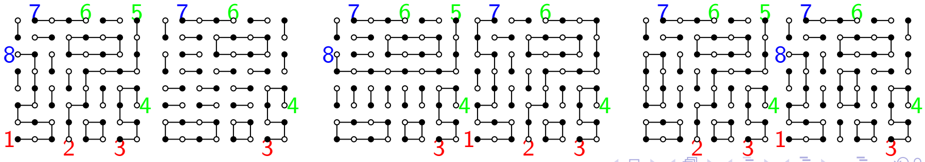

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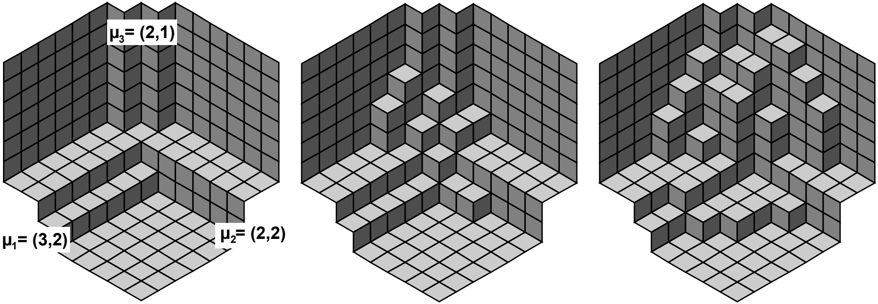
DT box counting

In DT theory, we count *plane partitions asymptotic to* (μ_1, μ_2, μ_3)



The DT topological vertex is $V(\mu_1, \mu_2, \mu_3) = \sum_{\pi \in P(\mu_1,\mu_2,\mu_3)} q^{w(\pi)}$

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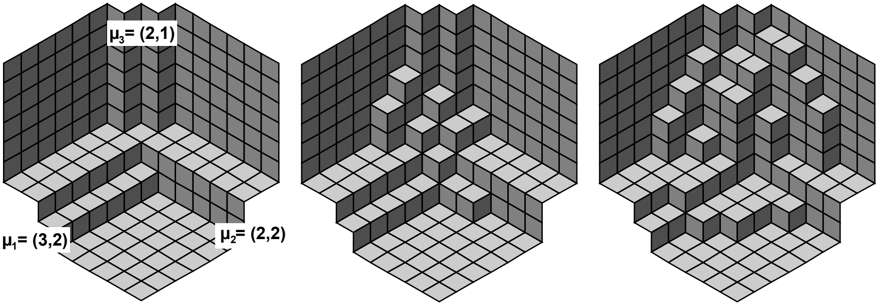

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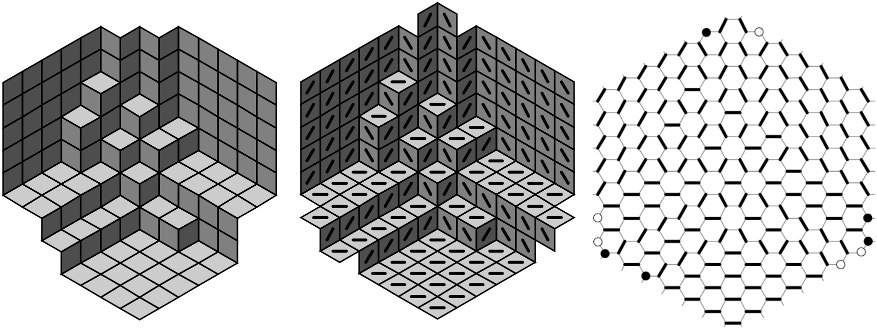
Theorem (J.-Webb-Young, Calabi-Yau case of Conj 4 from PT2009)

$$V(\mu_1, \mu_2, \mu_3) = M(q)W(\mu_1, \mu_2, \mu_3)$$

Dimer model interpretation



$\pi \in P(\mu_1, \mu_2, \mu_3)$ is equivalent to a dimer configuration on the honeycomb graph with some outer vertices removed.

The condensation recurrence in DT theory

Theorem (Kuo04, Theorem 5.1)

Let vertices a, b, c, and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

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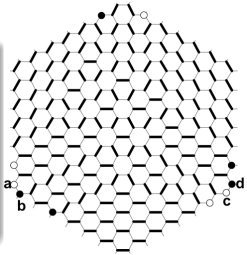

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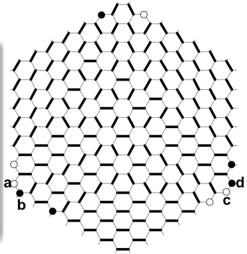

$$q^A V((3,1),(2,1),\mu_3)V((3,2),(2,2),\mu_3) = q^B V((3,2),(2,1),\mu_3)V((3,1),(2,2),\mu_3) + q^C V((2,1,1),(3),\mu_3)V((4),(1,1,1),\mu_3)$$

