

# Iterated foldings of discrete spaces and their limits: candidates for the role of Brownian map in higher dimensions

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Joint work with J.F. Marckert, [arXiv:1908.02259](https://arxiv.org/abs/1908.02259)

JCB 2020

1 – Planar maps, etc

2 – Notion of scaling limit

3 – The problem in higher dimensions

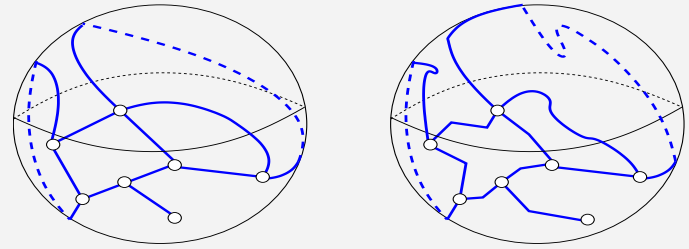
4 – The Cori-Vauquelin-Schaeffer bijection, etc

5 – Random feuilletages

1 – Planar maps, etc

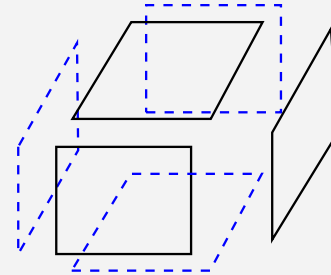
## Combinatorial maps:

Drawing on a surface without crossings



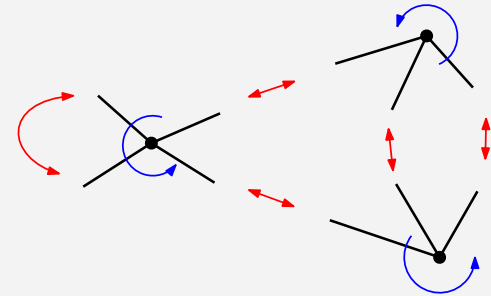
Gluing of polygons

Triangulations, quadrangulations, no restrictions...



Gluing of vertices

combinatorial encoding using permutations

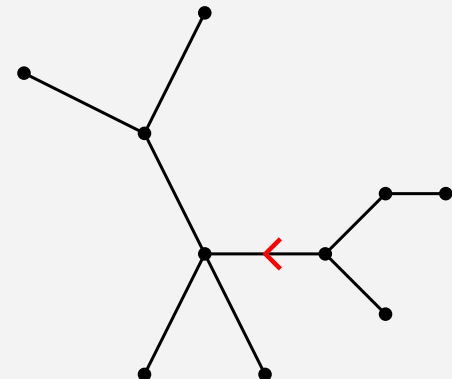


→ **Planar maps:** spherical topology  $V-E+F=2$

→ **Plane trees:**  $V=E+1$ ;  $F=1$

→ We consider all maps **rooted** (no symmetries).

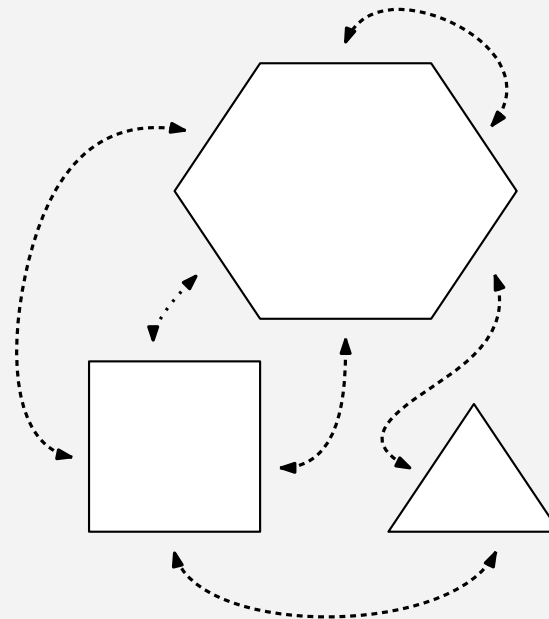
→ “**size**” = number of edges, or vertices, or faces...



## Random maps:

Choose a distribution for a given set of maps of the same size,

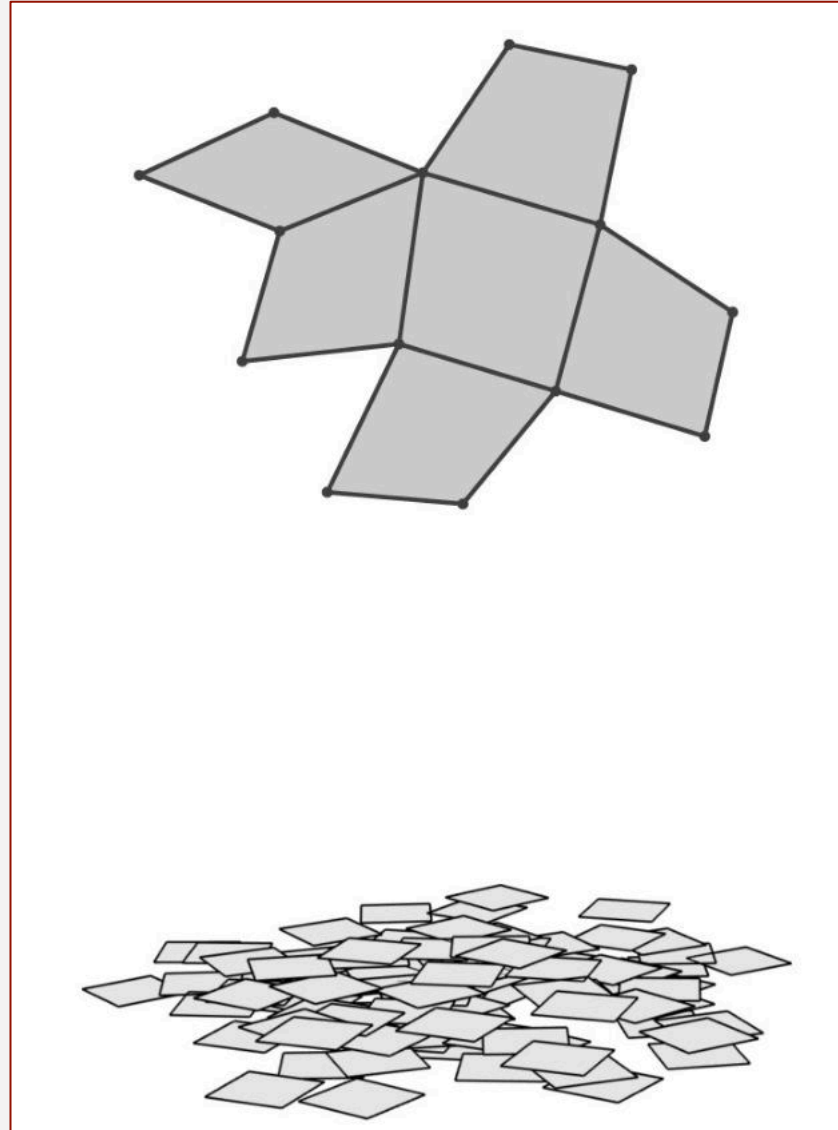
- Uniform distribution on **binary trees** with  $n$  edges
- Uniform distribution on **all trees** with  $n$  edges.
- Uniform distribution on planar maps made of  $n$  squares (**quadrangulations**)
- Same but for **all planar maps** with  $n$  faces
- Other kinds of distributions...



## Random maps:

→ Uniform distribution on planar maps made of  $n$  squares (quadrangulations)

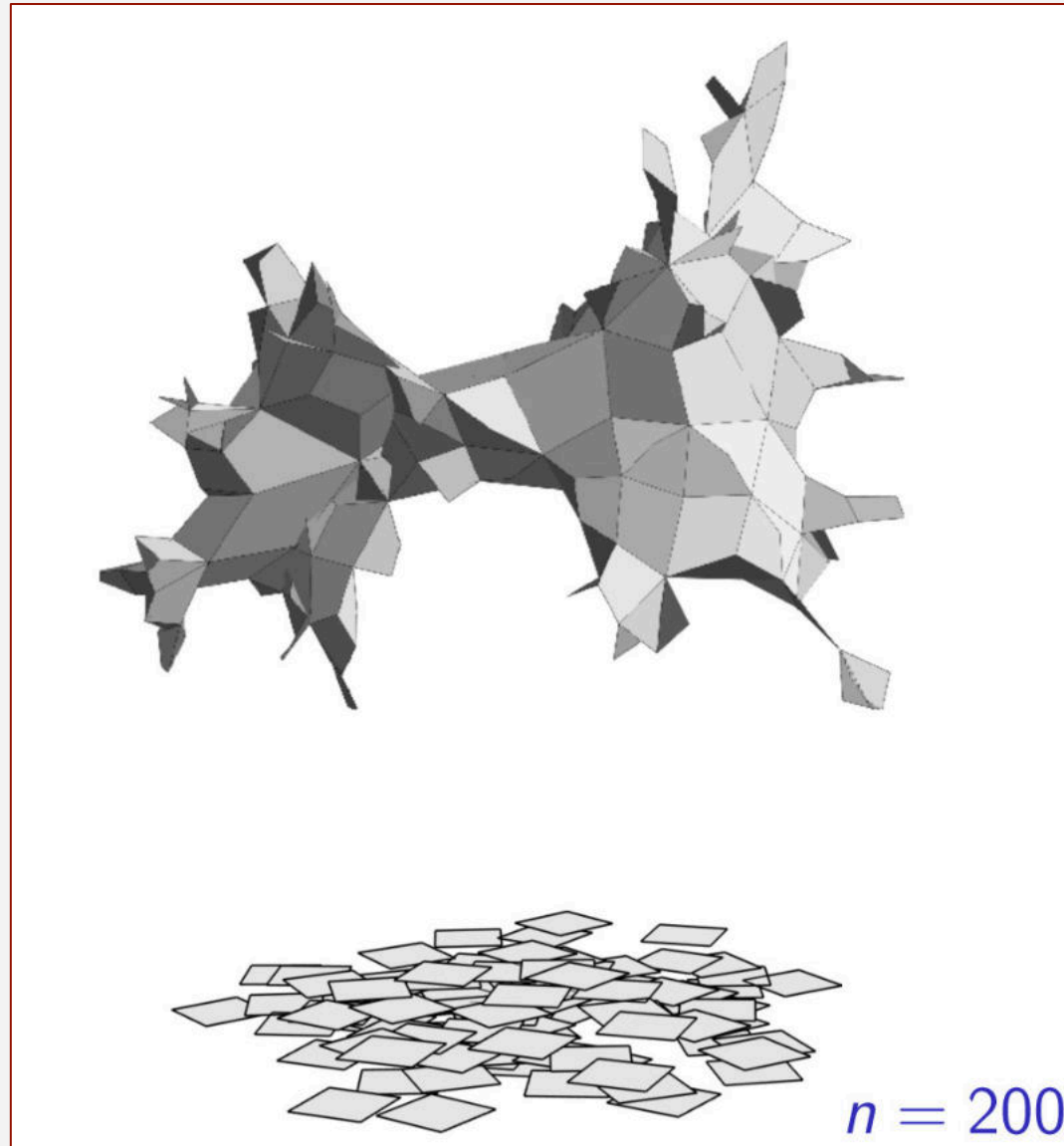
→ What does it  
typically look like??



