

Tightening Curves on Surfaces Monotonically and Applications

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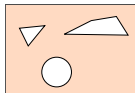
Université Gustave Eiffel, France



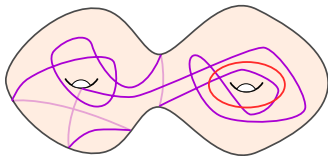
Picture by Quibe

Surfaces and embedded graphs

- A **surface** is a space that looks locally like the plane.
- In this talk we care about connected, orientable surfaces, which are classified by their number of holes (**genus**) and their number of boundaries.

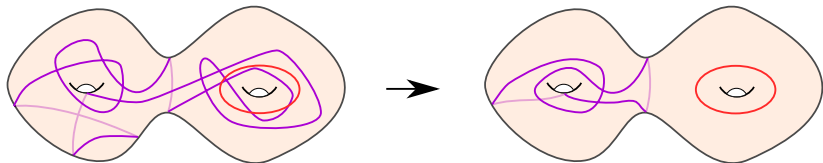


- A **closed curve** on a surface S is a continuous map $S^1 \rightarrow S$.
- We consider closed curves in **general position**, e.g., with a finite number of transverse intersections.



Tightening curves

- Given a family of closed curves on a surface, we want to *tighten* them, i.e., deform them continuously until they have a minimum number of crossings.

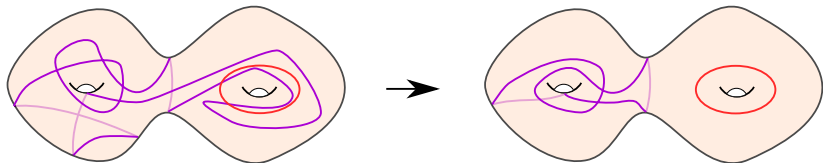


- This can be done using isotopies and a finite set of *homotopy moves*:



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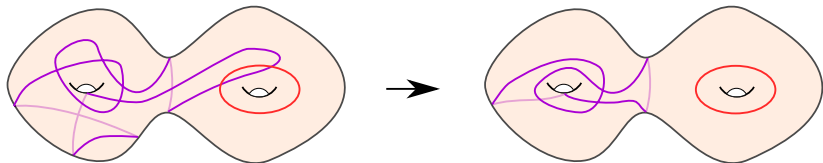


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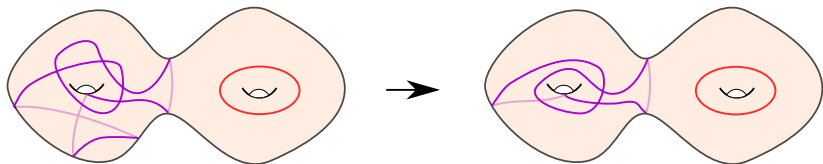


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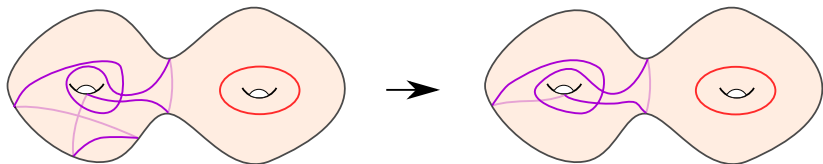


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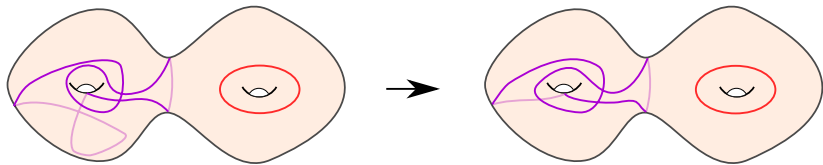


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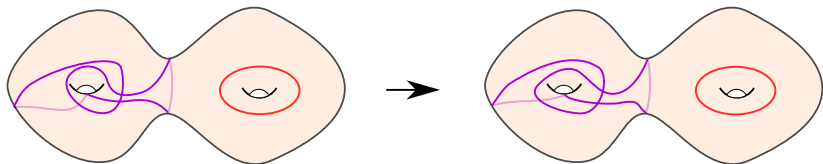


