## DIVIDED SYMMETRIZATION

 AND
# QUASISYMMETRIC FUNCTIONS 

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## Divided symmetrization

Let $f\left(x_{1}, \ldots, x_{n}\right)$ be a polynomial in $\mathbb{Q}\left[x_{1}, \cdots, x_{n}\right]$, then its divided symmetrization is defined by

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\left\langle f\left(x_{1}, \ldots, x_{n}\right)\right\rangle_{n}:=\quad \frac{f\left(x_{1}, \ldots, x_{n}\right)}{\prod_{1 \leq i \leq n-1}\left(x_{i}-x_{i+1}\right)}
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\left\langle x_{1}^{2}\right\rangle_{2}=\frac{x_{1}^{2}}{x_{1}-x_{2}}+\frac{x_{2}^{2}}{x_{2}-x_{1}}=x_{1}+x_{2} \\
\left\langle x_{2}^{2}\right\rangle_{3}=\frac{x_{2}^{2}}{\left(x_{1}-x_{2}\right)\left(x_{2}-x_{3}\right)}+\frac{x_{1}^{2}}{\left(x_{2}-x_{1}\right)\left(x_{1}-x_{3}\right)}+\frac{x_{2}^{2}}{\left(x_{3}-x_{2}\right)\left(x_{2}-x_{1}\right)}+ \\
+\frac{x_{3}^{2}}{\left(x_{1}-x_{3}\right)\left(x_{3}-x_{2}\right)}+\frac{x_{3}^{2}}{\left(x_{2}-x_{3}\right)\left(x_{3}-x_{1}\right)}+\frac{x_{1}^{2}}{\left(x_{3}-x_{1}\right)\left(x_{1}-x_{2}\right)}=-2
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Lemma: $\left\langle f\left(x_{1}, \ldots, x_{n}\right)\right\rangle_{n}$ is a polynomial in $x_{1}, \cdots, x_{n}$.
(Proof: $\operatorname{Sym}_{n}\left(\frac{f}{\Delta_{n}}\right)=\frac{\operatorname{Anti}_{n}(f)}{\Delta_{n}}$ with $\Delta_{n}=\prod_{1 \leq i<j \leq n}\left(x_{i}-x_{j}\right)$.)

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## Lemma: $\left\langle f\left(x_{1}, \ldots, x_{n}\right)\right\rangle_{n}$ is a polynomial in $x_{1}, \cdots, x_{n}$.

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Let $f$ be homogeneous of degree $\operatorname{deg}(f)=d$. Then $\left\langle f\left(x_{1}, \ldots, x_{n}\right)\right\rangle_{n}=0$ if $d<n-1$, while $\left\langle f\left(x_{1}, \ldots, x_{n}\right)\right\rangle_{n}$ has degree $d-n+1$ if $d \geq n-1$.

Let $R_{n}$ be the component of $\mathbb{Q}\left[x_{1}, \ldots, x_{n}\right]$ of degree $n-1$.

## Origin story

From now on we study $\langle\cdot\rangle_{n}: R_{n} \rightarrow \mathbb{Q}$.
The space $R_{n}$ has a monomial basis given by $\mathbf{x}^{\mathbf{c}}:=x_{1}^{c_{1}} \cdots x_{n}^{c_{n}}$ where $\sum_{i} c_{i}=n-1$, thus has dimension $\binom{2 n-2}{n-1}$.

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Permutahedra: Let $\left(z_{1}, \ldots, z_{n}\right)$ in $\mathbb{R}^{n}$. Define $P_{n}\left(z_{1}, \ldots, z_{n}\right)$ to be the convex hull of all points $\left(z_{\sigma(1)}, \ldots, z_{\sigma(n)}\right)$ with $\sigma \in S_{n}$. Let $V_{n}\left(z_{1}, \ldots, z_{n}\right)$ be the (suitably normalized) volume of $P_{n}\left(z_{1}, \ldots, z_{n}\right)$.

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Theorem [Postnikov '06]
$V_{n}\left(z_{1}, \ldots, z_{n}\right)=\left\langle\left(z_{1} x_{1}+z_{2} x_{2}+\cdots+z_{n} x_{n}\right)^{n-1}\right\rangle_{n}$.
Expanding this, $V_{n}\left(z_{1}, \ldots, z_{n}\right)$ is thus a polynomial in $z_{1}, \ldots, z_{n}$ with coefficients given by the $\left\langle\mathbf{x}^{\mathbf{c}}\right\rangle_{n}$.

$$
\left\langle x^{c}\right\rangle_{n}=\ldots
$$

The evaluation of $\left\langle\mathbf{x}^{\mathbf{c}}\right\rangle_{n}$ is also due to Postnikov. It was reformulated and reproved by Petrov (2016), and we give this version here:

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Given $\mathbf{c} \in \mathbb{N}^{n}$ s.t. $\sum_{i} c_{i}=n-1$, first attach to it a set $S_{\mathbf{c}} \subseteq[n-1]$ as follows: build a path by attaching to $c_{i}$ the step $\left(1, c_{i}-1\right)$, and record the points with negative $y$-coordinates.

$$
\begin{aligned}
& \mathbf{c}=(0,3,0,0,0,1,3,0) \\
& S_{\mathbf{c}}=\{1,4,5,6\}
\end{aligned}
$$


$\beta_{\mathbf{c}}:=\#\left\{w \in S_{n}\right.$ such that for all $i, w_{i}>w_{i+1}$ iff $\left.i \in S_{\mathbf{c}}\right\}$.