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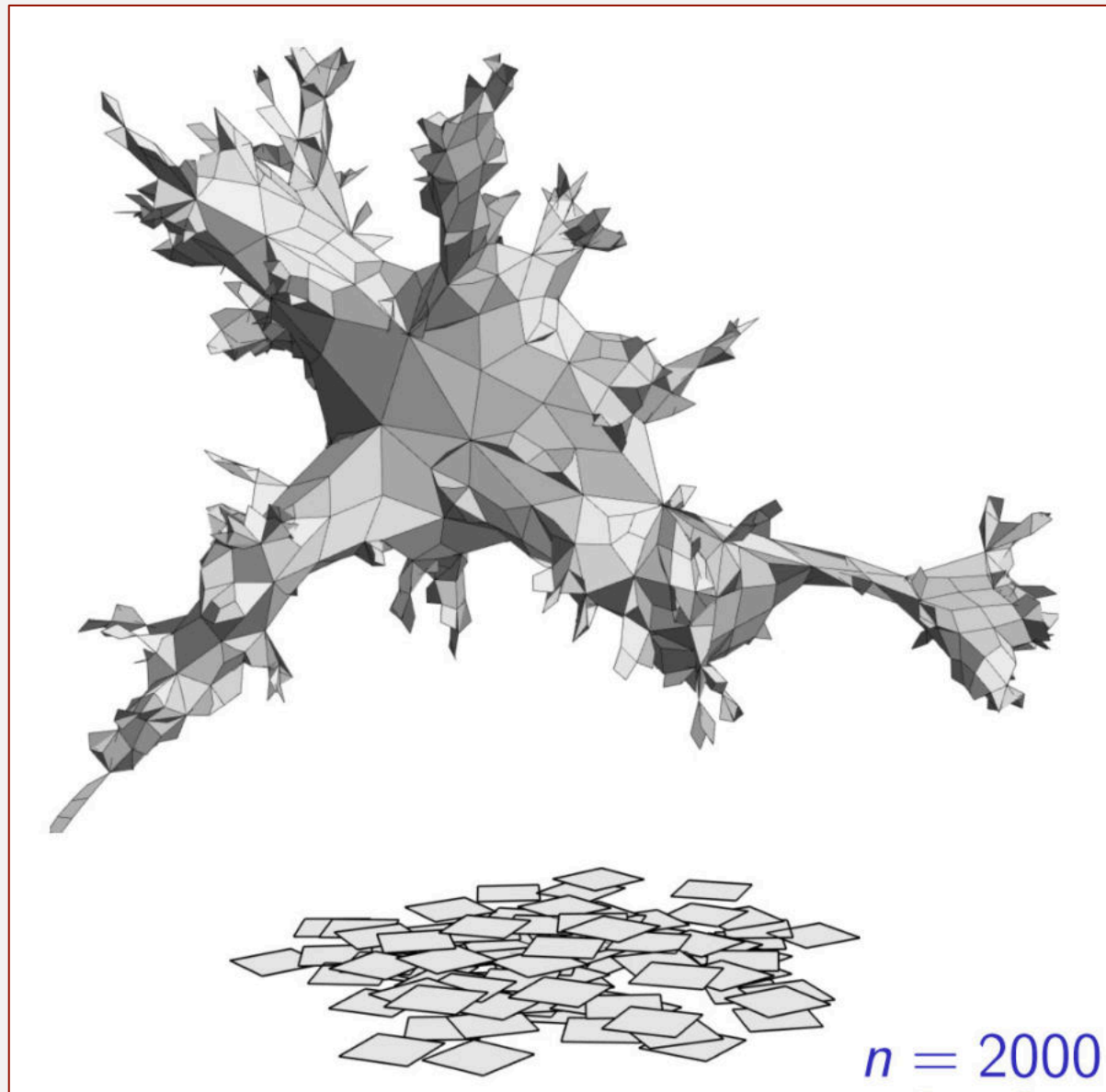
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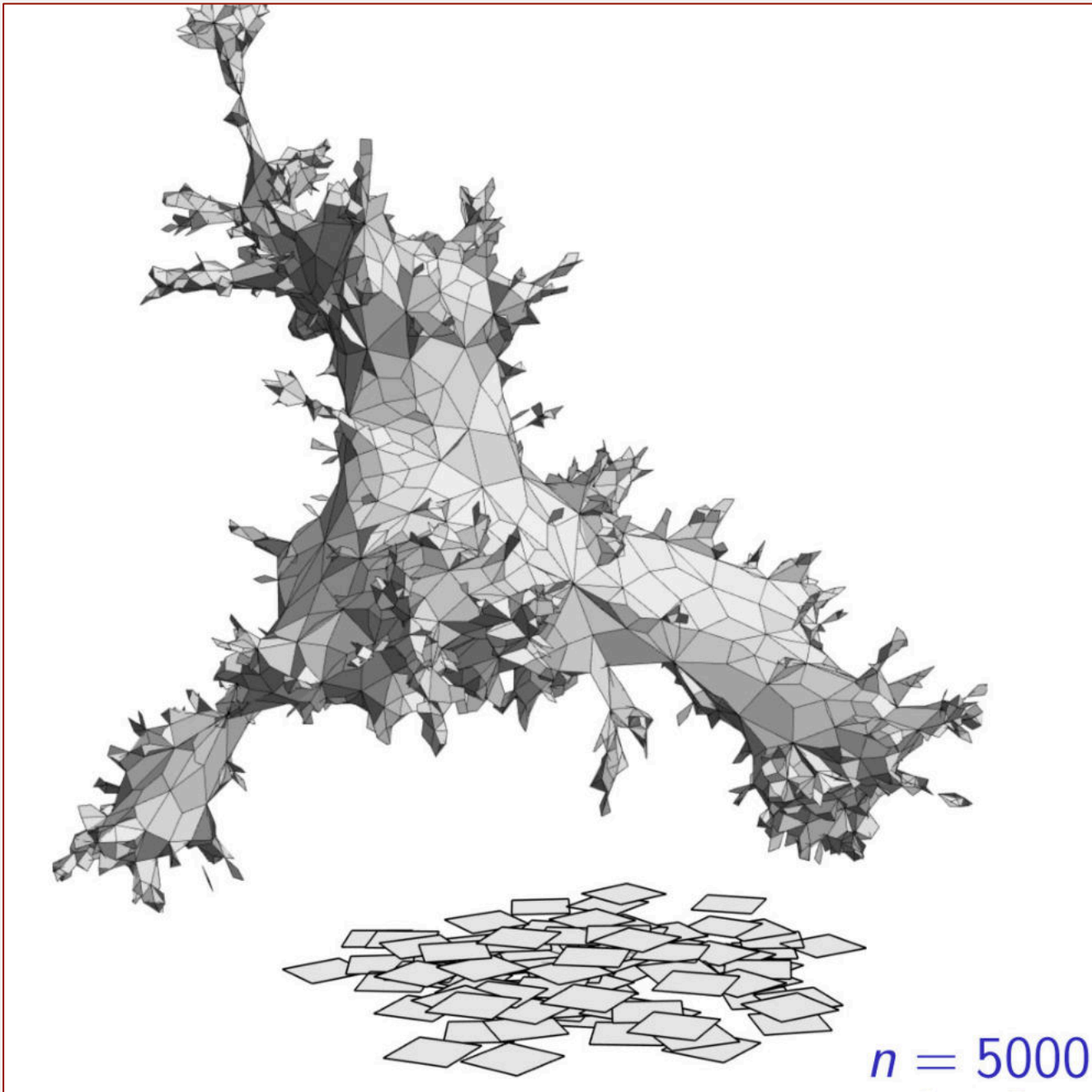


## Random maps:

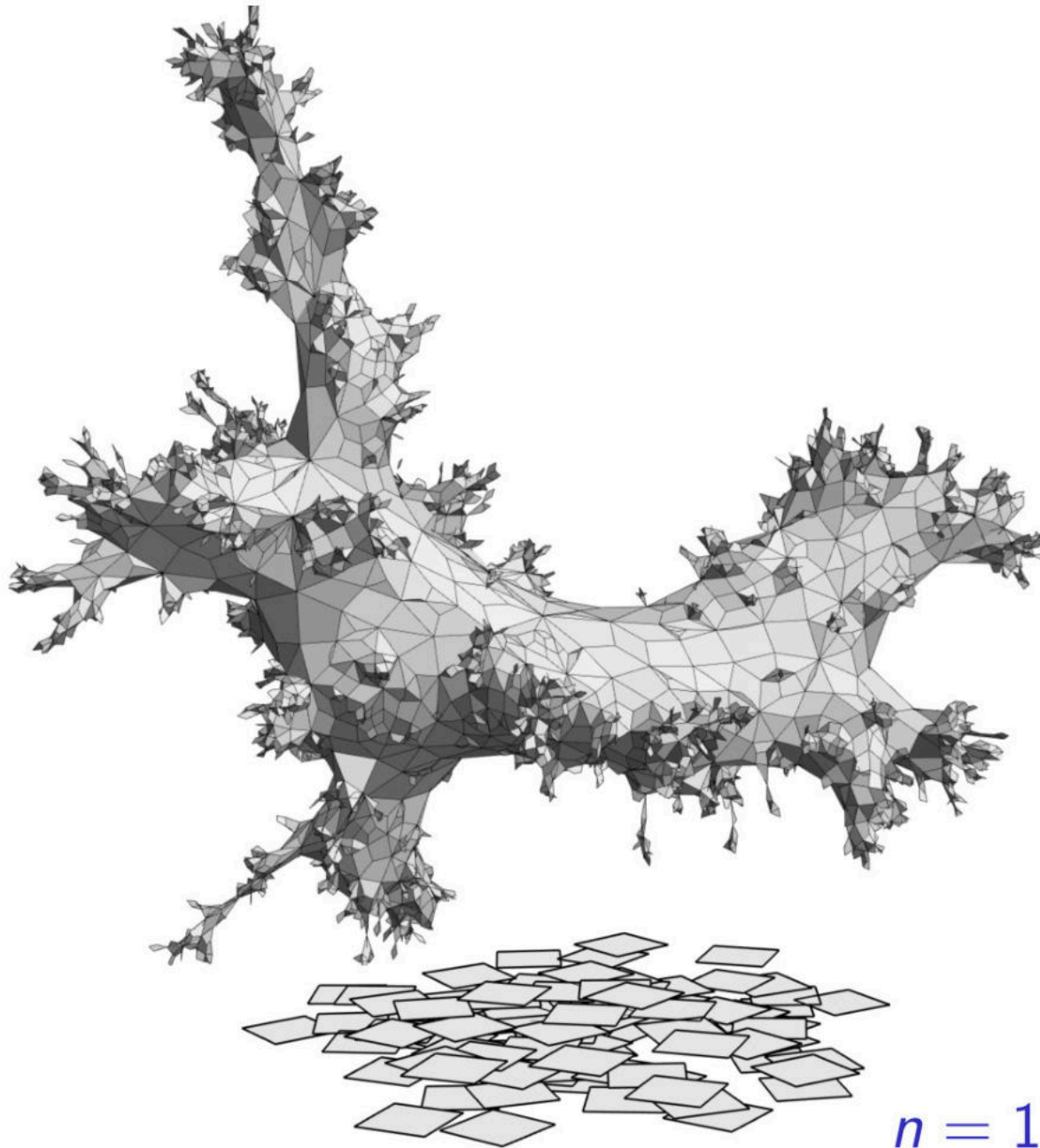
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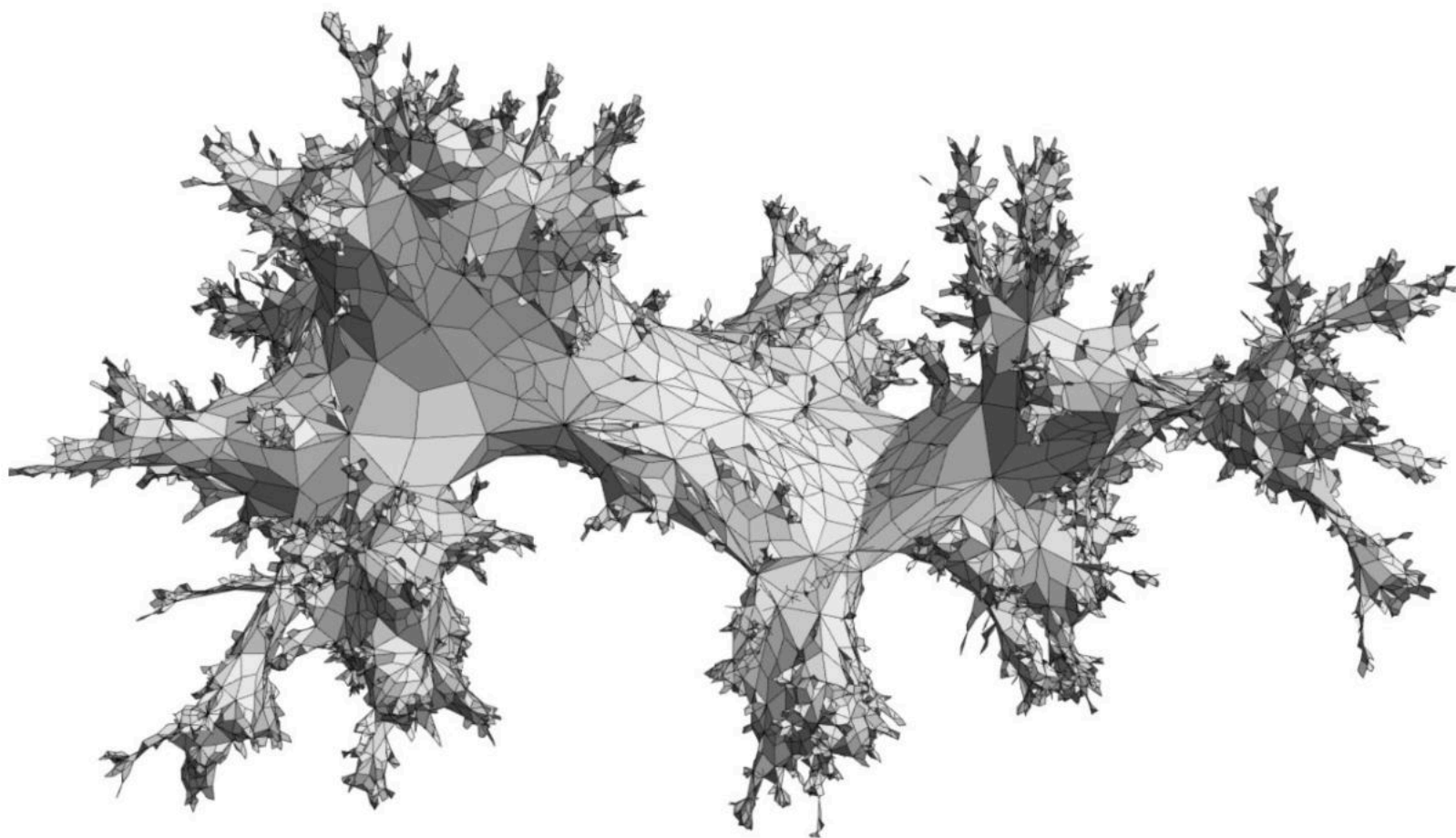




