

Variations on meanders

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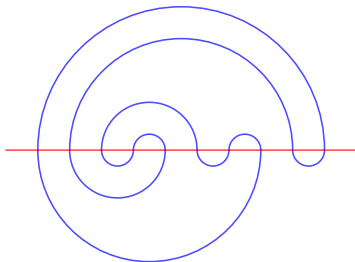
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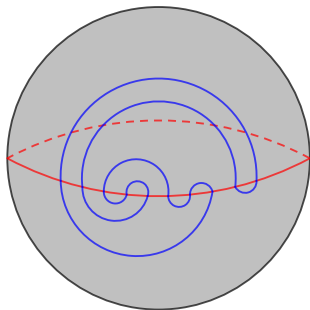
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Meanders



topological configuration of a line and a simple closed curve in the plane, intersecting transversally



topological configuration of a pair of tranverse simple closed curves on the 2-sphere
(+ marking)

Conjectural asymptotics

Conjecture (Di Francesco-Golinelli-Guitter, '97)

The number of meanders with $2n$ crossings is

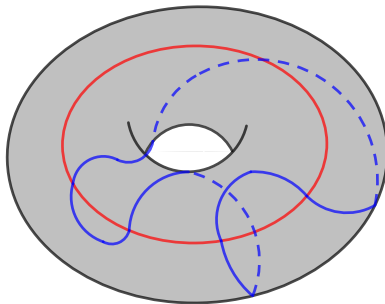
$$M_n \sim CR^n n^{-\alpha}$$

with $C > 0$, $R \simeq 12.2628$, $\alpha = \frac{29+\sqrt{145}}{12}$.

Jensen Guttman '00: extensive numerics non infirming the conjecture

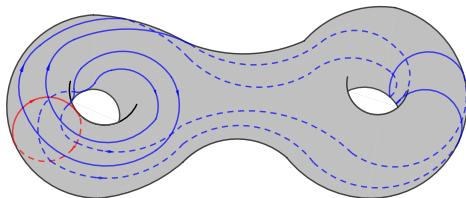
Albert-Paterson '05: $11.380 < R < 12.901$

Higher genus meanders



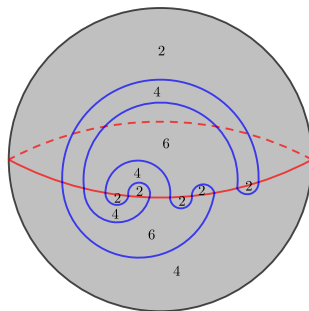
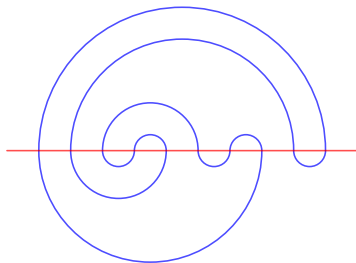
Topological configuration of a pair of tranverse simple closed curves on a higher genus surface, such that the corresponding faces are topological disks.

Oriented meander



Higher genus meander with a coherent orientation of the curves.

Results: counting for a fixed number of bigons



Results: counting for a fixed number of bigons

$M_{n,p}$: Plane meanders with $2n$ crossings and p bigons

Theorem (DGZZ, '17)

$$\sum_{n \leq N} M_{n,p} = A_p N^{2p-5} + o(N^{2p-5})$$

with

$$A_p = \frac{2}{p!(p-4)!(2p-5)} \left(\frac{2}{\pi^2}\right)^{p-3} \binom{2p-4}{p-2}^2$$

and

$$A_p \sim \left(\frac{32e^2}{\pi^2 p^2}\right)^p \text{ as } p \rightarrow \infty.$$

Results: counting for a fixed number of bigons

$M_{n,p}^g$: Genus g meanders with $2n$ crossings and p bigons

Theorem (DGZZ, '20)

- Explicit formula for $\sum_{n \leq N} M_{n,p}^g$ in term of algebraic quantities

-

$$\sum_{n \leq N} M_{n,p}^g = A_{g,p} N^{6g-5+2p} + o(N^{6g-6+2p})$$

with

$$A_{g,p} \sim b_g \left(\frac{32}{\pi^2} \right)^p p^{7g/2-6}. \text{ as } p \rightarrow \infty$$

Results: counting for a fixed number of bigons

M_n^{g+} : Oriented genus g meanders with $2n$ crossings

Theorem (DGZZ, '20)