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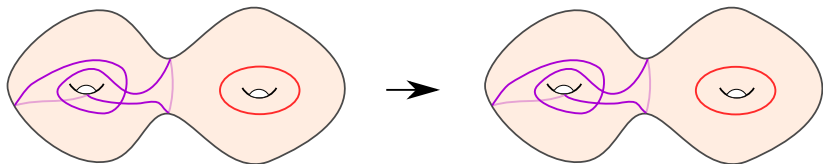


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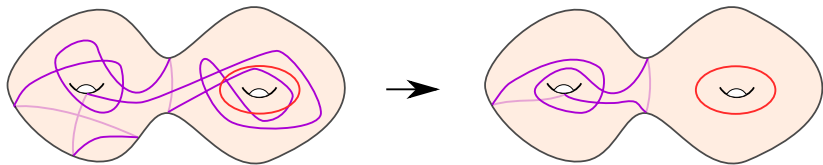
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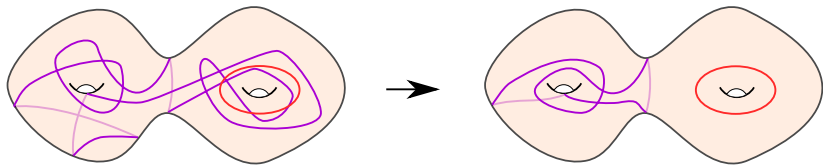
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- 2 How many moves are needed?
- 3 Is it ever needed to increase the number of crossings?

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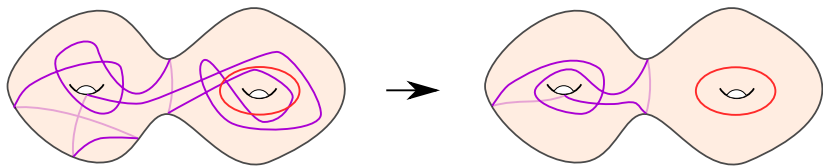
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Theorem (Hass and Scott '94, de Graaf and Schrijver '97)

There exists a *monotone* sequence of homotopy moves to tighten any collection of closed curves.



Questions

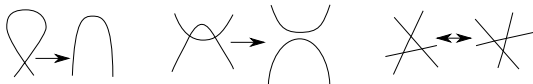


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- 2 How many *monotone* moves are needed?
- 3 Is it ever needed to increase the number of crossings? No.

Theorem (Hass and Scott '94, de Graaf and Schrijver '97)

*There exists a **monotone** sequence of homotopy moves to tighten any collection of closed curves.*



Theorem (Chang, dM '20)

$\tilde{O}(n^5/g^2 + gn^3)$ monotone homotopy moves are sufficient to tighten any collection of curves with n crossings on a surface* of genus g .

These moves can be computed in the same time, thus giving a polynomial-time algorithm.

Previously known:

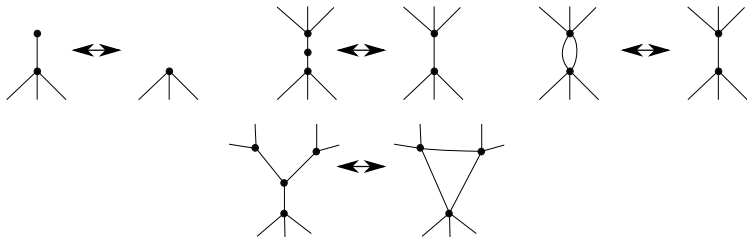
- Exponential bound on monotone homotopy moves.
- Polynomial-time algorithm to compute minimal position of a *single* curve [Despré Lazarus '17].
- Polynomial bound on homotopy moves for a *single* curve but not monotone [Chang Erickson Letscher dM Schleimer Sedgwick Tillmann Thurston '18].

*: except the torus (!)