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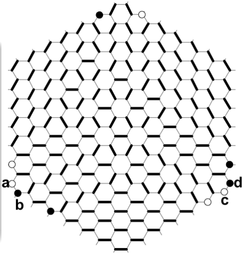

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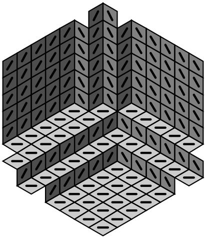

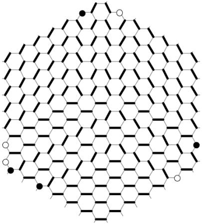

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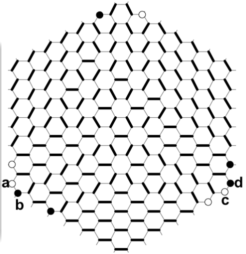

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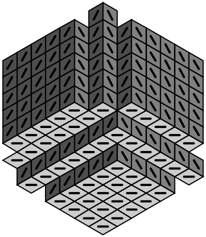

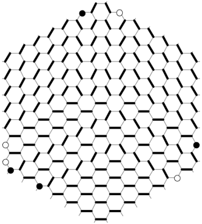

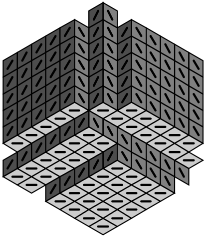

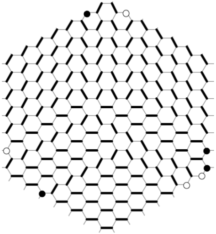

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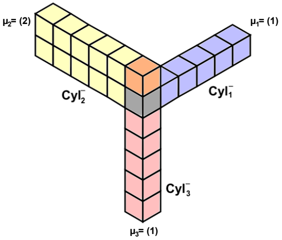
PT box counting



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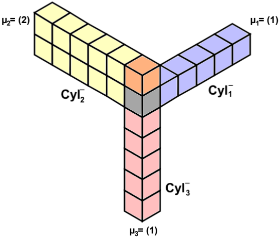
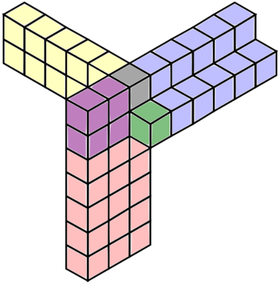
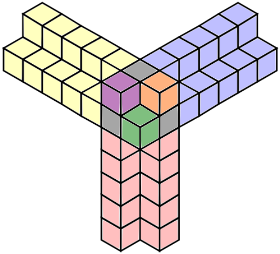
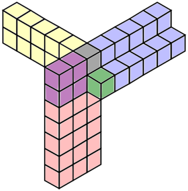


Define the following regions:

- $\mathrm{I}^- = \mathrm{Cyl}_1^- \cup \mathrm{Cyl}_2^- \cup \mathrm{Cyl}_3^-$
- $\mathrm{III} = \mathrm{Cyl}_1 \cap \mathrm{Cyl}_2 \cap \mathrm{Cyl}_3$
- $\mathrm{II} = \bigcup_{i \neq j \neq k} (\mathrm{Cyl}_i \cap \mathrm{Cyl}_j \setminus \mathrm{Cyl}_k)$

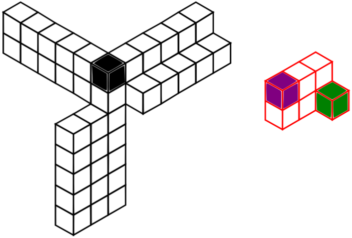






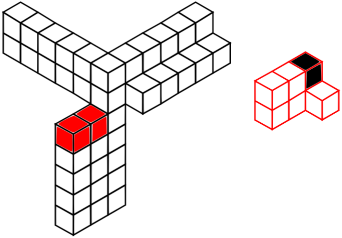
Definition

If $A \subseteq \mathrm{I}^- \cup \mathrm{III}$ and $B \subseteq \mathrm{II} \cup \mathrm{III}$ are finite sets of boxes, then (A, B) is an AB *configuration* if each of A and B satisfies the condition for plane partitions (with gravity pulling in the opposite direction).