Picture  
T. Budd



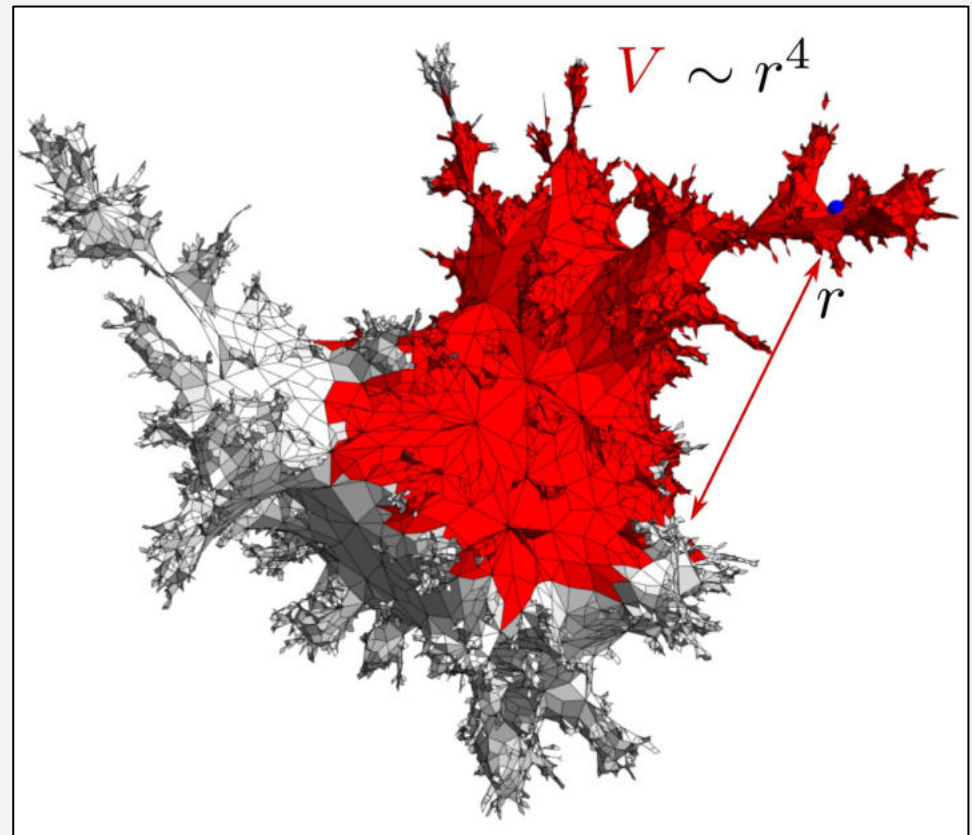
Picture  
T. Budd



$n = 20000$

## Random maps of large size exhibit common features → Universality classes

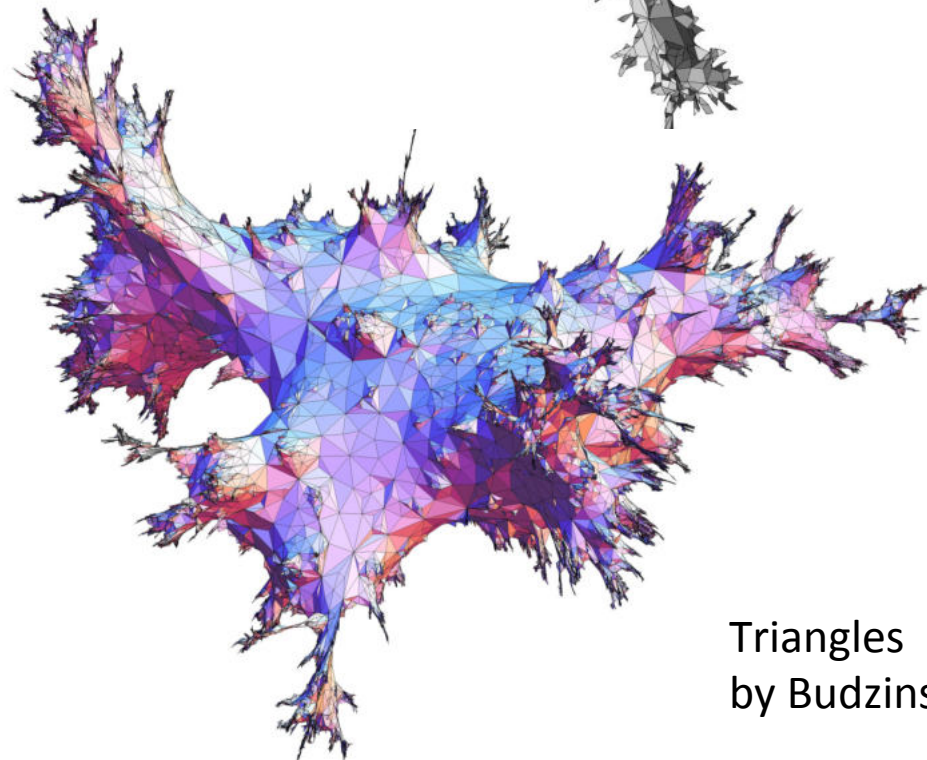
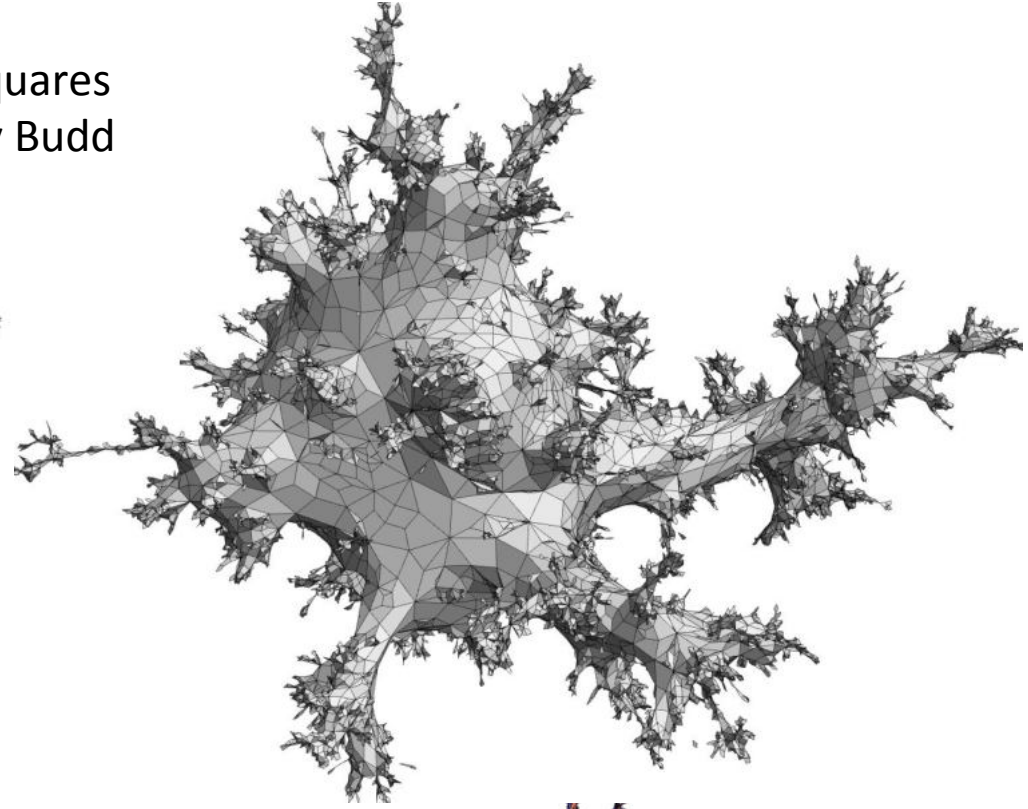
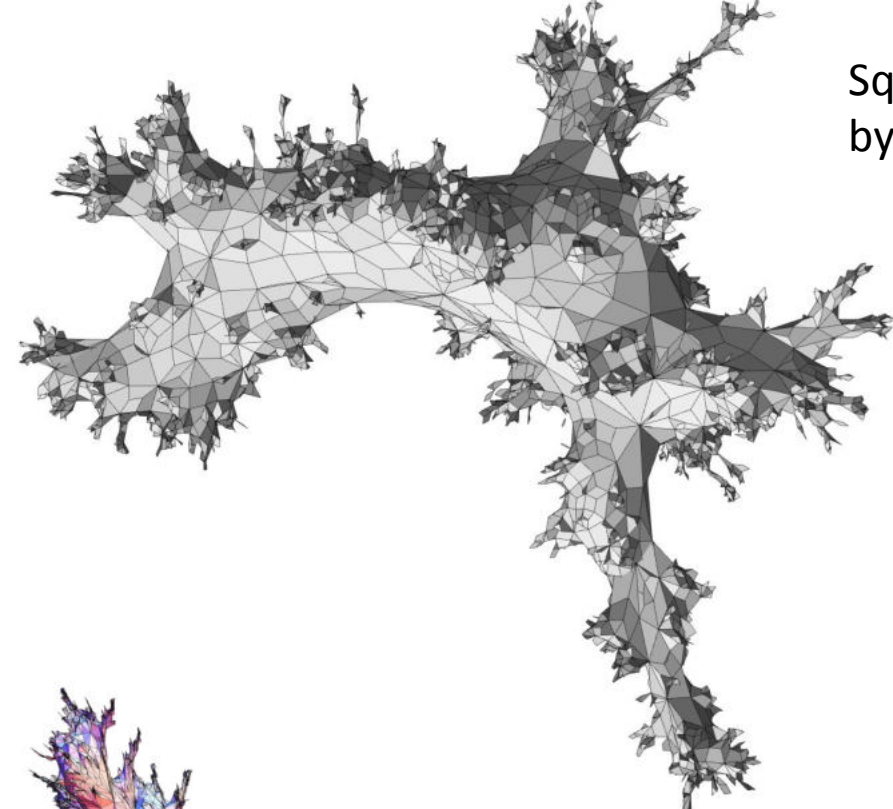
	Asymptotic enumeration	Typical diameter
Rooted plane trees	$N_T(n) \sim k_T r_T^{-n} n^{-3/2}$	$n^{1/2} \rightarrow d_H=2$
Rooted planar maps	$N_M(n) \sim k_M r_M^{-n} n^{-5/2}$	$n^{1/4} \rightarrow d_H=4$



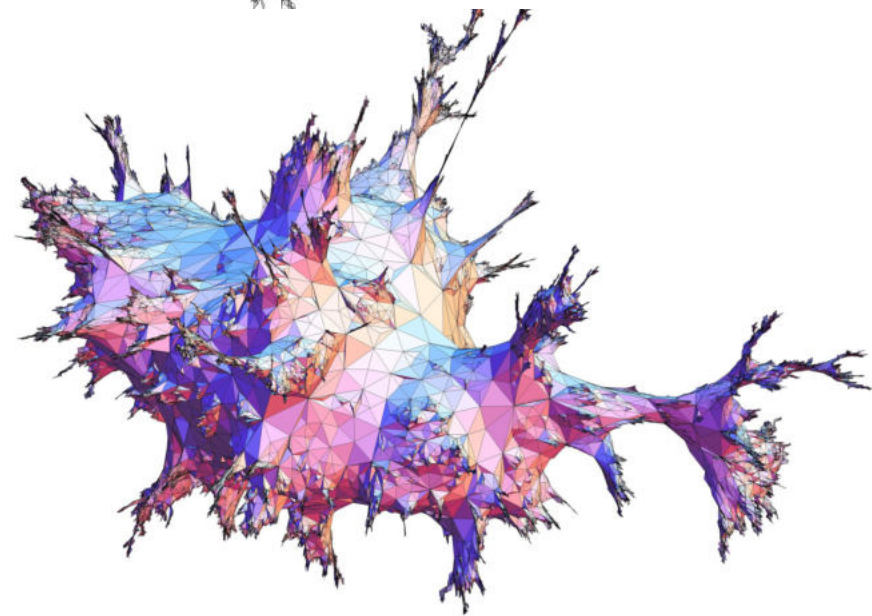
Picture  
T. Budd



Squares  
by Budd



Triangles  
by Budzinski



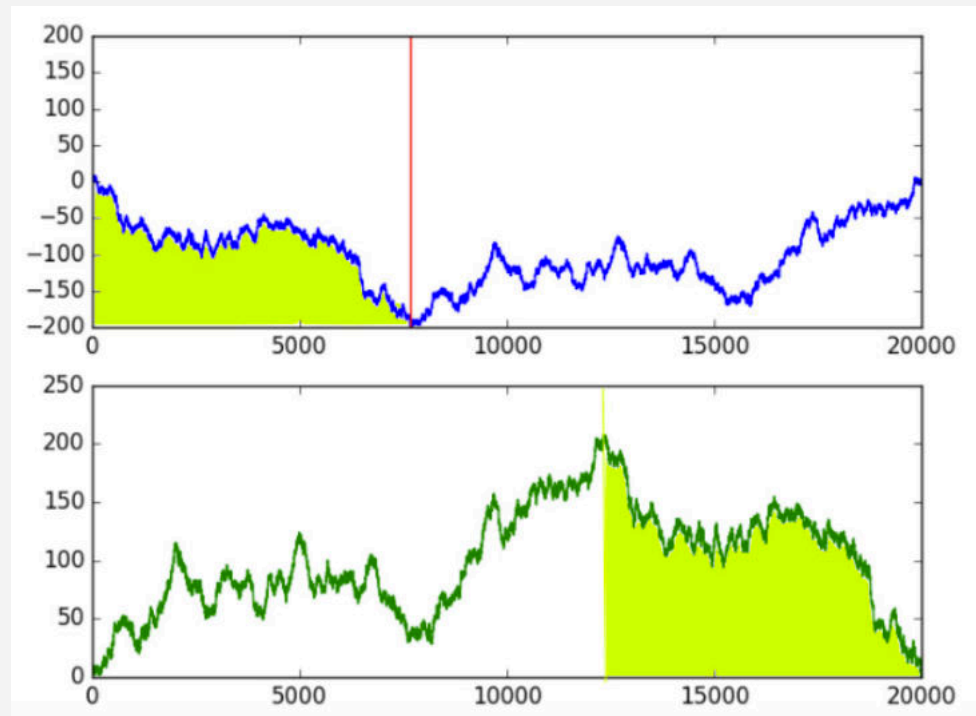
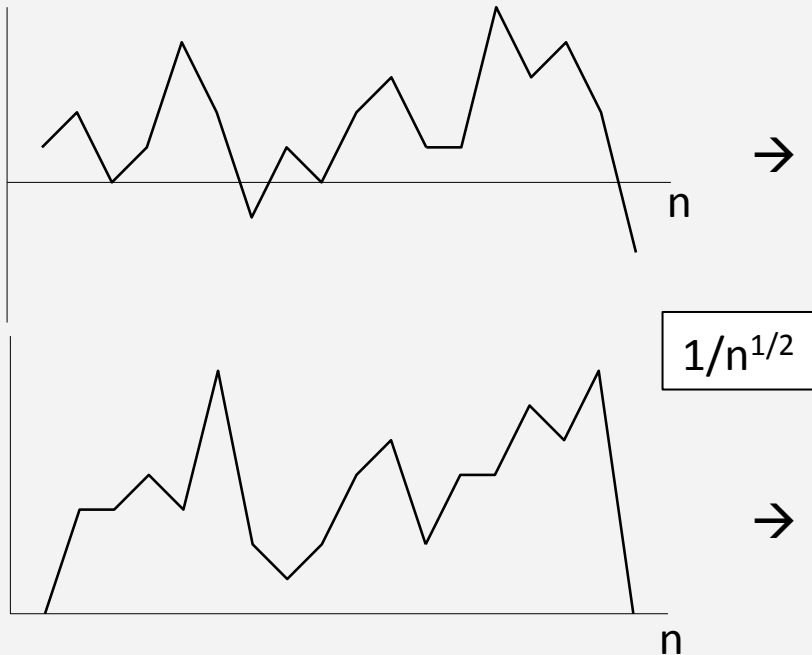
## 2 – Notion of scaling limit

## Random walks and excursions: Donsker's theorem

Random walk  $S_n = \sum_{i=1}^n X_i$ , where  $X_1, \dots, X_n$  are i.i.d. random variables with mean 0 and variance 1

$$W_n(t) = S_{[nt]} / n^{1/2} \quad t \text{ in } [0,1]$$

→  $W_n$  converges in distribution to a standard Brownian motion as  $n \rightarrow \infty$



Picture O. Safsari

## Scaling limit:

Take a random map of size  $n$

+ A notion of distance  $d$  on the maps (e.g. *graph distance* but not necessarily)

→ Random metric space  $G_n$

Suppose that the diameter behaves asymptotically as  $d(G_n) \sim_{n \rightarrow \infty} d_n$  (e.g.  $d_n = n^\alpha$ ), and normalize the distances in the map by  $d_n$ .

