DIVIDED SYMMETRIZATION AND QUASISYMMETRIC FUNCTIONS

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Let $f(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{Q}[x_1, \cdots, x_n]$, then its **divided symmetrization** is defined by

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$$\langle x_1^2 \rangle_2 = \frac{x_1^2}{x_1 - x_2} + \frac{x_2^2}{x_2 - x_1} = x_1 + x_2$$

$$\langle x_2^2 \rangle_3 = \frac{x_2^2}{(x_1 - x_2)(x_2 - x_3)} + \frac{x_1^2}{(x_2 - x_1)(x_1 - x_3)} + \frac{x_2^2}{(x_3 - x_2)(x_2 - x_1)} +$$

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Lemma: $\langle f(x_1, \dots, x_n) \rangle_n$ is a polynomial in x_1, \dots, x_n . (Proof: $\operatorname{Sym}_n(\frac{f}{\Delta_n}) = \frac{\operatorname{Anti}_n(f)}{\Delta_n}$ with $\Delta_n = \prod_{1 \le i < j \le n} (x_i - x_j)$.)

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Let f be homogeneous of degree $\operatorname{deg}(f) = d$. Then
 $\langle f(x_1, \ldots, x_n) \rangle_n = 0$ if $d < n - 1$, while $\langle f(x_1, \ldots, x_n) \rangle_n$ has
degree $d - n + 1$ if $d \ge n - 1$.

Let R_n be the component of $\mathbb{Q}[x_1, \ldots, x_n]$ of degree n-1.

Origin story

From now on we study $\langle \cdot \rangle_n : R_n \to \mathbb{Q}$.

The space R_n has a monomial basis given by $\mathbf{x}^{\mathbf{c}} \coloneqq x_1^{c_1} \cdots x_n^{c_n}$ where $\sum_i c_i = n - 1$, thus has dimension $\binom{2n-2}{n-1}$.

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Permutahedra: Let (z_1, \ldots, z_n) in \mathbb{R}^n . Define $P_n(z_1, \ldots, z_n)$ to be the convex hull of all points $(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ with $\sigma \in S_n$. Let $V_n(z_1, \ldots, z_n)$ be the (suitably normalized) volume of $P_n(z_1, \ldots, z_n)$.

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Theorem [Postnikov '06] $V_n(z_1,...,z_n) = \langle (z_1x_1 + z_2x_2 + \cdots + z_nx_n)^{n-1} \rangle_n.$

Expanding this, $V_n(z_1, \ldots, z_n)$ is thus a polynomial in z_1, \ldots, z_n with coefficients given by the $\langle \mathbf{x}^{\mathbf{c}} \rangle_n$.

 $\langle \mathbf{x}^{\mathbf{c}} \rangle_n = \dots$

The evaluation of $\langle \mathbf{x}^{\mathbf{c}} \rangle_n$ is also due to Postnikov. It was reformulated and reproved by Petrov (2016), and we give this version here:

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Given $\mathbf{c} \in \mathbb{N}^n$ s.t. $\sum_i c_i = n - 1$, first attach to it a set $S_{\mathbf{c}} \subseteq [n-1]$ as follows: build a path by attaching to c_i the step $(1, c_i - 1)$, and record the points with negative y-coordinates.



 $\beta_{\mathbf{c}} \coloneqq \# \{ w \in S_n \text{ such that for all } i, w_i > w_{i+1} \text{ iff } i \in S_{\mathbf{c}} \}.$

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 $\beta_{\mathbf{c}} \coloneqq \# \{ w \in S_n \text{ such that for all } i, w_i > w_{i+1} \text{ iff } i \in S_{\mathbf{c}} \}.$ $Proposition \langle \mathbf{x}^{\mathbf{c}} \rangle_n = (-1)^{|S_{\mathbf{c}}|} \beta_{\mathbf{c}}.$

For $\mathbf{c} = (0, 2, 0)$, $S_{\mathbf{c}} = \{1\}$ and $\beta_{\mathbf{c}} = \#\{213, 312\} = 2$, so $\langle x_2^2 \rangle_3 = -2$.

