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Classification of P-oligomorphic permutation groups Conjectures of Cameron and Macpherson

Justine Falque joint work with Nicolas M. Thiéry

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JCB, February 7th of 2020

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Profile of a permutation group, a finite example

• Permutation group G

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Profile of a permutation group, a finite example

• Permutation group $G \rightarrow \text{induced action on } subsets$

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Profile of a permutation group, a finite example

- Permutation group $G \rightarrow$ induced action on *subsets*
- Orbit of *n*-subsets = orbit of *degree* n

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Profile of a permutation group, a finite example

- $\bullet \ {\rm Permutation \ group} \ G \quad \rightarrow \quad {\rm induced \ action \ on} \ subsets$
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Profile of G

 $\varphi_G(n) = \# \text{ orbits of } degree \ n$

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Example

n	φ_G	n	φ_G
0		5	
1		6	
2		7	
3		8	
4		> 8	



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0	1	5	3
1	1	6	
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n	φ_G	n	φ_G
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2	3	7	1
3	3	8	
4	6	> 8	



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2	3	7	1
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Series of the profile

 $P_{G_3}(z) = 1 + 1z + 3z^2 + 3z^3 + 6z^4 + 3z^5 + 3z^6 + 1z^7 + 1z^8$

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 $P_{G_3}(z) = 1 + 1z + 3z^2 + 3z^3 + 6z^4 + 3z^5 + 3z^6 + 1z^7 + 1z^8$

G infinite $\rightarrow \mathcal{H}_G(z) = \sum_n \varphi_G(n) z^n$

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Hypothesis

G is P-oligomorphic: φ_G is bounded by a polynomial in n

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Hypothesis

G is *P*-oligomorphic : φ_G is bounded by a polynomial in *n*

Example

$$\mathcal{H}_{\mathfrak{S}_{\infty}}(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}$$

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$$\mathcal{H}_{\mathfrak{S}_{\infty}}(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}$$

Conjecture 1 - Cameron, 70's G P-oligomorphic $\Rightarrow \varphi_G(n) \sim an^k$

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Orbit algebra

Orbit algebra (Cameron, 80's) Structure of graded algebra $\mathcal{A}_G = \bigoplus_n \mathcal{A}_n$ on the orbits

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Orbit algebra

Orbit algebra (Cameron, 80's)

Structure of graded algebra $\mathcal{A}_G = \bigoplus_n \mathcal{A}_n$ on the orbits

• vector space formally spanned by the orbits of G (*i.e.* of basis indexed by the orbits)

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Structure of graded algebra $\mathcal{A}_G = \bigoplus_n \mathcal{A}_n$ on the orbits

- vector space formally spanned by the orbits of G (*i.e.* of basis indexed by the orbits)
- combinatorial description of the product

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- $\dim(\mathcal{A}_n) = \varphi_G(n)$

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- dim $(\mathcal{A}_n) = \varphi_G(n)$, so $\mathcal{H}_G(z) = \sum_n \dim(\mathcal{A}_n) z^n$

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Hilbert series of the graded algebra

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Conjecture of Macpherson

Example.

 $\mathcal{A}_{\mathfrak{S}_{\infty}} \simeq \mathbb{Q}[X]$

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Conjecture of Macpherson

Example. $\mathcal{A}_{\mathfrak{S}_{\infty}} \simeq \mathbb{Q}[X]$

Conjecture 2 (stronger) - Macpherson, 85 G P-oligomorphic $\Rightarrow \mathcal{A}_G$ is finitely generated





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Conjecture 2 (stronger) - Macpherson, 85 G P-oligomorphic $\Rightarrow \mathcal{A}_G$ is finitely generated

Theorem (Thiéry, F. 2018)

The orbit algebra of a P-oligomorphic group is finitely generated, and Cohen-Macaulay.

In particular, its profile is polynomial in the strong sense.

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Block systems

Block system

• Set partition of the domain into blocks

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- Set partition of the domain into blocks
- such that G acts by permutation on the blocks

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Example

Block systems of C_4 Not a block system \rightarrow 3 2

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The (closed) primitive P-oligomorphic groups



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The (closed) primitive P-oligomorphic groups

Macpherson:

G $P\text{-oligomorphic with no (non trivial) blocks <math display="inline">\Rightarrow \varphi_G(n) = 1 \ \forall n$



 \mathfrak{S}_{∞}

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The (closed) primitive P-oligomorphic groups

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Theorem (Classification, Cameron) Only 5 closed groups such that $\varphi_G(n) = 1 \quad \forall n$

 \mathfrak{S}_∞

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The (closed) primitive *P*-oligomorphic groups Macpherson:

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Theorem (Classification, Cameron) Only 5 closed groups such that $\varphi_G(n) = 1 \quad \forall n$

- $\operatorname{Aut}({\mathbb Q})$: automorphisms of the rational chain
- $\operatorname{Rev}(\mathbb{Q})$: generated by $\operatorname{Aut}(\mathbb{Q})$ and one reflection
- $\operatorname{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the circular order
- $\operatorname{Rev}(\mathbb{Q}/\mathbb{Z})$: generated by $\operatorname{Aut}(\mathbb{Q}/\mathbb{Z})$ and a reflection
- \mathfrak{S}_{∞} : the symmetric group

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- $\operatorname{Rev}(\mathbb{Q}/\mathbb{Z})$: generated by $\operatorname{Aut}(\mathbb{Q}/\mathbb{Z})$ and a reflection
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Well known, nice groups (called *highly homogeneous*). In particular, their orbit algebra is finitely generated.



