Universality for local limits of high genus maps

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joint work with Thomas Budzinski



image : N. Curien

Work supported by the grant ERC – Stg 716083 – "CombiTop"

Definition : maps

Map = gluing of polygons along their edges to create a (compact, connected, oriented) surface Genus g of the map = genus of the surface = # of handles

Rooted = an oriented edge is distinguished



Definition : triangulations and bipartite maps

Triangulations = only triangles Bipartite maps = polygons have bicolored vertices





Asymptotic properties of random maps

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Local limit:

$$T_n \to \mathbb{T}$$
 iff for all finite $t : \mathbb{P}(t \subset T_n) \to \mathbb{P}(t \subset \mathbb{T})$

The UIPT

Let T_n be a random uniform triangulation of the sphere with 2n triangles

[Angel, Schramm '02] : the sequence T_n converges to the Uniform Infinite Planar Triangulation (UIPT).



High genus triangulations

Let $\frac{g_n}{n} \to \theta$ with $\theta \in [0, \frac{1}{2}[$. Let (T_n) be a sequence of random triangulations, such that T_n is drawn uniformly among all triangulations of genus g_n with 2n triangles.

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Conjecture [Benjamini, Curien '12]

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Theorem [Budzinski, L. '18+] : the conjecture of Benjamini and Curien is true.



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Universality : same phenomenon for different models

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High genus case : Budzinski-L. '20+ (in progress)

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For bipartite maps, a similar peeling algorithm/spatial Markov property exists

Basic tools II

Enumeration : recurrence formula for triangulations [Goulden-Jackson '08]

$$(n+1)\tau(n,g) = 4n(3n-2)(3n-4)\tau(n-2,g-1) + 4(3n-1)\tau(n-1,g) +4\sum_{i+j=n-2}\sum_{g_1+g_2=g}(3i+2)(3j+2)\tau(i,g_1)\tau(j,g_2)$$

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For bipartite maps with prescribed face degrees : [L. '19]

$$\binom{n+1}{2}B_g(\mathbf{f}) = \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\\mathbf{s},\mathbf{t}\neq\mathbf{0}\\g_1+g_2+g^*=g}} (1+n_1)\binom{v_2}{2g^*+2}B_{g_1}(\mathbf{s})B_{g_2}(\mathbf{t}) + \sum_{g^*\geq 0} \binom{v+2g^*}{2g^*+2}B_{g-g^*}(\mathbf{f})$$

 $B_g(\mathbf{f}) =$ number of bipartite maps of genus g with f_i faces of size 2i, $\mathbf{f} = (f_1, f_2, \dots)$

The proof

1) In every subsequence, there is a converging subsubsequence (aka tightness)

- 2) What are the possible limits ?
- 3) There is only one possible limit

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For bipartite maps : similar strategy of proof, but in HARD MODE

1) Tightness — how do we prove it ?

As with the UIPT, equivalent to show that **the degree of the root vertex is a.s. finite**

Key tool : bounded ratio lemma : for $\frac{g}{n}<\frac{1}{2}-\varepsilon$, there is a constant C_{ε} s.t. :

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For bipartite maps : for $\frac{g}{n} < \frac{1}{2} - \varepsilon$ and $\frac{f_i}{n} > \varepsilon$:

$$\frac{B_g(\mathbf{f})}{B_g(\mathbf{f}-\mathbf{1}_i)} < \tilde{C}_{\varepsilon}$$

For $\frac{g}{n} < \frac{1}{2} - \varepsilon$, $\frac{\tau(n,g)}{\tau(n-1,g)} < C_{\varepsilon}$

There are $n + 2 - 2g \ge 2\varepsilon n$ vertices, and the average degree is $\frac{6n}{n+2-2g} \le \frac{3}{\varepsilon}$ \rightarrow there are $\ge \varepsilon n$ vertices of degree $\le \frac{6}{\varepsilon}$

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Thus
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For bipartite maps : way more complicated, less local, more "destructive"



One-endedness and planarity



Proven by using the recurrence formulas (and the bounded ratio lemma)

2) The potential limits

Weak spatial Markov property:

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Theorem: [Budzinski-L. '19]

if $\mathbb T$ is a random infinite planar one ended triangulation satisfying the weak Markov property, then $\mathbb T$ is a PSHT with a random parameter Λ

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For bipartite maps, same theorem, but this time we deal with an infinity of parameters instead of just one.

Two holes argument :

The hyperbolicity is uniform all around the map



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Finishing the proof with a local observable : the inverse root degree (explicitly calculable in the PSHT and in high genus triangulations)

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Two holes argument for bipartite maps : The hyperbolicity and the proportion of faces of each size are uniform

Finishing the proof : completely different argument, way more complicated

Bonus : asymptotics

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$$\frac{B_{g_n}(\mathbf{f})}{B_{g_n}(\mathbf{f}-\mathbf{1}_i)} \to c_i(\theta, (\alpha_j)_{j\geq 0})$$

$$B_{g_n}(\mathbf{f}) = n^{2g_n} \exp\left(nf(\theta, (a_j)_{j\ge 1} + o(n))\right)$$

Thank you !