*Why would I care about monotone homotopy moves for
multiple curves?*

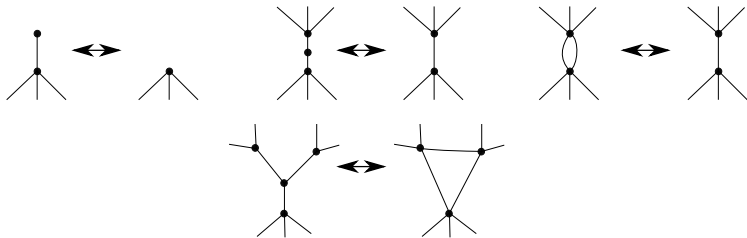
Electrical moves

Take a graph G embedded on a surface. Allow it to be modified with facial *electrical moves*:

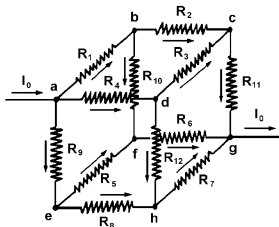


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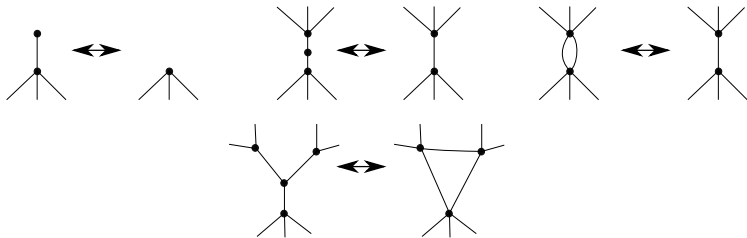


- These moves are useful for *resistor networks* [Kennelly 1899],

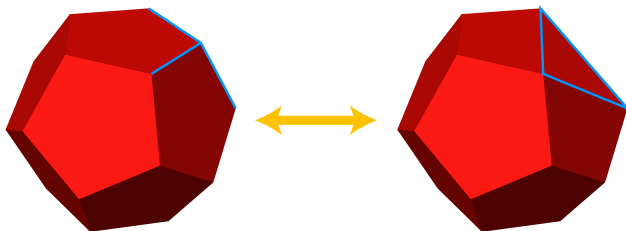


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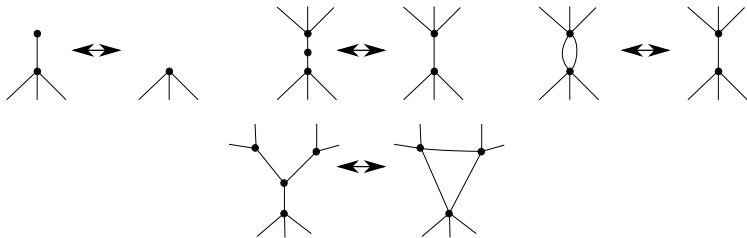


- These moves are useful for *polyhedra* [Steinitz 1916],

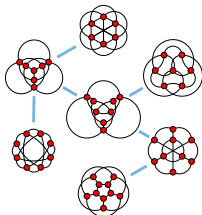


Electrical moves

Take a graph G embedded on a surface. Allow it to be modified with facial *electrical moves*:



- These moves are useful for *structural graph theory* [Robertson Seymour Thomas 1993]...



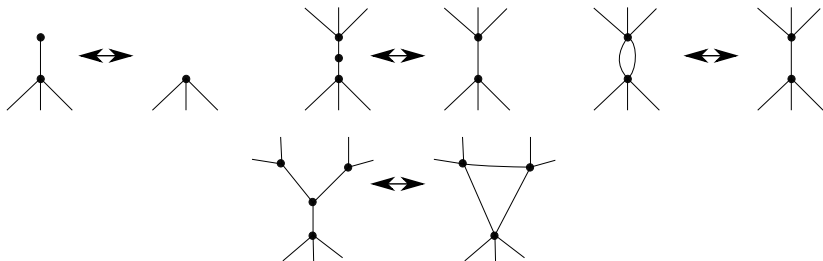
Minimizing graphs with electrical moves

How to make a graph as small as possible with electrical moves?

Previous work: planar graphs with at most 4 terminals [Demasi Mohar 2015].

Topological approach:

- Consider the *medial graph*.
- Vertices have degree four \rightarrow family of curves.
- Almost like homotopy moves!