If a is any vector in \mathbb{N}^m , let \mathbf{a}^+ be the composition obtained by deleting all 0's in a, so $(0,3,0,0,1,2,0,0,0)^+ = (3,1,2)$.

A polynomial $P(x_1, \ldots, x_m)$ is **quasisymmetric** if the coefficients of $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ in P are equal whenever $\mathbf{a}^+ = \mathbf{b}^+$.

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Example M_{α} = the sum of all $\mathbf{x}^{\mathbf{a}}$ such that $\mathbf{a}^{+} = \alpha$ for a given composition α . Thus $M_{(3,1,2)} = \sum_{i < j < k} x_{i}^{3} x_{j} x_{k}^{2}$.

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Let f be a quasisymmetric function ($f \in QSym$). We denote by $f(x_1, \dots, x_m)$ the quasisymmetric polynomial obtained by setting $x_i = 0$ for i > m. We then denote by $f(1^j)$ the value of $f(x_1, \dots, x_j)$ at $(1, \dots, 1)$.

Theorem [N.-Tewari '19] For any
$$f \in \operatorname{QSym}^{(n-1)}$$
,

$$\sum_{j\geq 0} f(1^j)t^j = \frac{\sum_{m=0}^{n-1} \langle f(x_1, \dots, x_m) \rangle_n t^m}{(1-t)^n}.$$

Remark This gives an expression for $\langle f(x_1, \cdots, x_m) \rangle_n$ as a linear combination of the values $f(1^j)$ for $j \leq m$.

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Proof sketch: 1) By linearity, it is enough to prove the formula for $f = M_{\alpha}$.

2) One can evaluate the l.h.s. and we get the following identity, equivalent to the theorem:

 $\langle M_{\alpha}(x_1, \ldots, x_m) \rangle_n = (-1)^{m-\ell(\alpha)} {\binom{n-1-\ell(\alpha)}{m-\ell(\alpha)}}$ 3) To prove this identity, one shows first that the l.h.s. only depends on $\ell(\alpha)$, and concludes by evaluating at a special α .

The preceding theorem is especially nice to apply in the case of F_{α} , the **fundamental quasisymmetric functions**.

Corollary If $|\alpha| = n - 1$, one has $\langle F_{\alpha}(x_1, \dots, x_m) \rangle_n = 0$ if $\ell(\alpha) < m$, and = 1 if $m = \ell(\alpha)$.

$$f = \sum_{\alpha} c_{\alpha} F_{\alpha} \in Qsym^{(n-1)} \Rightarrow \left\langle f(x_1, \dots, x_m) \right\rangle_n = \sum_{\ell(\alpha) = m} c_{\alpha}$$

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Origin story (bis) Our study of $\langle \cdot \rangle_n$ came from the investigation of the cohomology class of the Peterson variety. Its coefficient a_w on the Schubert class σ_w is given precisely by $\langle \mathfrak{S}_w \rangle_n$, where \mathfrak{S}_w is a *Schubert polynomial*. Now if w is a Grassmannian permutation, then \mathfrak{S}_w is a Schur polynomial $s_\lambda(x_1, \ldots, x_m)$. We can then apply the result above to show that a_w is the number of standard tableaux of shape λ with m-1 descents.

Quotienting by quasisymmetric polynomials

If $f \in R_n$ has a homogeneous, symmetric factor of positive degree, then $\langle f \rangle_n = 0$.

 \Rightarrow By linearity, $\langle \cdot \rangle_n$ vanishes on $R_n \cap I_n$ where $I_n \subset \mathbb{Q}[\mathbf{x}_n]$ is the ideal generated by homogeneous symmetric polynomials in x_1, \ldots, x_n of positive degree.

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By the work of Aval-Bergeron-Bergeron (2004), one has

$$R_n = (R_n \cap J_n) \oplus \operatorname{Vect}(\mathbf{x}^{\mathbf{c}} \mid S_{\mathbf{c}} = \emptyset).$$

Corollary Write $f \in R_n$ as f = g + h according to the decomposition above. Then $\langle f \rangle_n = h(1, ..., 1)$.