# Universality for local limits of high genus maps 

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image: N. Curien

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## Definition : maps

Map $=$ gluing of polygons along their edges to create a (compact, connected, oriented) surface
Genus $g$ of the map $=$ genus of the surface $=\#$ of handles Rooted $=$ an oriented edge is distinguished


## Definition : triangulations and bipartite maps

Triangulations $=$ only triangles
Bipartite maps $=$ polygons have bicolored vertices


## Asymptotic properties of random maps

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Map inclusion : $t \subset \mathbb{T}$


Local limit:
$T_{n} \rightarrow \mathbb{T}$ iff for all finite $t: \mathbb{P}\left(t \subset T_{n}\right) \rightarrow \mathbb{P}(t \subset \mathbb{T})$

## The UIPT

Let $T_{n}$ be a random uniform triangulation of the sphere with $2 n$ triangles
[Angel, Schramm '02] : the sequence $T_{n}$ converges to the Uniform Infinite Planar Triangulation (UIPT).


## High genus triangulations

Let $\frac{g_{n}}{n} \rightarrow \theta$ with $\theta \in\left[0, \frac{1}{2}[\right.$.
Let $\left(T_{n}\right)$ be a sequence of random triangulations, such that $T_{n}$ is drawn uniformly among all triangulations of genus $g_{n}$ with $2 n$ triangles.

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Conjecture [Benjamini, Curien '12]
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Theorem [Budzinski, L. '18+] : the conjecture of Benjamini and Curien is true.

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## Universality

Universality: same phenomenon for different models
For maps, a good "universal" model is bipartite maps with prescribed degrees : for all $i$, a fraction $\alpha_{i}$ of the faces have size $2 i$.

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High genus case : Budzinski-L. '20+ (in progress)

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Obtain precise asymptotics for $\tau(n, g)$ (the number of maps of genus $g$ with $2 n$ triangles) as $\frac{g}{n} \rightarrow \theta$ (and same for bipartite maps)

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## Basic tools

Spatial Markov property of the PSHT : $\mathbb{P}\left(t \subset \mathbb{T}_{\lambda}\right)=C_{p} \lambda^{v}$

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p=\text { perimeter, } v=\text { volume }
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For bipartite maps, a similar peeling algorithm/spatial Markov property exists

## Basic tools II

Enumeration : recurrence formula for triangulations [Goulden-Jackson '08]

$$
\begin{aligned}
(n+1) \tau(n, g)= & 4 n(3 n-2)(3 n-4) \tau(n-2, g-1)+4(3 n-1) \tau(n-1, g) \\
& +4 \sum_{i+j=n-2} \sum_{g_{1}+g_{2}=g}(3 i+2)(3 j+2) \tau\left(i, g_{1}\right) \tau\left(j, g_{2}\right)
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For bipartite maps with prescribed face degrees: [L. '19]

$$
\binom{n+1}{2} B_{g}(\mathbf{f})=\sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f} \\ \mathbf{s}, \mathbf{f}=\mathbf{0} \\ g_{1}+g_{2}+g^{*}=g}}\left(1+n_{1}\right)\binom{v_{2}}{2 g^{*}+2} B_{g_{1}}(\mathbf{s}) B_{g_{2}}(\mathbf{t})+\sum_{g^{*} \geq 0}\binom{v+2 g^{*}}{2 g^{*}+2} B_{g-g *}(\mathbf{f})
$$

$B_{g}(\mathbf{f})=$ number of bipartite maps of genus $g$ with $f_{i}$ faces of size $2 i$, $\mathbf{f}=\left(f_{1}, f_{2}, \ldots\right)$

## The proof

1) In every subsequence, there is a converging subsubsequence (aka tightness)
2) What are the possible limits ?
3) There is only one possible limit

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For bipartite maps : similar strategy of proof, but in HARD MODE

## 1) Tightness - how do we prove it ?

As with the UIPT, equivalent to show that the degree of the root vertex is a.s. finite

Key tool : bounded ratio lemma : for $\frac{g}{n}<\frac{1}{2}-\varepsilon$, there is a constant $C_{\varepsilon}$ s.t. :

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\frac{\tau(n, g)}{\tau(n-1, g)}<C_{\varepsilon}
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For bipartite maps: for $\frac{g}{n}<\frac{1}{2}-\varepsilon$ and $\frac{f_{i}}{n}>\varepsilon$ :

$$
\frac{B_{g}(\mathbf{f})}{B_{g}\left(\mathbf{f}-\mathbf{1}_{i}\right)}<\tilde{C}_{\varepsilon}
$$

## The bounded ratio lemma

For $\frac{g}{n}<\frac{1}{2}-\varepsilon, \frac{\tau(n, g)}{\tau(n-1, g)}<C_{\varepsilon}$
There are $n+2-2 g \geq 2 \varepsilon n$ vertices, and the average degree is
$\frac{6 n}{n+2-2 g} \leq \frac{3}{\varepsilon}$
$\rightarrow$ there are $\geq \varepsilon n$ vertices of degree
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Thus $\varepsilon n \tau(n, g) \leq \frac{6}{\varepsilon} \cdot 6 n \tau(n-1, g)$
For bipartite maps : way more complicated, less local, more "destructive"

## One-endedness and planarity



Proven by using the recurrence formulas (and the bounded ratio lemma)

## 2) The potential limits

Weak spatial Markov property:
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## Theorem: [Budzinski-L. '19]

if $\mathbb{T}$ is a random infinite planar one ended triangulation satisfying the weak Markov property, then $\mathbb{T}$ is a PSHT with a random parameter $\Lambda$

Proof involves the peeling algorithm and the Hausdorff moments method

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For bipartite maps, same theorem, but this time we deal with an infinity of parameters instead of just one.

## 3) Unicity of the potential limit

Two holes argument :
The hyperbolicity is uniform all around the map


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Two holes argument for bipartite maps :
The hyperbolicity and the proportion of faces of each size are uniform

Finishing the proof : completely different argument, way more complicated

## Bonus: asymptotics

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\frac{B_{g_{n}}(\mathbf{f})}{B_{g_{n}}\left(\mathbf{f}-\mathbf{1}_{i}\right)} \rightarrow c_{i}\left(\theta,\left(\alpha_{j}\right)_{j \geq 0}\right) \\
B_{g_{n}}(\mathbf{f})=n^{2 g_{n}} \exp \left(n f\left(\theta,\left(a_{j}\right)_{j \geq 1}+o(n)\right)\right.
\end{gathered}
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Thank you!