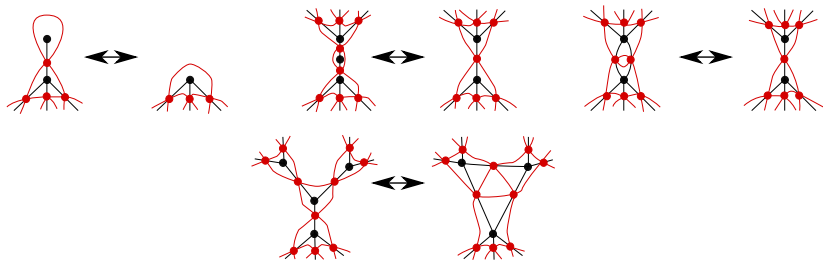
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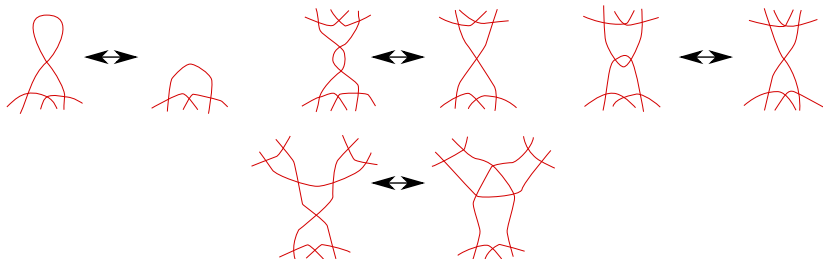
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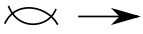



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Minimizing electrical graphs in polynomial time

One solution:

- 1 Use **monotone** homotopy moves to tighten the curves of the medial graph,
- 2 but instead of  \rightarrow , do  \rightarrow ,
- 3 restart the tightening with the new family of curves, loop to (1),
- 4 end when the family of curves that is tight.

Complexity

The number of crossings never increases, and, using our theorem, step (1) is polynomially long,
 \rightarrow polynomial number of moves.



That would not work with a non-monotone sequence of moves.

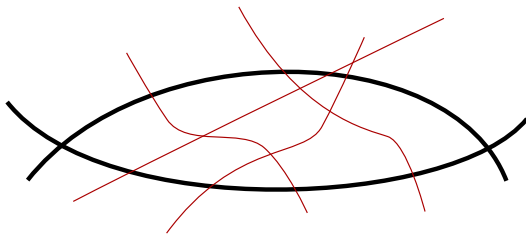
Theorem (Chang Cossarini Erickson '19)

If a family of curves is tight for homotopy moves, it is also tight for electrical moves.

Back to tightening curves monotonically

Friends...



- When tightening curves, empty monogons  and bigons  can be removed instantly.
- But what if there is stuff inside?

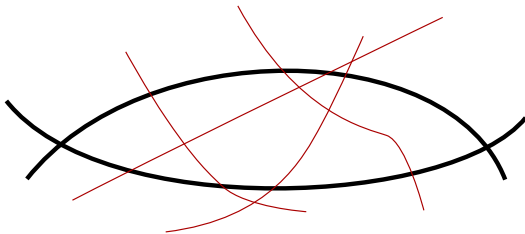


Theorem (Steinitz 1916)

A minimal bigon always has a triangle incident to one of the two curves, and thus can be emptied and removed in a linear number of moves.

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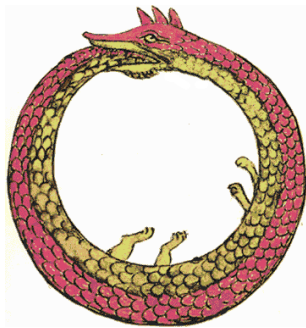
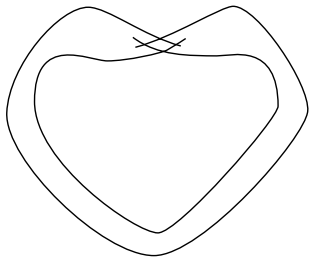
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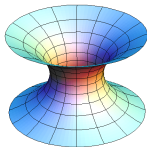
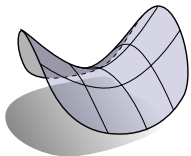
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But when the bigon is not embedded, things get tricky:



- A **hyperbolic metric** on a surface is a metric with constant negative curvature, i.e., points are locally isometric to saddles.