A B

Valid AB configuration



A B

Invalid AB configuration
(A is valid, B is not)

Labelled AB configurations

- Some AB configs can be labelled (following an intricate set of rules).
- Define

$$W(\mu_1, \mu_2, \mu_3) = \sum_{\text{labelled } AB \text{ configs}} q^{|A|+|B|}$$

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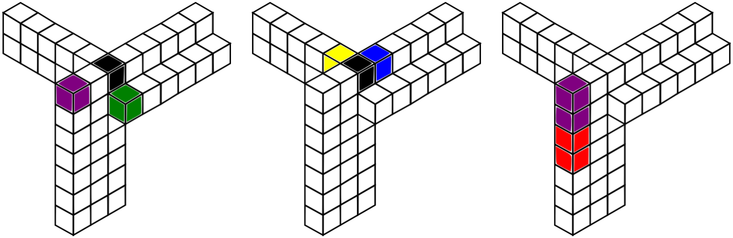
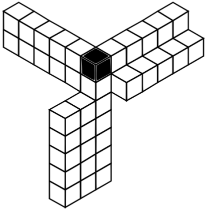
- We prove that labelled AB configurations are a discrete version of *labelled box configurations* used to define the PT topological vertex.



- The observation that a PT labelled box config can be split into an A config and a B config gives a tripartite double-dimer interpretation for the PT topological vertex $W(\mu_1, \mu_2, \mu_3)$

Connection to the tripartite double-dimer model



A configuration

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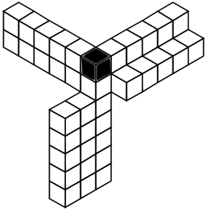

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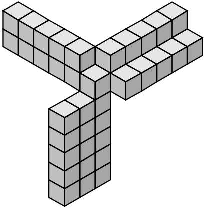

Draw boxes NOT in A

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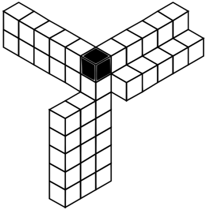

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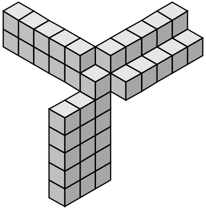

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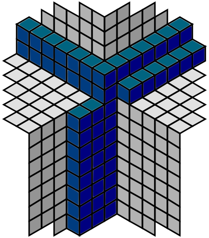
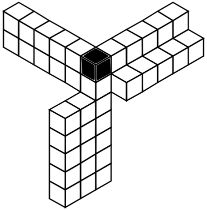


Extend to tiling of the plane

Connection to the tripartite double-dimer model



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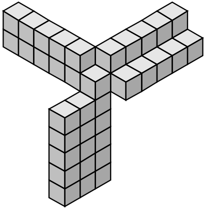

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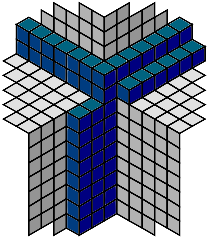
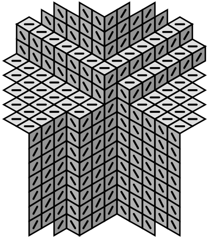
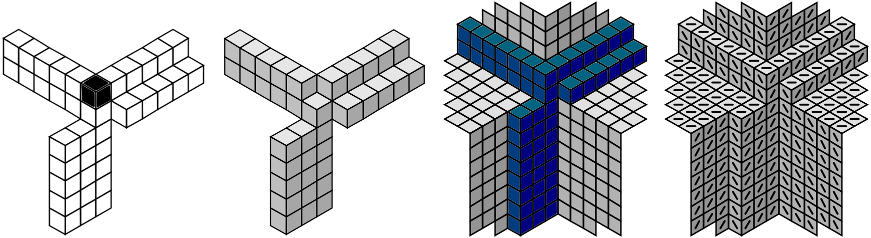


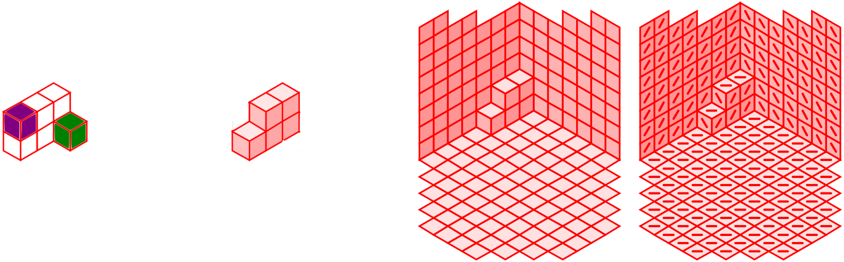


Extend to tiling of the plane

Connection to the tripartite double-dimer model



A configuration | Draw boxes NOT in A | Extend to tiling of the plane



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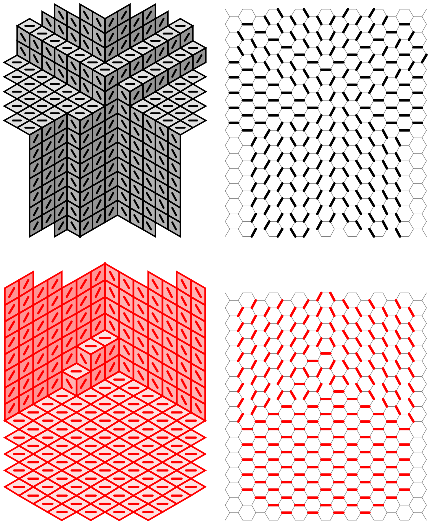