→ *Limit of the random metric space  $(G_n, d/d_n)$  is a **random compact continuum metric space***

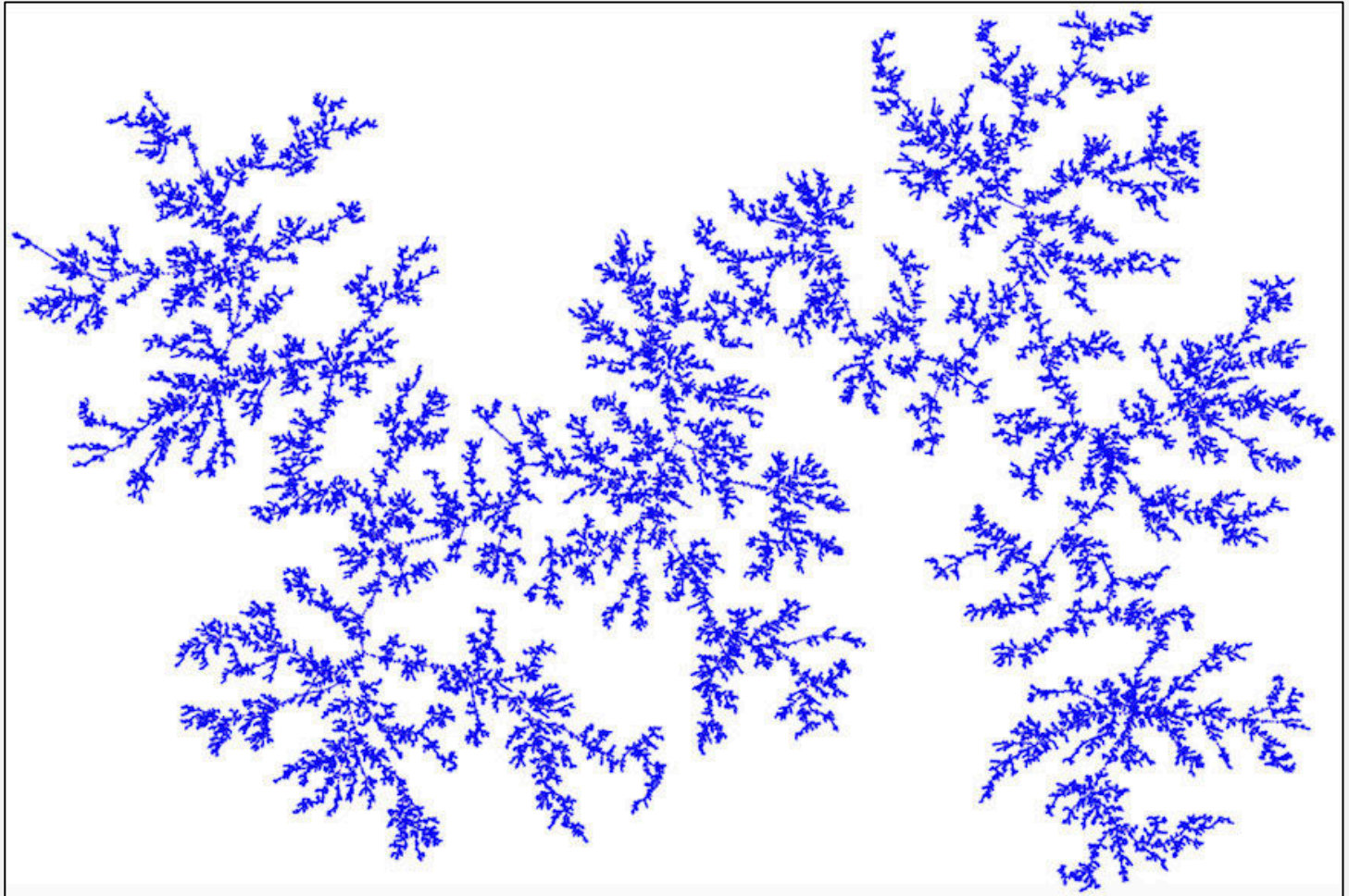
## Two levels:

- Definition of the limit using limits of random walks + gluing procedure
- Limit for the Gromov-Hausdorff distance, in the space of metric spaces



→ Example: trees

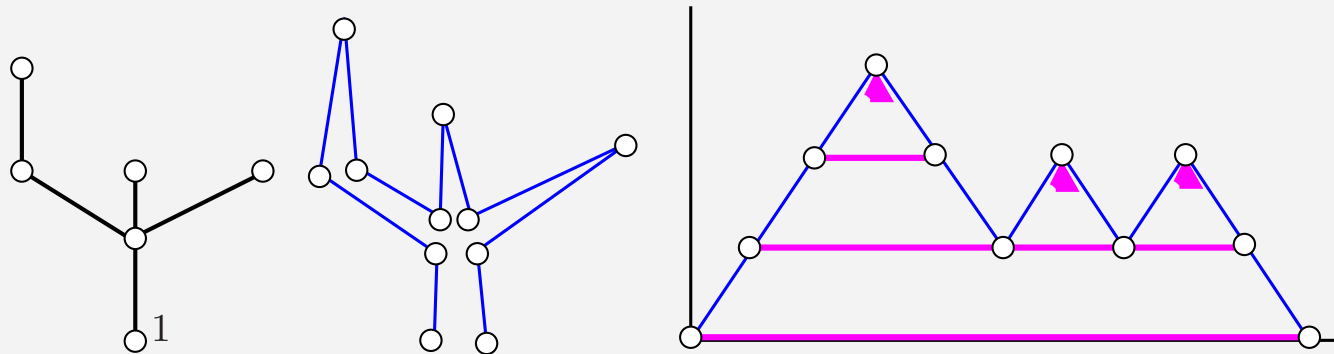
## Scaling limit of uniform trees: Aldous' continuum random tree (CRT)



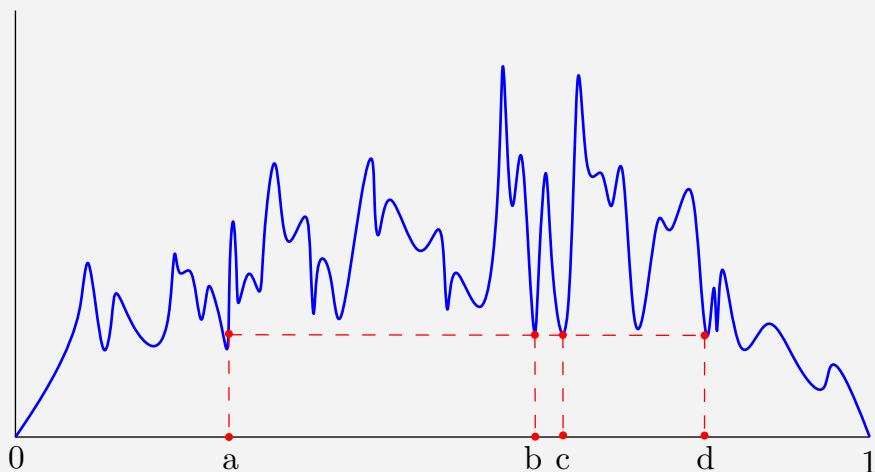
Picture:  
L. Ménard  
100000 edges

To define the scaling limit of random rooted plane trees, we need contour processes + real trees.

### Contour process:



Real trees: notion of tree for any continuous positive function  $f$  on  $[0,1]$  with  $f(0)=f(1)=0$



### Vertices:

$$x \sim_g y \Leftrightarrow g(x) = g(y) = \min_{[x,y]} g$$

### Distance:

$$D_f(x, y) = f(x) + f(y) - 2\min_{[x,y]} f$$

## Scaling limit of random trees: the continuum random tree (CRT)

Consider a uniform random rooted plane tree  $\mathbf{T}_n$  with  $n$  edges,  $\mathbf{c}_n$  its contour process

### Two levels:

1. CV (in law) of the normalized contour process to the Brownian excursion  $\mathbf{e}$  (Donsker)

$$\mathbf{c}_n(2nt)/n^{1/2} \rightarrow 2\mathbf{e}(t)$$

→ Definition of the CRT as real tree with contour  $2\mathbf{e}$ , and CV of (uniform) discrete random trees to the CRT in the space of real trees. [Aldous 97]

2. CV in law of  $(\mathbf{T}_n, d/n^{1/2})$  to the CRT in the space of metric-spaces for the Gromov Hausdorff distance [Aldous 97, Le Gall 10]

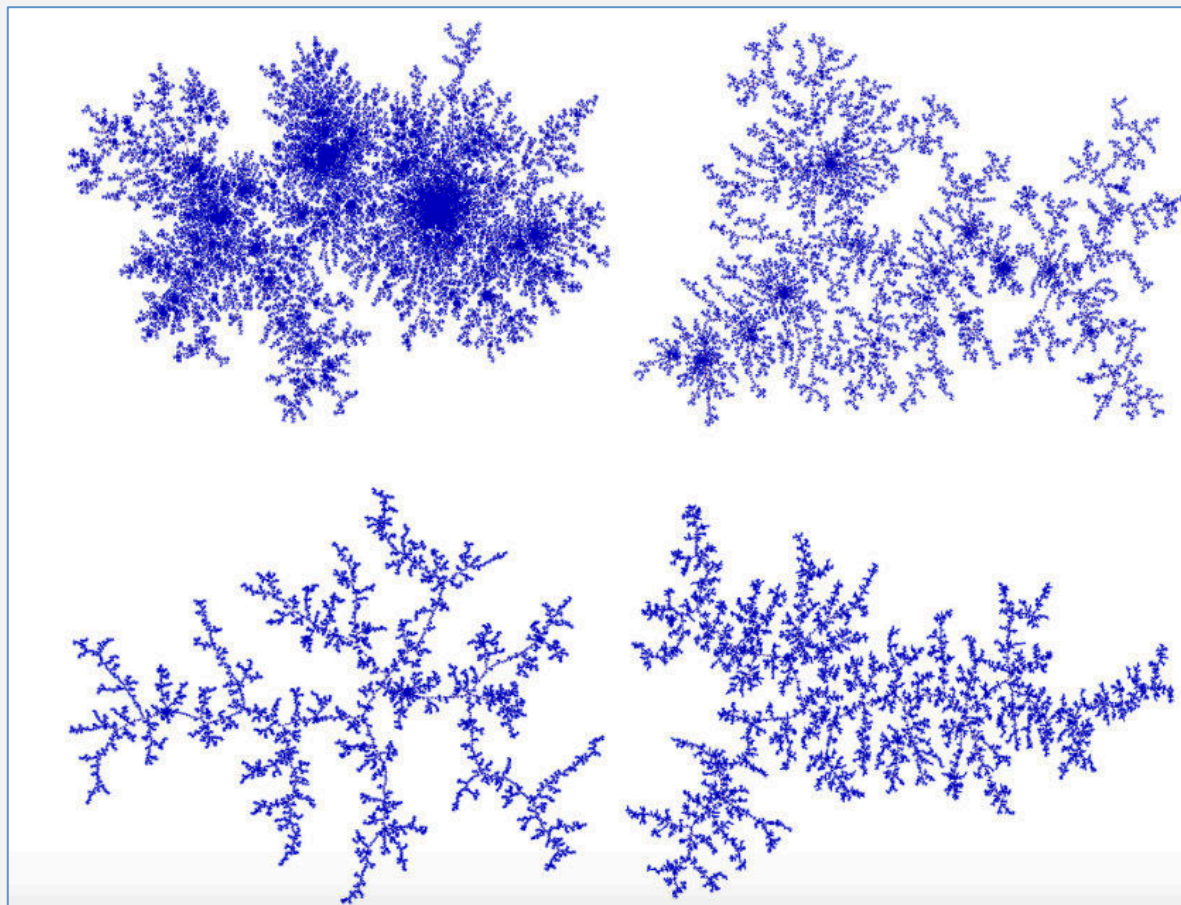
## Other scaling limits of “non-uniform” random trees

e.g.  $\alpha$ -stable trees.

For  $\alpha$  in  $(1, 2)$ , consider Galton-Watson trees with offspring distribution  $\eta(k)$  of mean 1 and such that  $\eta(k) \sim Ck^{-1-\alpha}$  as  $k \rightarrow \infty$ . Normalizing the graph distance by  $n^{1-1/\alpha}$ , they converge towards a compact random metric space called the  $\alpha$ -stable tree, of Hausdorff dimension  $d_H = \alpha/\alpha - 1$  in  $(2, \infty)$ .

Approximations of  $\alpha$ -stable trees for  $\alpha=1.1, 1.5, 1.9$ , and 2. The  $\alpha=2$  case corresponds to the CRT.

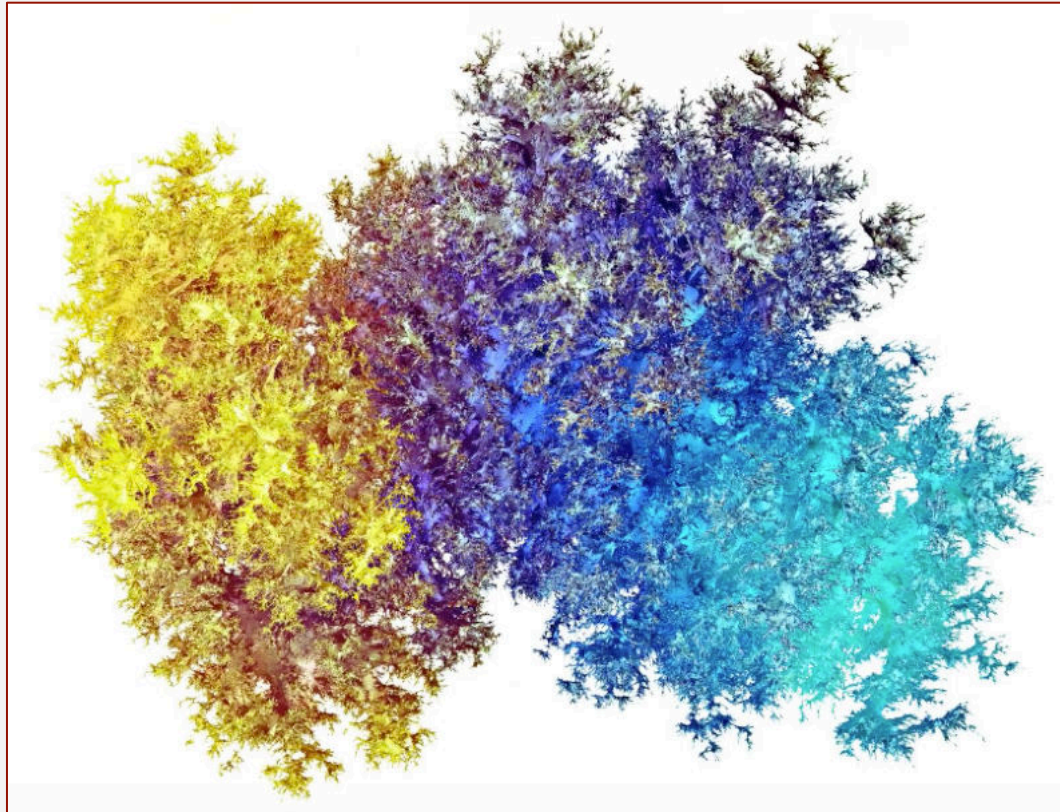
Pictures: Kortchemski.