Theorem

Any surface of genus g with b boundaries such that $2g + b > 2$ can be endowed with a hyperbolic metric.

In this talk, we focus on those surfaces with $2g + b > 2$.

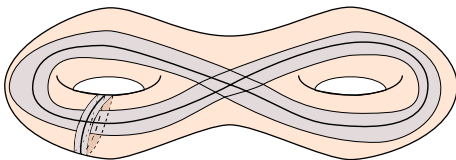
- A **geodesic** is a closed curve minimizing lengths locally.

Hyperbolic geometry helps a lot because, with such a metric

- Every closed curve is homotopic to a unique geodesic, and
- primitive Geodesics are in minimal position.

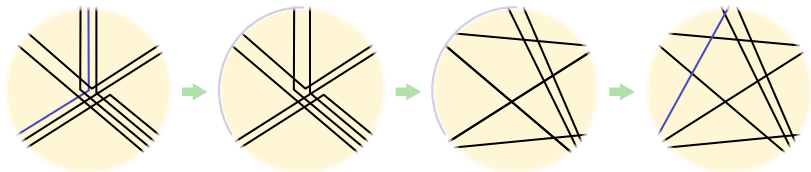
Our first step is thus to endow S with a (well chosen) hyperbolic metric and *straighten* the curves for that metric.

Pushing all the way to geodesics is too expensive but we can push curves to a small neighborhood of the geodesics.



- Everything outside of this small neighborhood will never be needed in the homotopy. \rightarrow Surface with boundary.

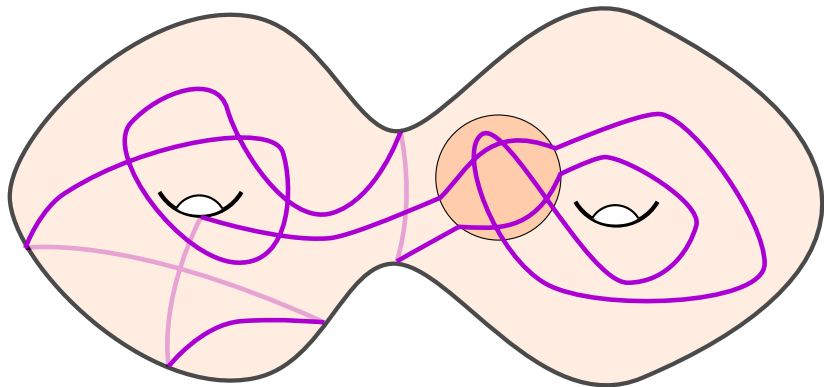
To push the curves towards their geodesics, we straighten paths iteratively in *disks*.



Theorem (Ringel 1956)

Paths in a Euclidean or hyperbolic disk can be straightened in a polynomial number of monotone homotopy moves.

Straightening



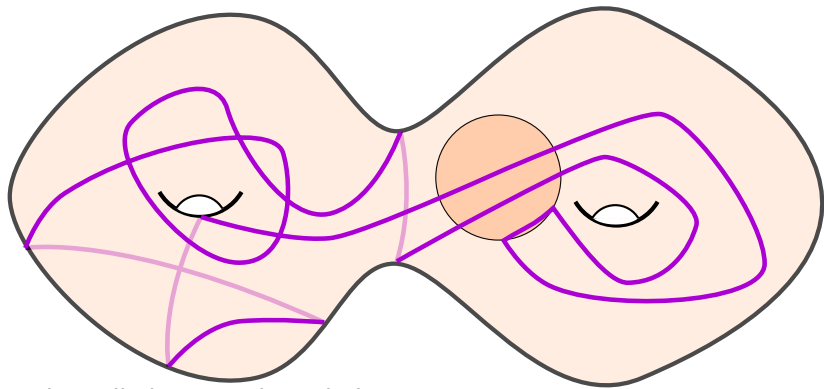
Disks well chosen + hyperbolic computations

- polynomial number of rounds to get to a small neighborhood of the geodesics that does not cover the whole surface.
- Reduction to the case of a surface with boundary.