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\text { Proposition }\left\langle\mathbf{x}^{\mathbf{c}}\right\rangle_{n}=(-1)^{\left|S_{\mathbf{c}}\right|} \beta_{\mathbf{c}}
$$

For $\mathbf{c}=(0,2,0), S_{\mathbf{c}}=\{1\}$ and $\beta_{\mathbf{c}}=\#\{213,312\}=2$, so $\left\langle x_{2}^{2}\right\rangle_{3}=-2$.

## Quasisymmetric polynomials

If $\mathbf{a}$ is any vector in $\mathbb{N}^{m}$, let $\mathbf{a}^{+}$be the composition obtained by deleting all 0 's in a, so $(0,3,0,0,1,2,0,0,0)^{+}=(3,1,2)$.
A polynomial $P\left(x_{1}, \ldots, x_{m}\right)$ is quasisymmetric if the coefficients of $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ in $P$ are equal whenever $\mathbf{a}^{+}=\mathbf{b}^{+}$.

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Example $M_{\alpha}=$ the sum of all $\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{a}^{+}=\alpha$ for a given composition $\alpha$. Thus $M_{(3,1,2)}=\sum_{i<j<k} x_{i}^{3} x_{j} x_{k}^{2}$.

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Let $f$ be a quasisymmetric function $(f \in \mathrm{QSym})$. We denote by $f\left(x_{1}, \cdots, x_{m}\right)$ the quasisymmetric polynomial obtained by setting $x_{i}=0$ for $i>m$. We then denote by $f\left(1^{j}\right)$ the value of $f\left(x_{1}, \cdots, x_{j}\right)$ at $(1, \ldots, 1)$.

## Evaluation at quasisymmetric polynomials

Theorem [N.-Tewari' '19] For any $f \in \operatorname{QSym}^{(n-1)}$,

$$
\sum_{j \geq 0} f\left(1^{j}\right) t^{j}=\frac{\sum_{m=0}^{n-1}\left\langle f\left(x_{1}, \ldots, x_{m}\right)\right\rangle_{n} t^{m}}{(1-t)^{n}} .
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Remark This gives an expression for $\left\langle f\left(x_{1}, \cdots, x_{m}\right)\right\rangle_{n}$ as a linear combination of the values $f\left(1^{j}\right)$ for $j \leq m$.

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Remark This gives an expression for $\left\langle f\left(x_{1}, \cdots, x_{m}\right)\right\rangle_{n}$ as a linear combination of the values $f\left(1^{j}\right)$ for $j \leq m$.
Proof sketch: 1) By linearity, it is enough to prove the formula for $f=M_{\alpha}$.
2) One can evaluate the I.h.s. and we get the following identity, equivalent to the theorem:
$\left\langle M_{\alpha}\left(x_{1}, \ldots, x_{m}\right)\right\rangle_{n}=(-1)^{m-\ell(\alpha)}\binom{n-1-\ell(\alpha)}{m-\ell(\alpha)}$
3) To prove this identity, one shows first that the I.h.s. only depends on $\ell(\alpha)$, and concludes by evaluating at a special $\alpha$.

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The preceding theorem is especially nice to apply in the case of $F_{\alpha}$, the fundamental quasisymmetric functions.

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\begin{aligned}
& \text { Corollary If }|\alpha|=n-1 \text {, one has }\left\langle F_{\alpha}\left(x_{1}, \ldots, x_{m}\right)\right\rangle_{n}=0 \text { if } \\
& \ell(\alpha)<m \text {, and }=1 \text { if } m=\ell(\alpha) \text {. }
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f=\sum_{\alpha} c_{\alpha} F_{\alpha} \in \operatorname{symm}^{(n-1)} \Rightarrow\left\langle f\left(x_{1}, \ldots, x_{m}\right\rangle_{n}=\sum_{\ell(\alpha)=m} c_{\alpha}\right.
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Origin story (bis) Our study of $\langle\cdot\rangle_{n}$ came from the investigation of the cohomology class of the Peterson variety. Its coefficient $a_{w}$ on the Schubert class $\sigma_{w}$ is given precisely by $\left\langle\mathfrak{S}_{w}\right\rangle_{n}$, where $\mathfrak{S}_{w}$ is a Schubert polynomial.
Now if $w$ is a Grassmannian permutation, then $\mathfrak{S}_{w}$ is a Schur polynomial $s_{\lambda}\left(x_{1}, \ldots, x_{m}\right)$. We can then apply the result above to show that $a_{w}$ is the number of standard tableaux of shape $\lambda$ with $m-1$ descents.

## Quotienting by quasisymmetric polynomials

If $f \in R_{n}$ has a homogeneous, symmetric factor of positive degree, then $\langle f\rangle_{n}=0$.
$\Rightarrow$ By linearity, $\langle\cdot\rangle_{n}$ vanishes on $R_{n} \cap I_{n}$ where $I_{n} \subset \mathbb{Q}\left[\mathbf{x}_{n}\right]$ is the ideal generated by homogeneous symmetric polynomials in $x_{1}, \ldots, x_{n}$ of positive degree.

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Theorem [ N .-Tewari '19] The form $\langle\cdot\rangle_{n}$ vanishes on $R_{n} \cap J_{n}$ where the ideal $J_{n} \subset \mathbb{Q}\left[\mathbf{x}_{n}\right]$ is generated by homogeneous quasisymmetric polynomials in $x_{1}, \ldots, x_{n}$ of positive degree.

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By the work of Aval-Bergeron-Bergeron (2004), one has

$$
R_{n}=\left(R_{n} \cap J_{n}\right) \oplus \operatorname{Vect}\left(\mathbf{x}^{\mathbf{c}} \mid S_{\mathbf{c}}=\emptyset\right) .
$$

Corollary Write $f \in R_{n}$ as $f=g+h$ according to the decomposition above. Then $\langle f\rangle_{n}=h(1, \ldots, 1)$.