→ Example: planar maps



## Scaling limit of uniform random maps: the Brownian map



Movies by Benedikt Stufler

## Scaling limit of uniform random maps: the Brownian map

### Two levels:

1. Random planar maps  $\Leftrightarrow$  pairs of random trees + gluing procedure + distance  
(details later).

→ Definition of the Brownian map as random real trees + limit of gluing procedure + limit of distance. [Marckert, Mokkadem 2006]

2. CV in the space of metric-spaces in the Gromov-Hausdorff sense:

[Le Gall 13; Miermont 13; Bettinelli et al. 14; Abraham 16; Addario-Berry & Albenque 19]

Consider a uniform random rooted planar map  $\mathbf{M}_n$  with  $n$  faces and the graph distance  $d$  normalized by  $n^{1/4}$ . Then  $(\mathbf{M}_n, cd/n^{1/4})$  converges in law to the the Brownian map for the Gromov-Hausdorff distance.

The same is true (for a different  $c$  but the *same* distance on the Brownian map) for any  $p$ -angulation and for bipartite maps.



## Scaling limit of uniform random maps: the Brownian map

Different notions of dimensions...

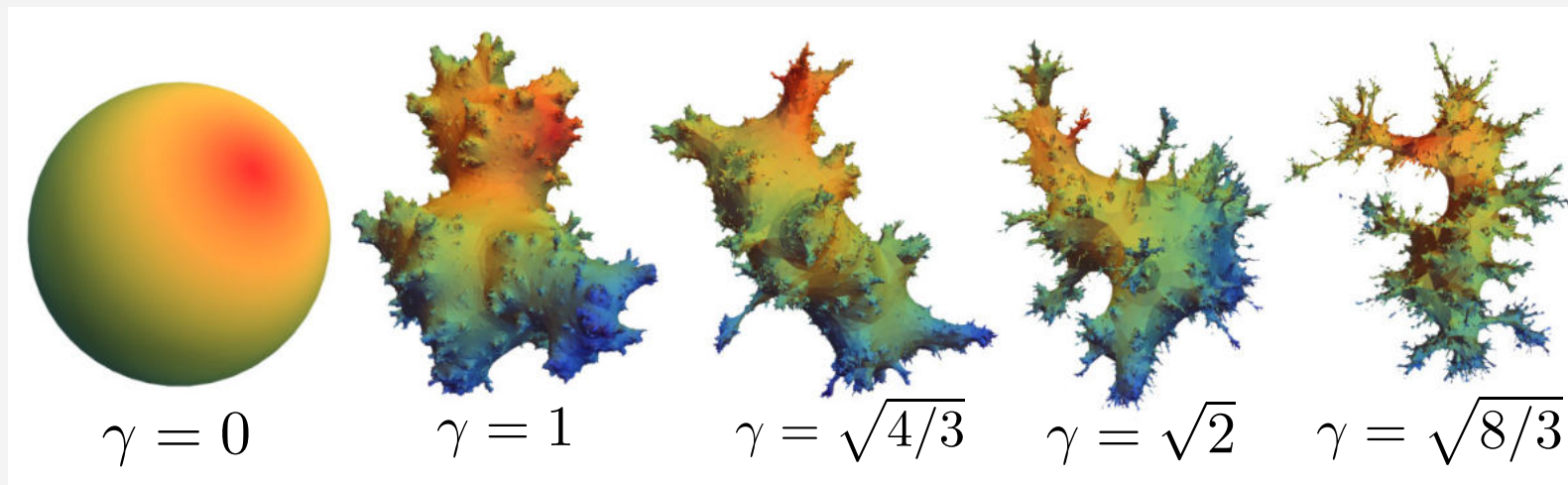
- Hausdorff dimension 4 (a.s.) [Kawai *et al.* 93; Ambjørn & Watabiki 95; Le Gall 07]
- Homeomorphic to the 2-sphere (a.s.) [Le Gall & Paulin 08; Miermont 08]
- Spectral dimension 2: [Ambjørn *et al.* 98; Rhodes & Vargas 13, Gwynne & Miller 17]

## Other universality classes of “non-uniform” random spheres

Without entering into details:

- **Stable spheres** [Le Gall, Miermont 09]. Scaling limits of random planar maps with large faces. Distances  $\sim n^{1/2\alpha}$  for  $\alpha$  in  $(1,2)$ , Hausdorff dimension  $2\alpha$ .
- **Statistical physics models on random planar maps** (Ising, tree-decorated, bipolar orientations...)

Pic. T.  
Budd



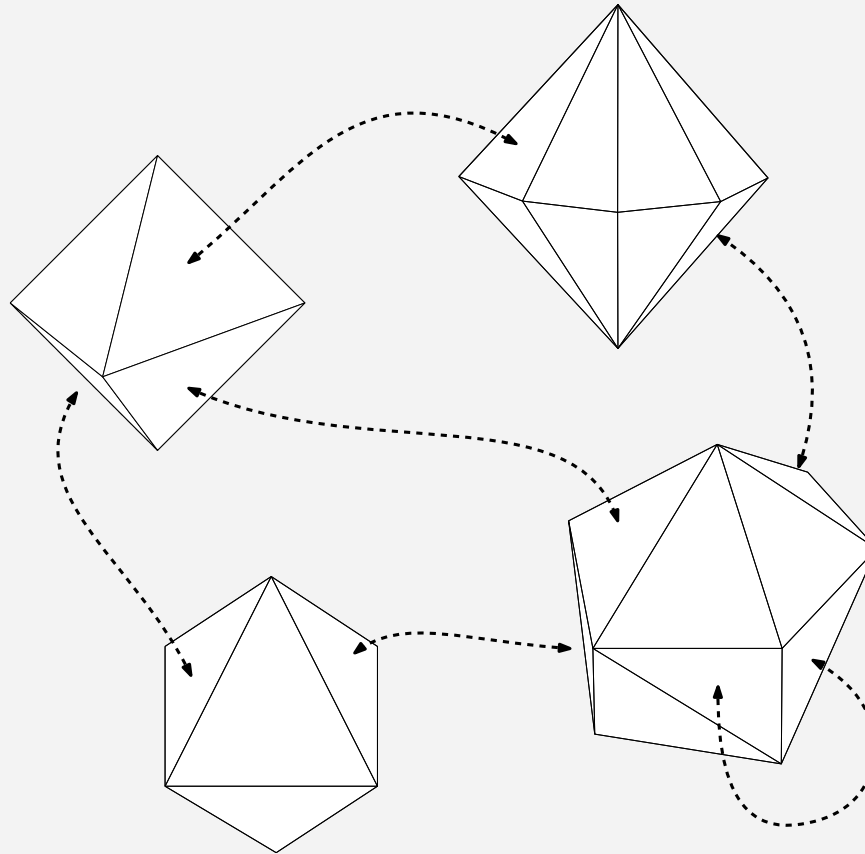
Maps selected according to number of spanning trees for  $\gamma=\sqrt{2}$ , Ising for  $\gamma=\sqrt{3}$ , uniform for  $\gamma=\sqrt{8/3}$ ...

- **Liouville quantum gravity measures on the unit sphere**, equivalent to the previous kind.

## 3 – The problem in higher dimensions

We want to build a compact random **uniform** continuum volume.

→ Natural to start from random gluings of  $n$  “polytopes” (tetrahedra...) with *uniform* distribution and **topology of the 3-sphere**:



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In 2D, spherical topology and maximizing the curvature is equivalent (Gauss-Bonnet) but not in higher  $D$

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→ No known way of producing anything else than random continuum **trees** or **surfaces** from random gluings of building blocks.

→ **We can try to explore new universality classes of random geometry by taking limits of well-motivated more general random graphs.**

→ We then face the difficult question of characterizing what we want...



## What are we looking for?

- Ability to define a scaling limit (diameter grows as a power law)?
- Typical diameter smaller than  $n^{1/4}$ ?
- ↓ • Finite Haus. dim., larger than 4? (we have 2 for trees, 4 for planar maps)
- Spectral dimension, larger than 2? (we have 4/3 for trees, 2 for planar maps)
- Scaling limit: not a tree, not a surface...
- ↓ • Locally looks like a 3-ball
- Well defined topology (topological dimension 3)?

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A proposal: we built a family of random graphs obtained by identifying many points on some random discrete 2-spheres [L. & Marckert 19].

There are different ways of encoding maps:

Drawing on a surface

→ Renders the fact that you have surfaces obvious

Gluing of polygons

→ Same, and uniformity

Gluing of vertices

→ Nice combinatorial encoding using permutations

No known way of producing new  
scaling limits in higher dimensions

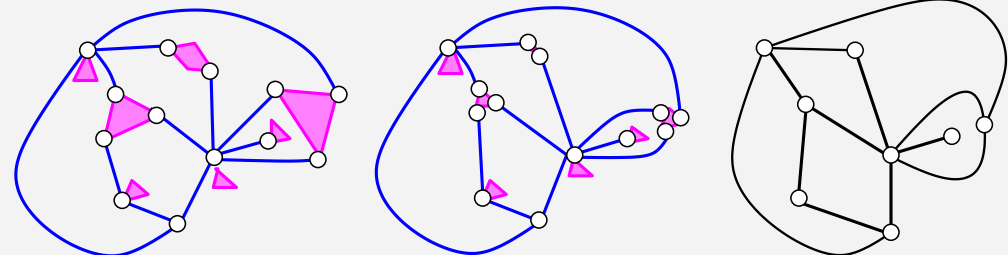
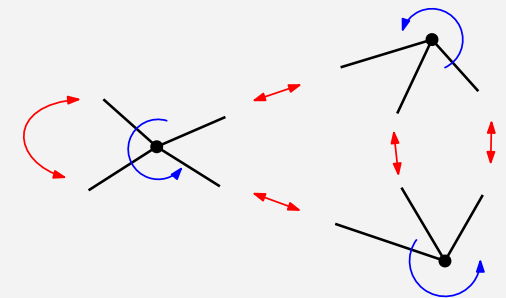
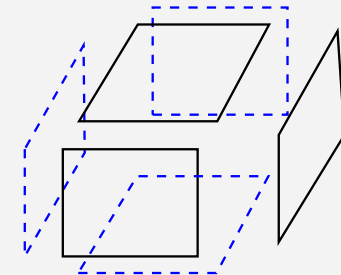
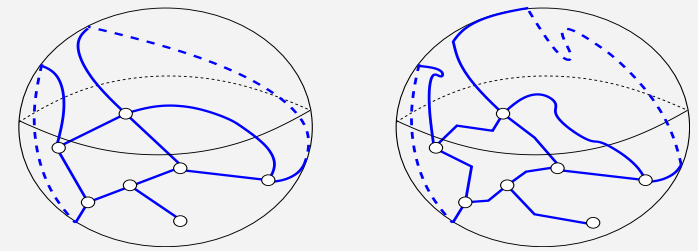
A tree + some corner identifications

⇔ a map + splitting some vertices

→ Nice bijections: enumeration but also distances in some cases

→ But: more difficult to see that it's a surface, to track the topology

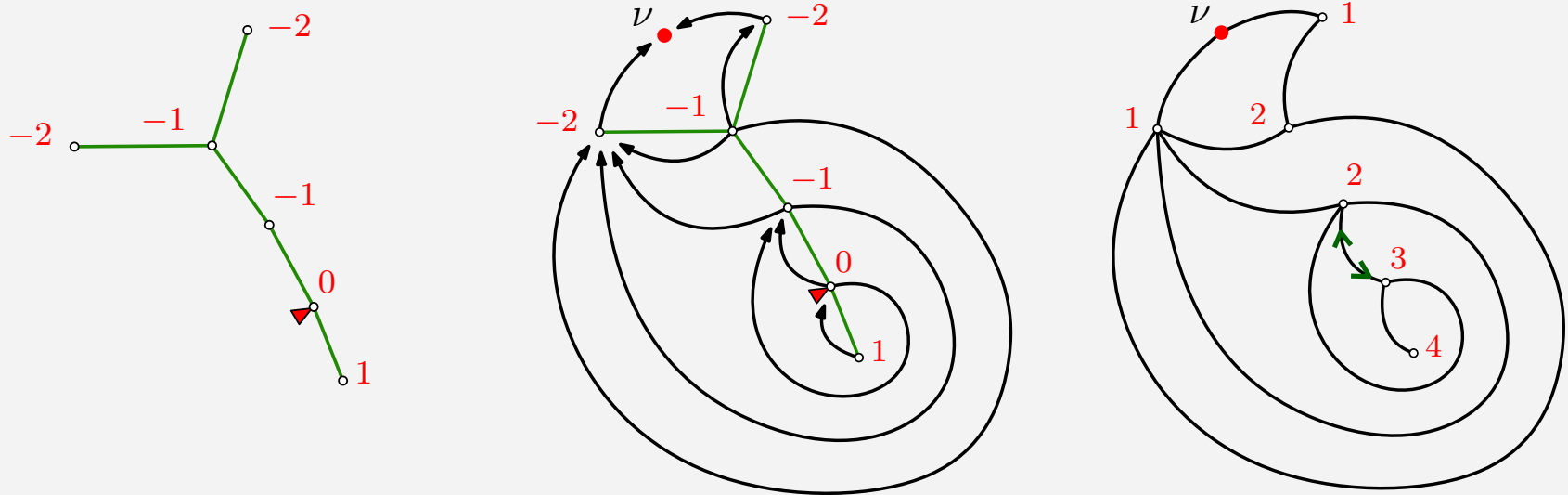
→ Some other things are also less obvious e.g. invariance by change of root



## 4 - The Cori-Vauquelin-Schaeffer bijection:

Distances in planar maps and construction of the Brownian map

**Cori-Vauquelin-Schaeffer's bijection** between labeled trees and (rooted pointed) planar quadrangulations:

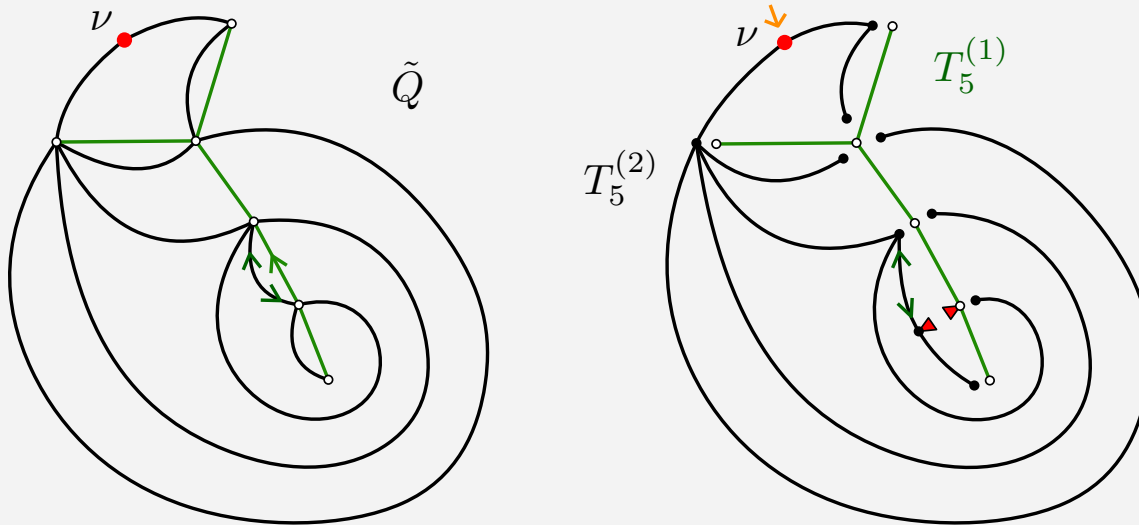


The CVS bijection is a powerful tool giving **control on the distances** in planar maps: it allows proving that the **diameter** of random planar quadrangulations is  $\sim n^{1/4}$ .