Caveat

Does not work for the torus.

Straightening



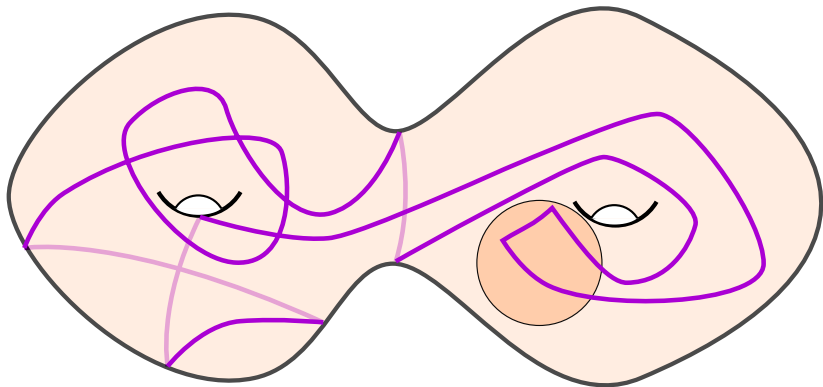
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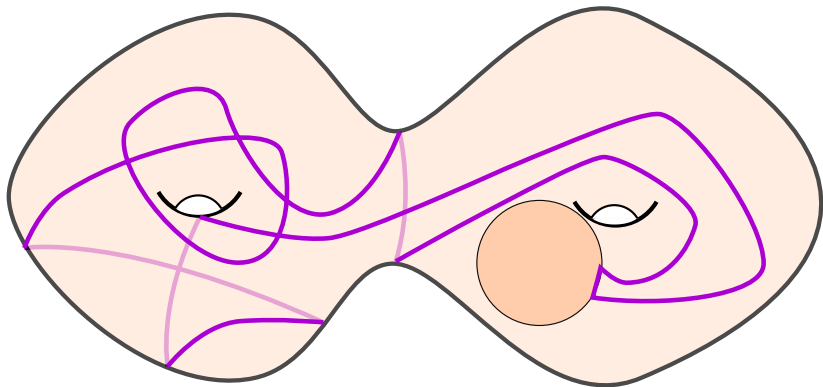
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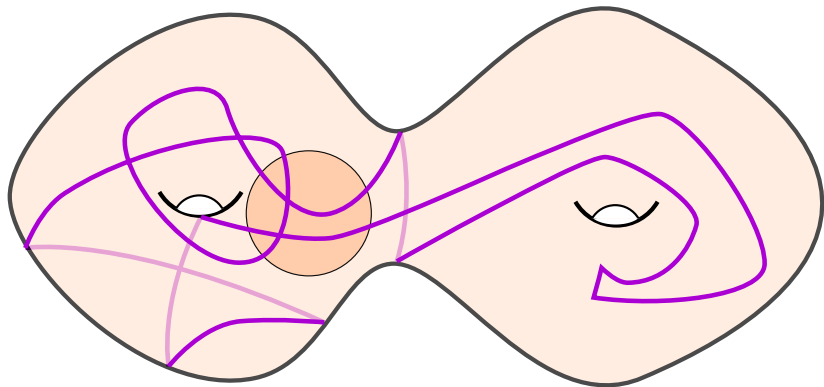
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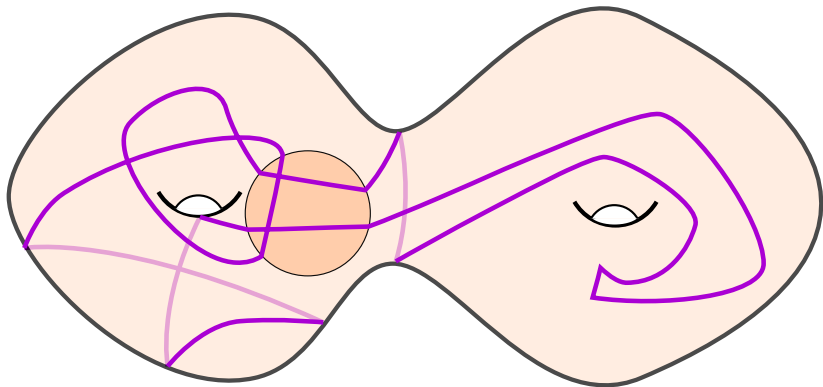