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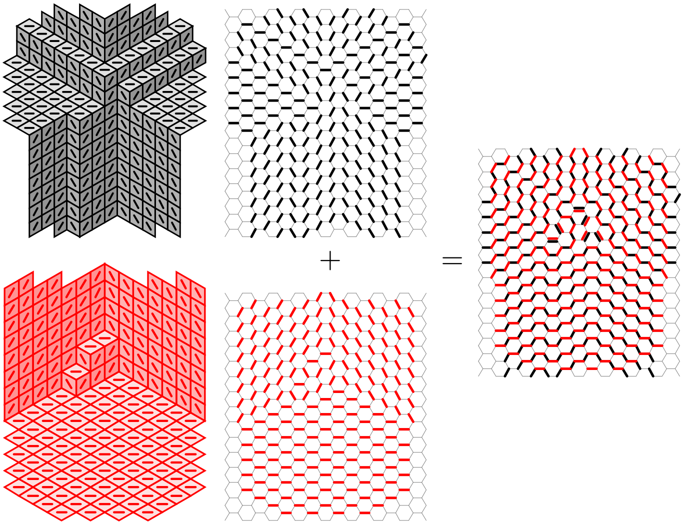

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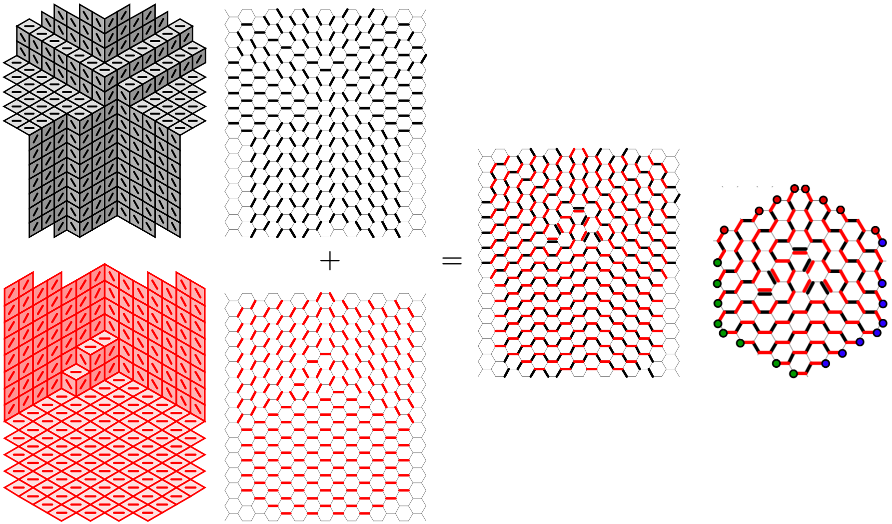

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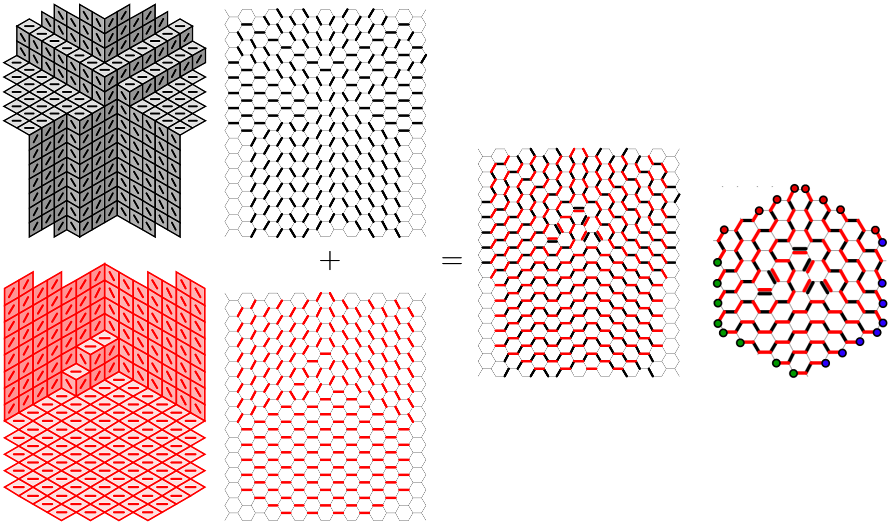

Apply double-dimer condensation by *adding* nodes.

Thank you for listening!

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