It also allows **constructing the Brownian map**.

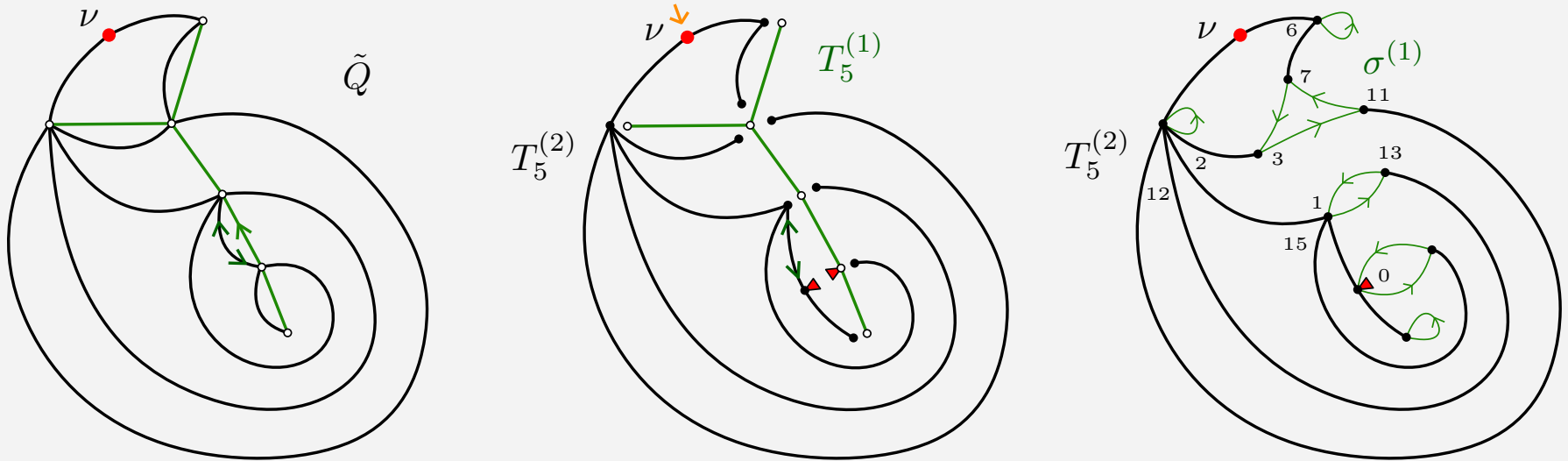
**Cori-Vauquelin-Schaeffer's bijection** between labeled trees and (rooted pointed) planar quadrangulations.

→ It can be reformulated in terms of identifications on a second tree:  
 [Marckert, Mokrakadem 2006]



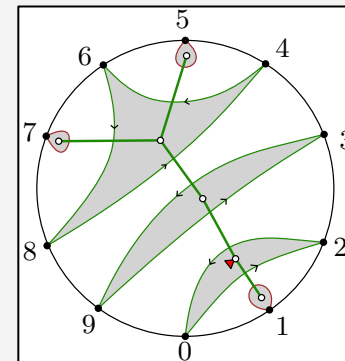
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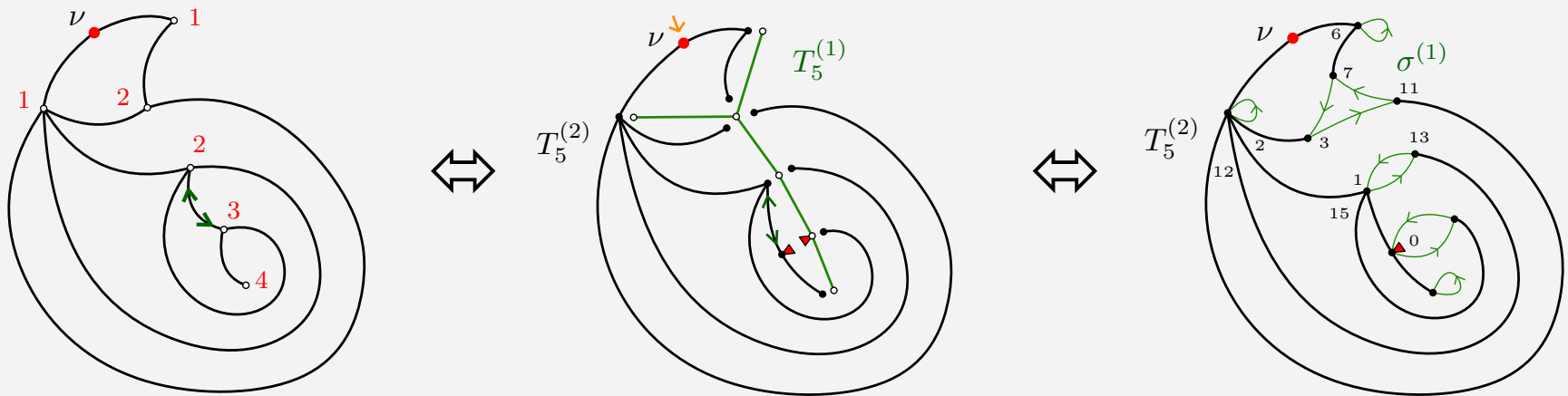
→ It can be reformulated in terms of identifications on a second tree:  
 [Marckert, Mokkadem 2006]



Gluing the vertices of  $T_2$  using  $T_1$  **preserves the distances** to the pointed vertex:

Distances to  $\nu$  in this second tree are the distances to  $\nu$  in the quadrangulation





A **Uniform random planar quadrangulation** (root. point.) is

- A **uniform tree**  $T_1$ , **vertices** give the vertices of the map
- A **non-uniform tree**  $T_2$ , diameter  $\sim n^{1/4}$ , **edges** give the edges of the map
- **Distance**: distance in  $T_2$  + free jumps on the vertices of  $T_1$

**Distances**: typical diameter of the quadrangulation is that of  $T_2$  :  $\sim n^{1/4}$

**Construction of the Brownian map**: Random trees  $\rightarrow$  random walks

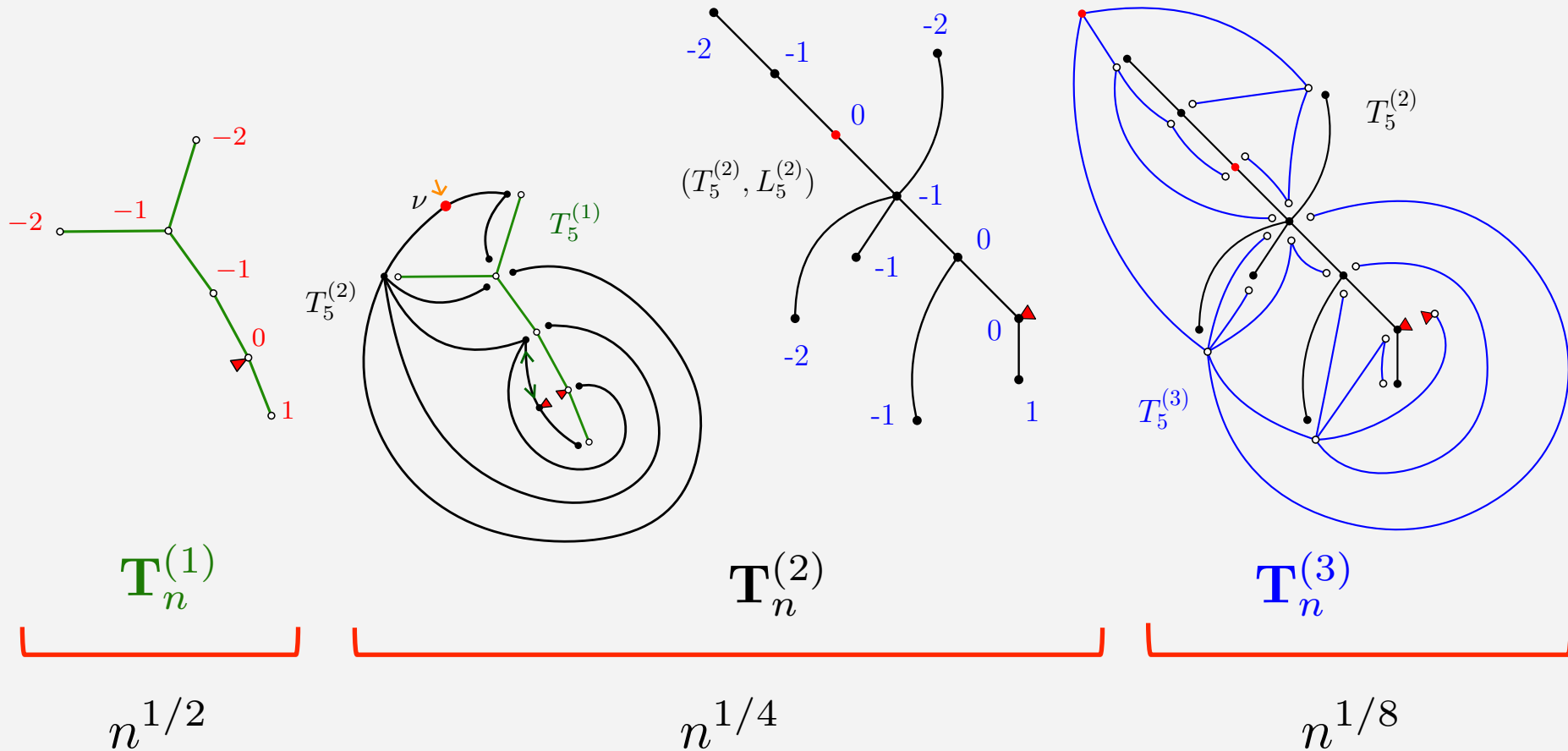
- Gluing procedure defined on the walks
- Distances defined on the walks

$\rightarrow$  The Brownian map defined by taking their limits, using the limiting gluing procedure ( $T_2$  quotiented by the vertices of  $T_1$ ), and using the limiting distance.



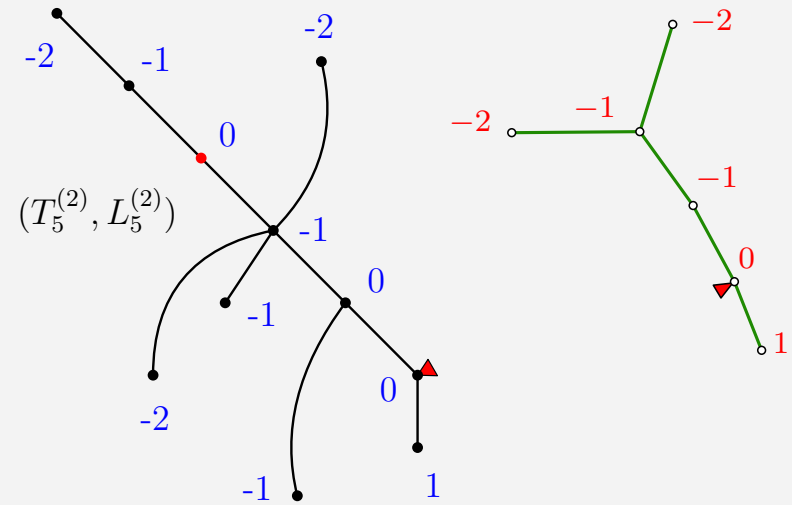
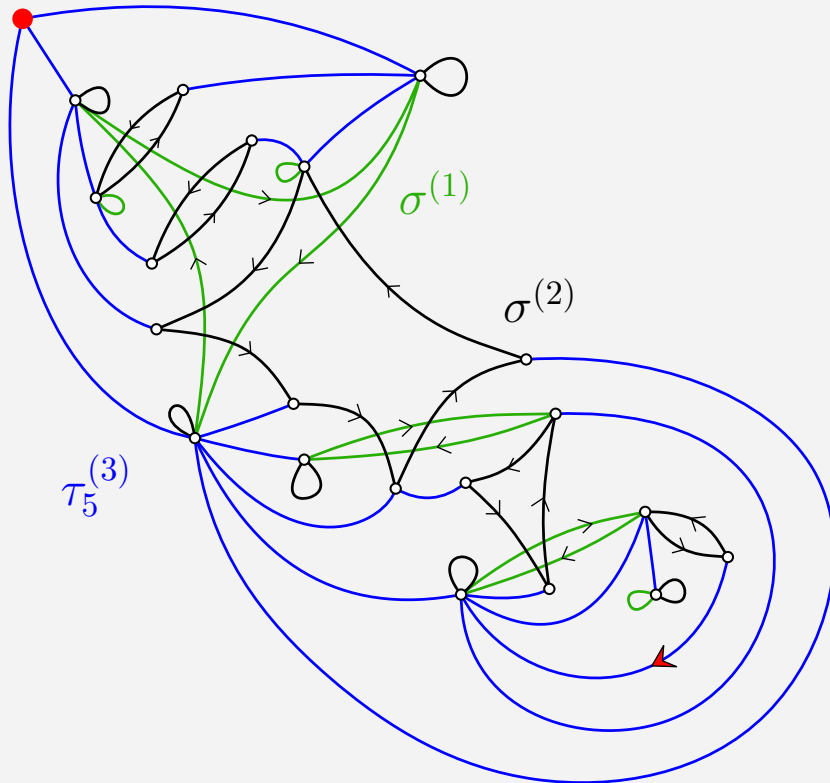
## 4 – Random feuilletages

Iterate this construction: label the second tree, build a third tree, quotiented by the vertices of the two first trees!