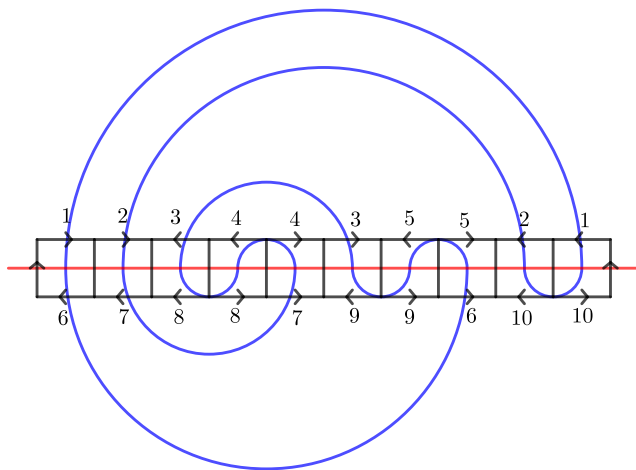
- Explicit formula for $\sum_{n \leq N} M_n^{g+}$ in term of algebraic quantities (+ recursion)

- $$\sum_{n \leq N} M_n^{g+} = A_g N^{4g-2} + o(N^{4g-2})$$

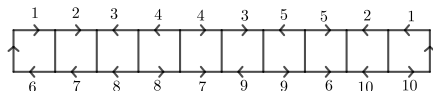
with

$$A_g \sim \frac{1}{g^{7/2}} \left(\frac{e^2}{16g^2} \right)^g \text{ as } g \rightarrow \infty.$$

Outline of the proof: meanders as square-tiled surfaces

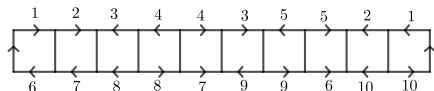


Outline of the proof: meanders as square-tiled surfaces



A **square-tiled surface** is a surface glued from squares (vertical \leftrightarrow vertical, horizontal \leftrightarrow horizontal). They form a special family of flat surfaces with conical singularities (half-translation surfaces).

Outline of the proof: meanders as square-tiled surfaces



A **square-tiled surface** is a surface glued from squares (vertical \leftrightarrow vertical, horizontal \leftrightarrow horizontal). They form a special family of flat surfaces with conical singularities (half-translation surfaces).

Meanders with $2n$ crossings correspond to marked square-tiled surfaces of genus 0 with $2n$ squares and exactly one horizontal cylinder, one vertical cylinder.

Bigons correspond to singularities of angle π .

Outline of the proof: square-tiled surfaces and volumes of moduli spaces

- Such square-tiled surfaces correspond to special points (integer points) in the family $\mathcal{Q}_{0,p}$ of flat surfaces of genus 0 with p singularities of angle π . (moduli space for half-translation surfaces / quadratic differentials)
- $\text{Vol } \mathcal{Q}_{0,p} = c \lim_{n \rightarrow \infty} \frac{\text{Card}\{\text{SQT with } \leq n \text{ squares, genus 0 and } p \text{ angles } \pi\}}{n^{2p-6}}$
- Let $\text{cyl}_1(0, p)$ be the same limit for SQT with one horizontal cylinder, and $\text{cyl}_{1,1}(0, p)$ the same limit for SQT for one horizontal cylinder and one vertical cylinder.

Theorem (DGZZ '17)

$$\frac{\text{cyl}_{1,1}(0, p)}{\text{cyl}_1(0, p)} = \frac{\text{cyl}_1(0, p)}{\text{Vol } \mathcal{Q}(0, p)}$$

Outline of the proof: computation of volumes and recent advances

- Eskin-Okounkov \sim '00: algorithm to compute volumes of moduli space of small dimension.
- Athreya-Eskin-Zorich '12: closed formulas for $\text{Vol } \mathcal{Q}_{0,p}$
- DGZZ, Chen-Möller-Sauvaget '18-19: explicit formulas for volumes in terms of algebraic quantities
- Andersen-Borot-Charbonnier-Delecroix-Giacchetto-Lewanski-Wheeler '19: topological recursion for volumes
- Aggarwal, Sauvaget, Chen-Möller-Zagier, '18-19: Large genus asymptotics of volumes (oriented case)
- Chen-Möller-Sauvaget '19: asymptotics for fixed genus and $p \rightarrow \infty$
- ...

Perspectives

- Meanders with fixed combinatorics (number of k -gons)
- Probability for a pair of arc systems to form a meander
- Conjectures on the number of connected components of random multicurves
- "4-meanders"