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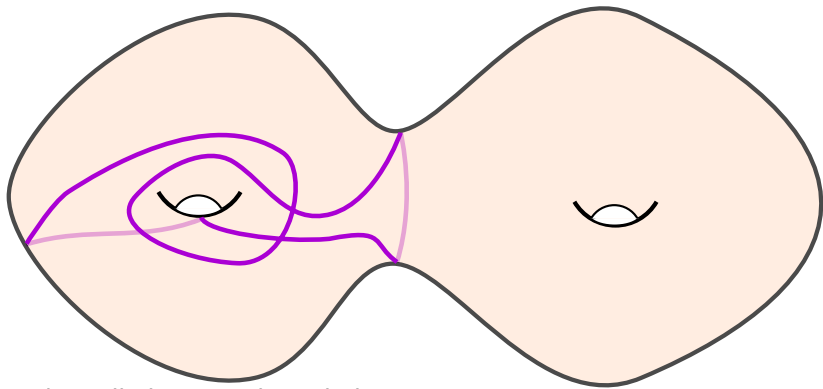
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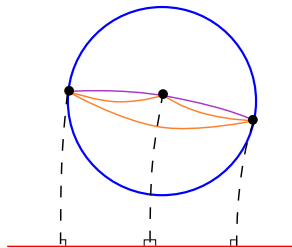
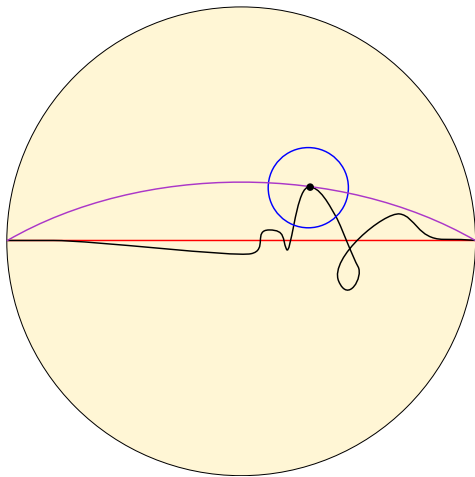
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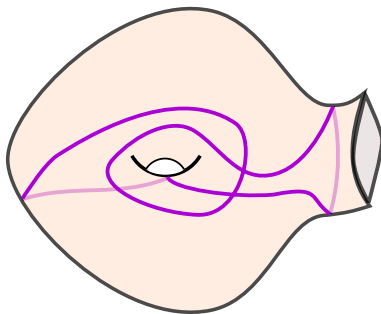
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Hyperbolic considerations