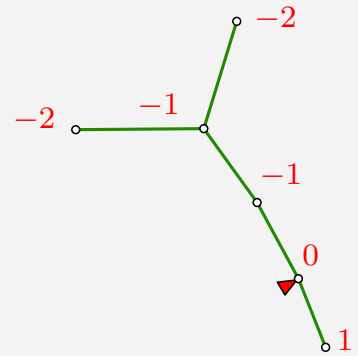
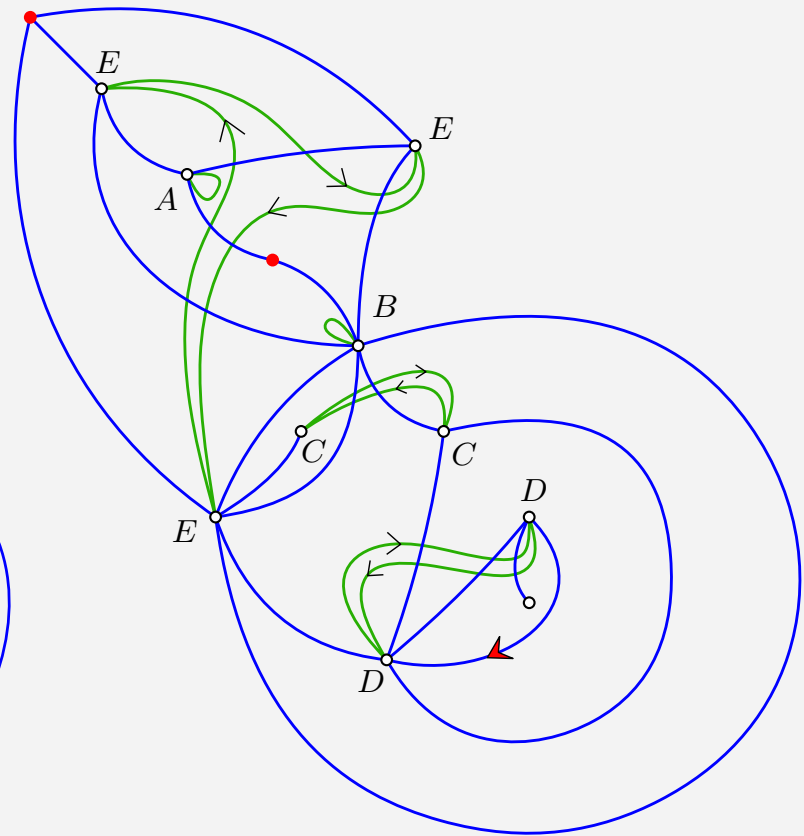
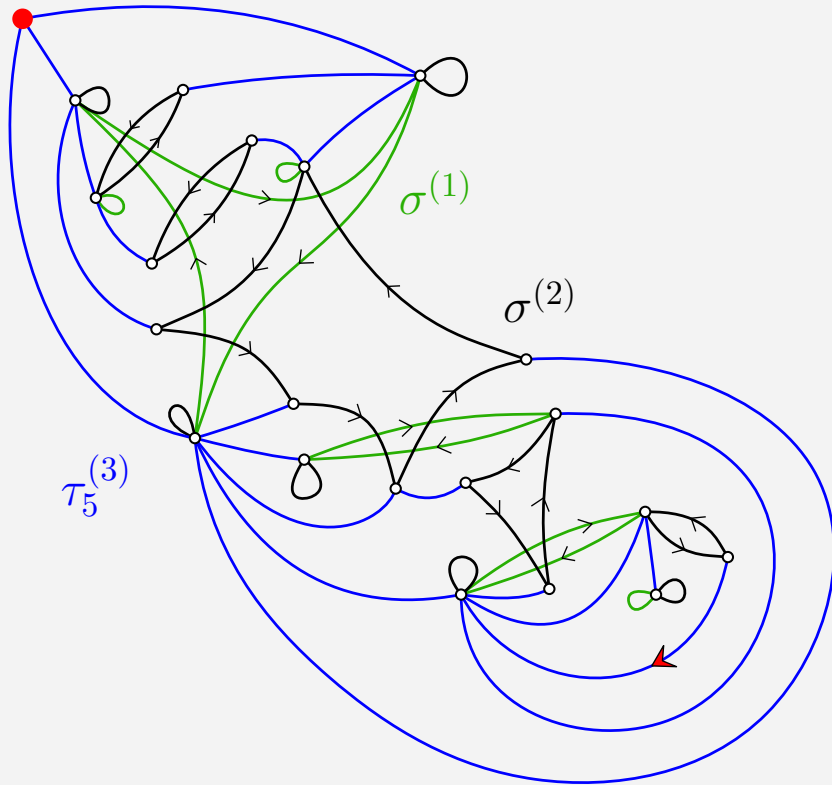


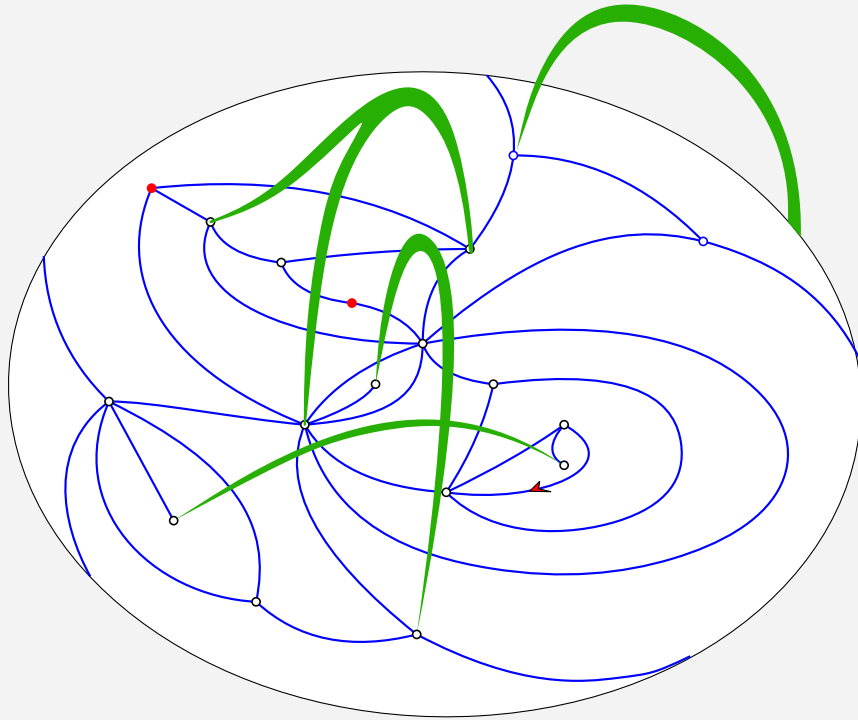
Typical asymptotic diameter

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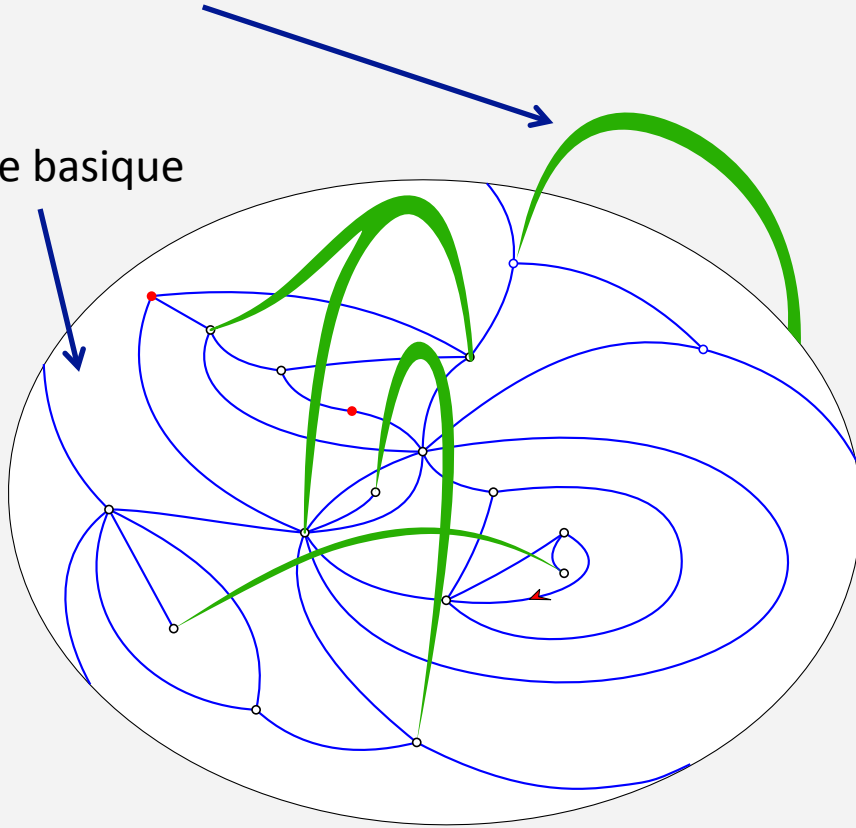


“La pâte feuilletée, ou feuilletage, est un type de pâte obtenue par abaissage et pliages successifs de couches alternant une pâte basique (farine, eau et sel) et du beurre.”



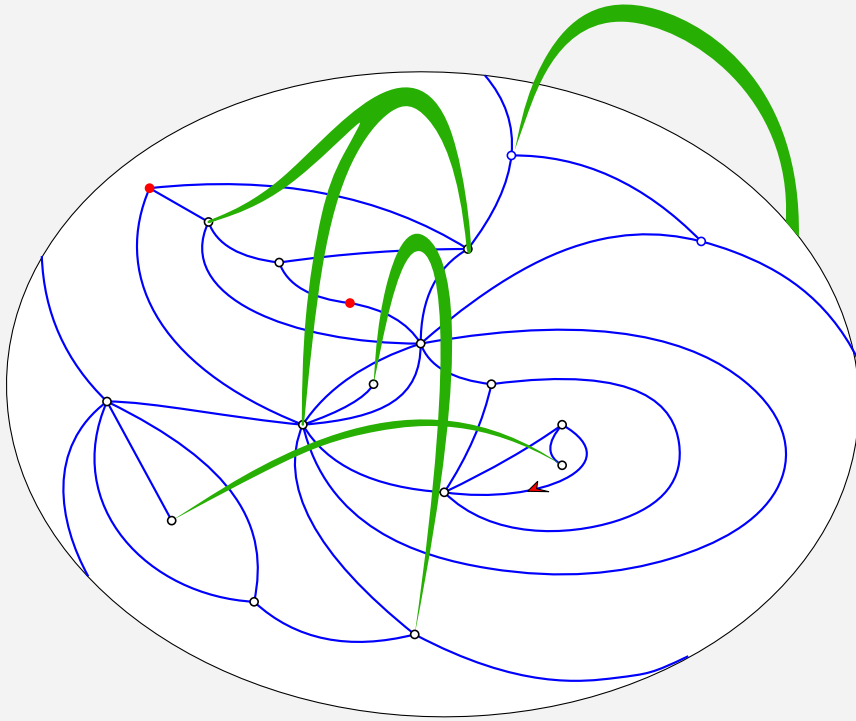
Beurre non passé

Pâte basique



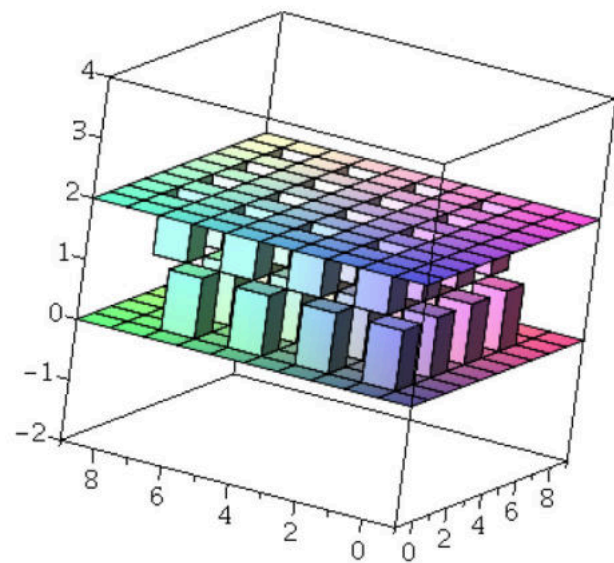
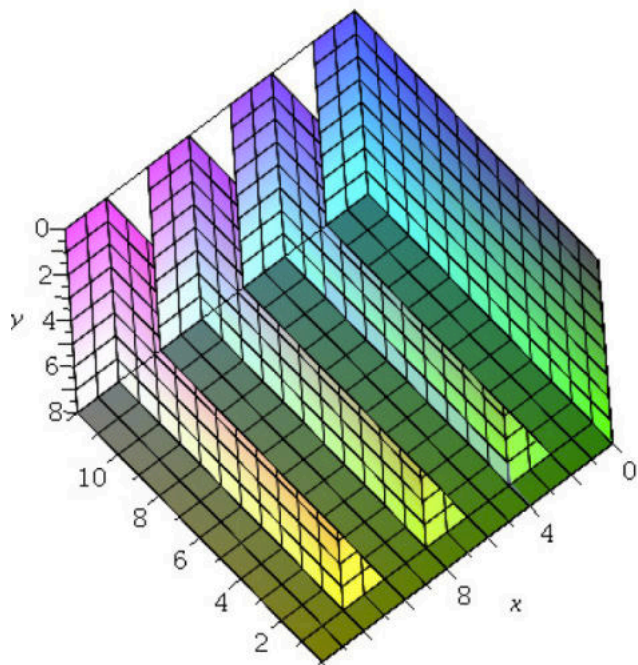
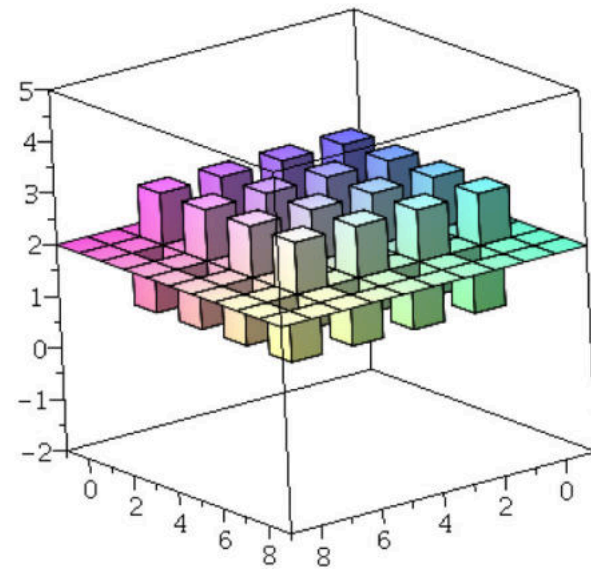
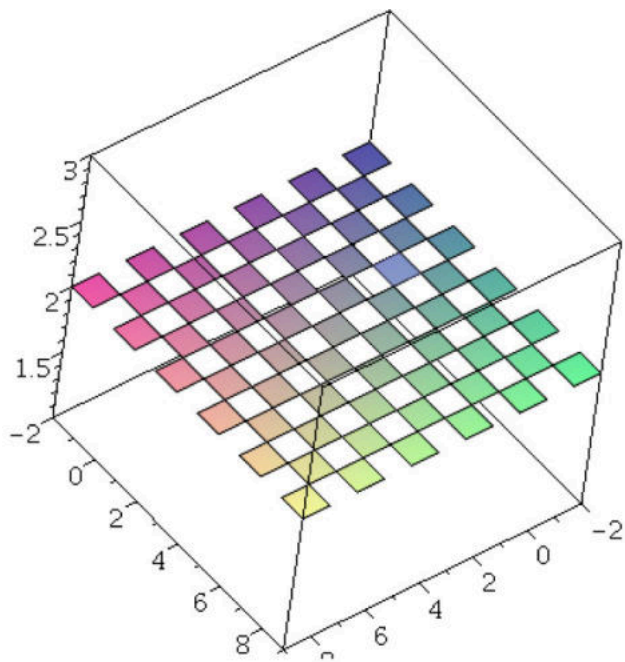
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- Sequence of random graphs with asymptotic diameter  $O(n^{1/8})$
- Asymptotically not going to give random trees or surfaces of any genus





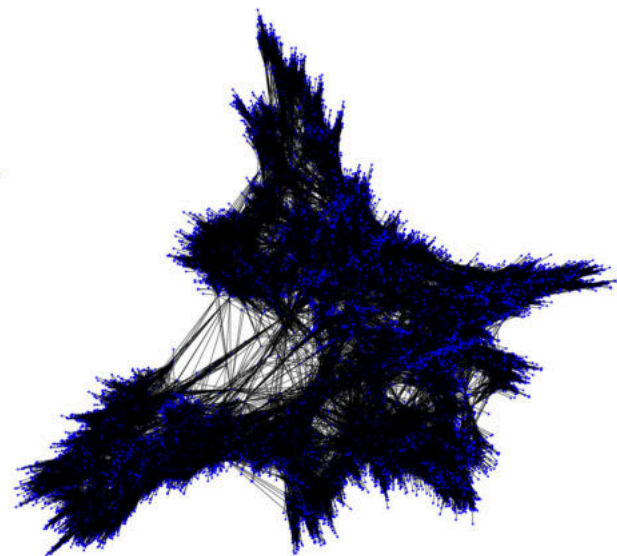
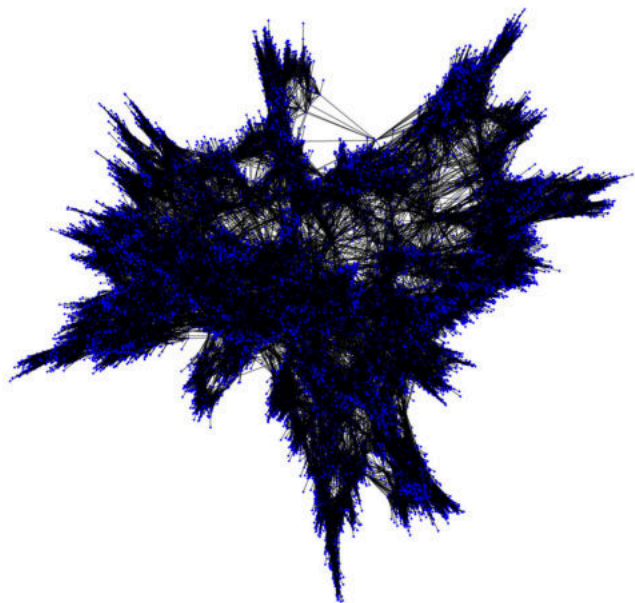
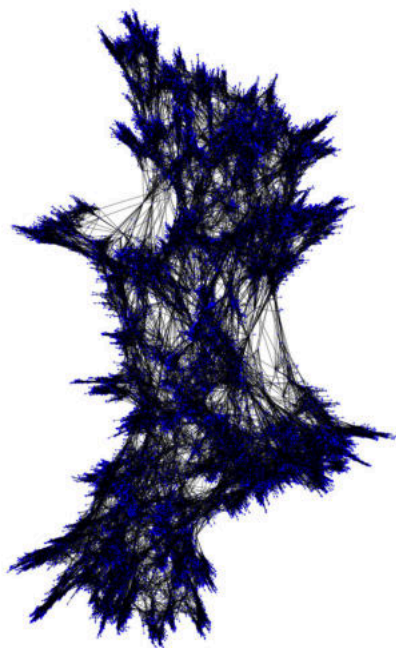
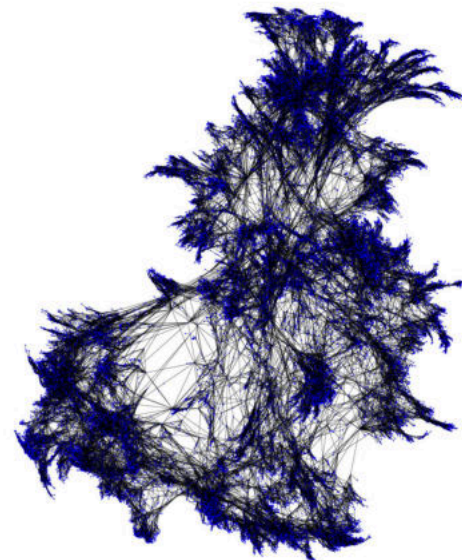
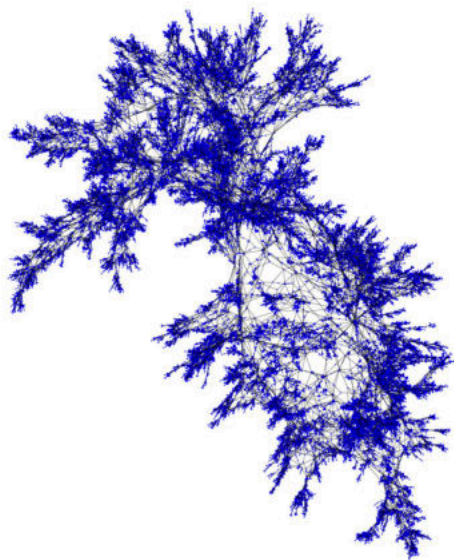
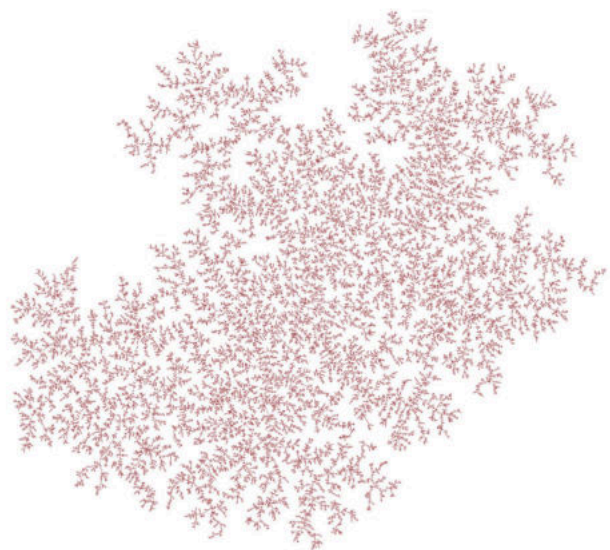


→ Iteration

# D-Random feuilletages:

→ Iterative construction to obtain a family indexed by a parameter  $D$

- **$D=1$ :** discrete objects are uniform rooted plane **trees** with  $n+1$  vertices.  
CV to the **continuum random tree** in terms of Gromov-Hausdorff.  
Typical diameter  $\sim n^{1/2}$  (Hausdorff dimension **2**).
- **$D=2$ :** discrete objects are uniform root. point. planar quadrangulations with  $n+2$  vertices.  
CV to the **Brownian map** in terms of Gromov-Hausdorff.  
Typical diameter  $\sim n^{1/4}$  (Hausdorff dimension **4**).
- **$D=3$ :** discrete objects obtained by a series of foldings of a discrete 2-sphere with  $n+3$  vertices, rooted and pointed 2 times.  
CV to a **continuum space** with **typical diameter**  $O(n^{1/8})$  (Hausdorff dimension **8** or more).
- **$D$ :** discrete objects obtained by  $D-2$  series of foldings of a discrete 2-sphere with  $n+D$  vertices, rooted and pointed  $D-1$  times.  
CV to a **continuum space** with Hausdorff dimension  **$2^D$**  or more).



## Asymptotic enumeration:

From the CVS bijection, for rooted pointed and non-pointed planar quadrangulations:

$$m_n^{\bullet(2)} = 2 \times 3^n C_n \quad \text{and} \quad (n+2)m_n^{(2)} = 2 \times 3^n C_n$$

$$m_n^{\bullet(2)} \sim 2\pi^{-1/2} \cdot 12^n \cdot n^{\gamma_2-1} \quad \text{and} \quad m_n^{(2)} \sim 2\pi^{-1/2} \cdot 12^n \cdot n^{\gamma_2-2}$$

$$\gamma_2 = -1/2$$

Not entirely clear for instance how the feuilletages are pointed, but in any case, this factor is asymptotically of order  $n^{D-1}$ , so that we have asymptotically:

$$m_n^{\bullet(D)} \sim c_D \cdot \lambda_D^n \cdot n^{\gamma_D+D-3} \quad \text{and} \quad m_n^{(D)} \sim c_D \cdot \lambda_D^n \cdot n^{\gamma_D-2}$$

$$\gamma_D = \frac{3}{2} - D \quad \longrightarrow \propto 4 \times 3^{\sum_{j=0}^{D-2} 2^j}$$

This generalizes the  $\gamma_1=1/2$  for trees and  $\gamma_2=-1/2$  for planar maps and more generally:

$$\gamma_{2,L} = \frac{3}{2}L + \frac{1}{2} \quad \gamma_{2,g} = \frac{5}{2}g - \frac{1}{2}$$

A combinatorial encoding:

$$\mathcal{F}_N[D] = (C_N, \sigma^{(D)}, \dots, \sigma^{(1)})$$

$$C_N = \{0 < 1 < \dots < N - 1\}$$

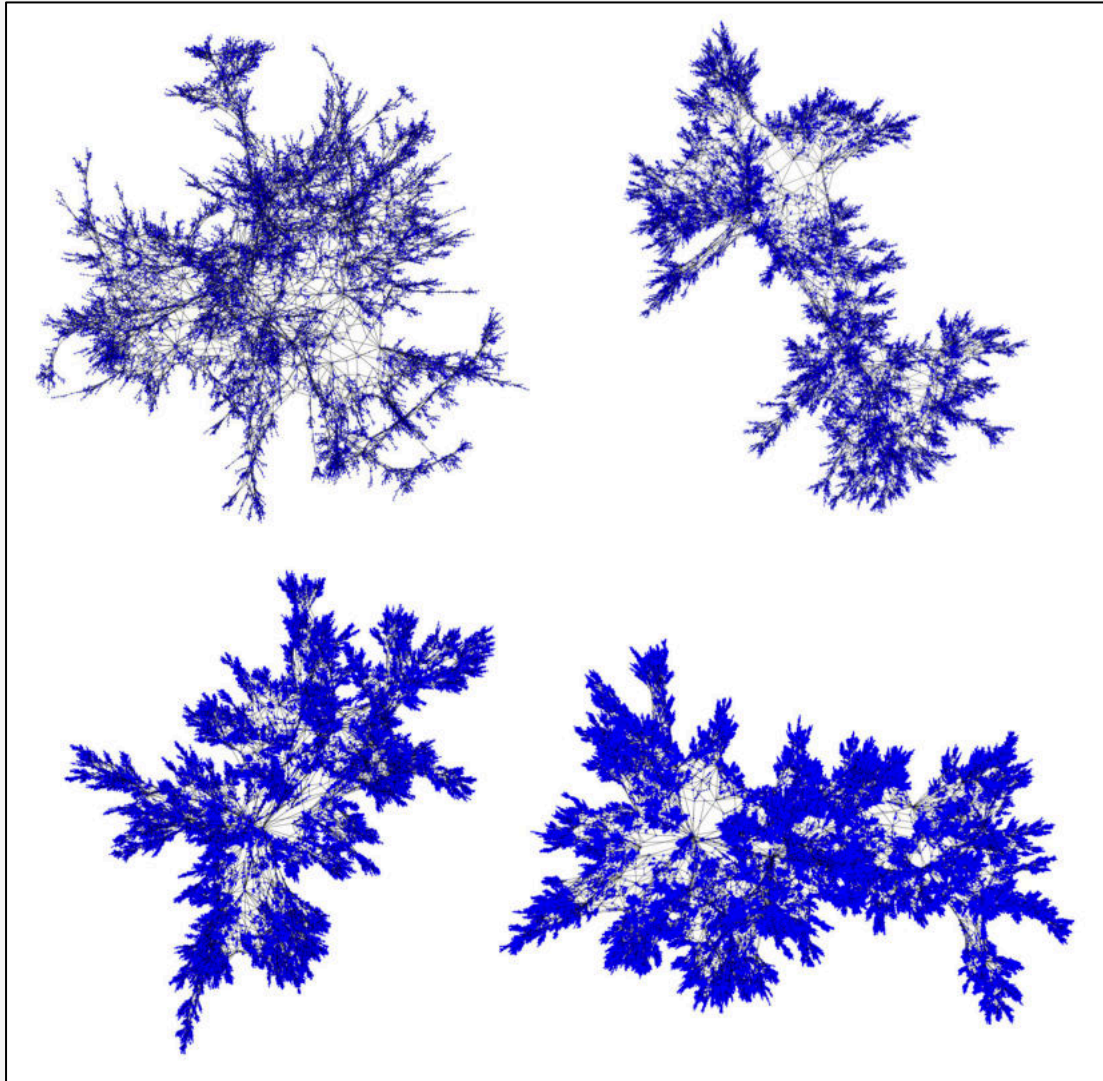
$\sigma^{(j)}$  nested non-crossing partitions on  $C_N$

$\sigma^{(j)}$  + parity gives a tree ( $\mathbf{T}_n^{(j)}$  of scale  $n^{1/2^j}$ , for the right distributions)

$(\sigma^{(j)}, \sigma^{(j-1)})$  + parity gives a planar map (random planar quad. of scale  $n^{1/2^j}$  for the right dist.

# Questions...

- We conjecture that the asymptotic diameter is of order  $n^{1/2^D}$  (we know it is  $\leq$ ) and that the Hausdorff dimension is  $2^D$  (*we know it is  $\geq$* ) ... needs to be proven.
- Convergence in the sense of Gromov-Hausdorff (second level)?
- What can we say about the topology? (hard)
- Can we obtain the same scaling limit from a model of random  $D$ -dimensional triangulations? (Universality class?)
- Spectral dimension?



Thank you!