In hyperbolic geometry, a curve equidistant to a geodesic is **not** a geodesic. This is what allows us to quantify our progress.



Disks well chosen + hyperbolic computations

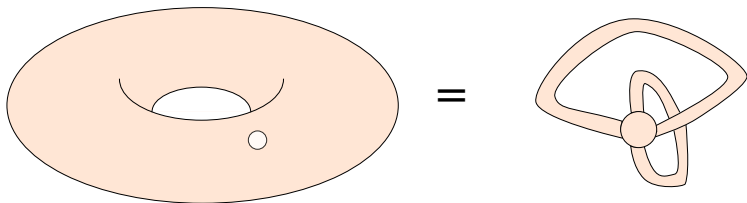
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Surfaces with boundary

Topologically, a surface with boundary is like the neighborhood of a graph.



- Curves on the surface correspond to walks on the graph.
- We can tinker the curves so that the walks contain no spurs and all the crossings happen in the *clusters* (=vertices) of the graph.

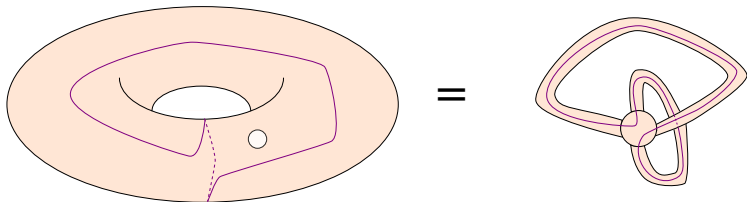
These walks are *topological* invariants of the curves.

Main tool

Cluster and pipe expansions from *graph drawing* [Cortese di Battista Patrignani Pizzonia '09]

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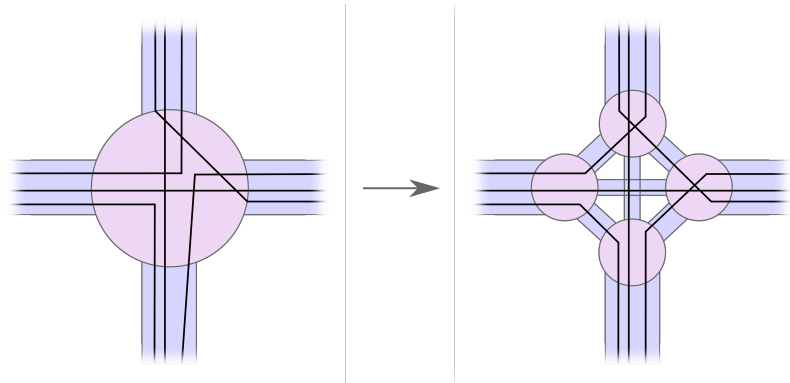
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Cluster expansions

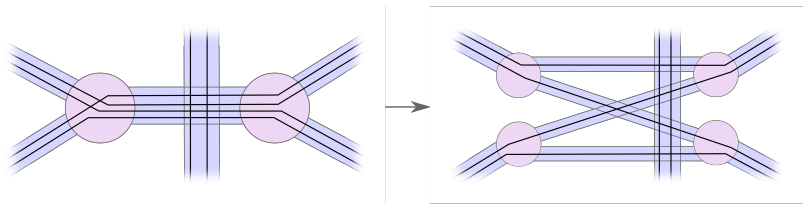
Straighten the curve in the clusters, and *expand* the clusters.



Morally, it is safe to do so because the walks force the situation at the cluster.

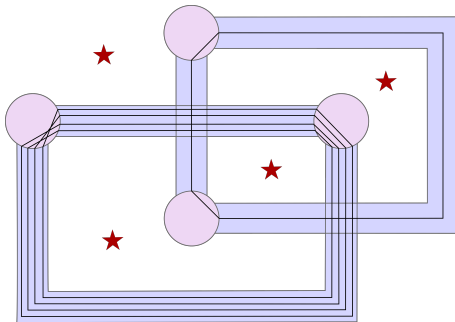
Pipe expansions

Then *expand* the *pipes* (=edges).



Some potential drops at each step...

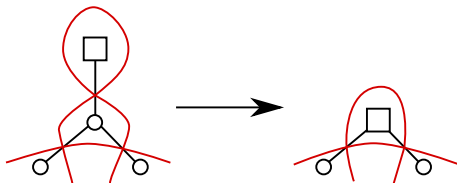
...and after a polynomial number of rounds, the curves do not branch at the clusters anymore, i.e., look like braids.



Now, the situation is roughly the same as for curves in an annulus, for which a quadratic bound exists [Geck and Pfeiffer '93, de Graaf and Schrijver '97].

Concluding words

Monster in the closet: *terminal-leaf reductions*



Two open problems:

- 1 Solve the case of the torus.
- 2 Obtain tight bounds, notably:

Feo-Provan Conjecture

In the plane, the number of monotone homotopy moves to tighten a collection of closed curves/electrical moves to reduce a graph to a point is $\Theta(n^{3/2})$.

The non-monotone lower bound and upper bound are known [Chang Erickson '17].