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An infinite example: $\mathfrak{S}_{\infty} \wr \mathfrak{S}_3$

 $\mathfrak{S}_{\infty}\wr\mathfrak{S}_{3}$



Wreath product

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An infinite example: $\mathfrak{S}_{\infty} \wr \mathfrak{S}_3$



Wreath product

$$\mathfrak{S}_{\infty}\wr\mathfrak{S}_3 \simeq \mathfrak{S}_{\infty}^3\rtimes\mathfrak{S}_3$$



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An infinite example: $\mathfrak{S}_{\infty} \wr \mathfrak{S}_3$

 \mathfrak{S}_3 \mathfrak{S}_∞ \mathfrak{S}_{∞} \mathfrak{S}_{∞} Wreath product

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Subset of "shape" 2, 3, 2

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Subset of "shape" 2, 3, 2 $\rightarrow x_1^2 x_2^3 x_3^2$

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Subset of "shape" 2,3,2 $\rightarrow x_1^2 x_2^3 x_3^2$

Orbits of subsets

 \leftrightarrow symmetric polynomials in x_1, x_2, x_3

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Subset of "shape" 2,3,2 $\rightarrow x_1^2 x_2^3 x_3^2$

Orbits of subsets \leftrightarrow symmetric polynomials in x_1, x_2, x_3

$$\mathcal{A}_{\mathfrak{S}_{\infty}\wr\mathfrak{S}_{3}} \simeq \operatorname{Sym}_{3}[X] = \mathbb{Q}[X]^{\mathfrak{S}_{3}}$$

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An infinite example: $\mathfrak{S}_{\infty} \wr \mathfrak{S}_3$

 \mathfrak{S}_3 \mathfrak{S}_{∞} \mathfrak{S}_{∞}

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Orbits of subsets \leftrightarrow symmetric polynomials in x_1, x_2, x_3

$$\mathcal{A}_{\mathfrak{S}_{\infty}\wr\mathfrak{S}_3} \simeq \operatorname{Sym}_3[X] = \mathbb{Q}[X]^{\mathfrak{S}_3}$$

Examples

Integer partitions; combinations; P-partitions... (with optional length and/or height restrictions)



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Further examples

For H subgroup of \mathfrak{S}_m :

•
$$G = \mathfrak{S}_{\infty} \wr H$$
:
 $\mathcal{A}_G \simeq \mathbb{Q}[X_1, \dots, X_m]^H$, the algebra of invariants of H



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Further examples

For H subgroup of \mathfrak{S}_m :

• $G = \mathfrak{S}_{\infty} \wr H$: $\mathcal{A}_G \simeq \mathbb{Q}[X_1, \dots, X_m]^H$, the algebra of invariants of H

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 \cdots ()()()()() \cdots

• $G = H \wr \mathfrak{S}_{\infty}$: $\mathcal{A}_G \simeq \mathbb{Q}[(X_o)_{o \in \operatorname{orb}(H)}]$ polynomial algebra generated by $\operatorname{orb}(H)$

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What block system to choose?

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One superblock

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What block system to choose?

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 $\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$

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What block system to choose?



 $\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$

If G is P-oligomorphic, either:

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What block system to choose?



 $\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$

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 $\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$

If G is P-oligomorphic, either:

• $M < \infty$

Nested block system

One superblock

Classification 00000 Bonus

What block system to choose?



 $\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$

If G is P-oligomorphic, either:

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or

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Nested block system

One superblock

Classification 00000 Bonus

What block system to choose?



 $\implies G \leq \mathfrak{S}_M \wr \mathfrak{S}_N$

If G is P-oligomorphic, either:

• $M < \infty$ $\longrightarrow \varphi_G(n) \ge O(n^{M-1})$

or

• $N < \infty$

Nested block system

One superblock

Classification 00000 Bonus

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Nested block system

One superblock

Classification 00000 Bonus

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- $M < \infty$ $\longrightarrow \varphi_G(n) \ge O(n^{M-1})$ or
- $N < \infty$ $\longrightarrow \varphi_G(n) \ge O(n^{N-1})$

Better have big finite blocks and/or "small" infinite ones...

Profile, conjectures 0000000	Nested block system 00000000	One superblock 0000	Classification 00000	Bonus	
	Lat	tices			

Partially ordered set (poset) with notions of join \lor and meet \land : any subset has a unique supremum (resp. infimum).

Profile, conjectures ooooooo	Nested block system 00000000	One superblock 0000	Classification 00000	Bonus	
	Lat	tices			

Partially ordered set (poset) with notions of join \lor and meet \land : any subset has a unique supremum (resp. infimum).

Not a lattice:



Profile, conjectures 0000000	Nested block system 00000000	One superblock 0000	Classification 00000	Bonus	
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Profile, conjectures	Nested block system	One superblock	Classification	Bonus	
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Profile, conjectures	Nested block system	One superblock	Classification	Bonus	
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Profile, conjectures	Nested block system	One superblock	Classification	Bonus	
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Not a lattice:





Nested block system 00000000

One superblock

Classificatio

Bonus

Lattices of block systems

Lattice of partitions \rightarrow lattice on block systems

Nested block system

One superblock

Classification 00000 Bonus

Lattices of block systems



Nested block system

One superblock

Classification 00000 Bonus

Lattices of block systems





Non trivial fact

- {Systems with $< \infty$ blocks only} = sublattice with maximum
- {Systems with ∞ blocks only} = sublattice with minimum

Nested block system

One superblock

Classification 00000 Bonus

Lattices of block systems





Non trivial fact

- {Systems with $< \infty$ blocks only} = sublattice with maximum
- {Systems with ∞ blocks only} = sublattice with minimum

Remark. If G is P-oligomorphic, both of them are actually finite!

Nested block system 0000000

One superblock 0000 Classification 00000 Bonus

The nested block system

Idea

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Nested block system 0000000

One superblock 0000 Classification 00000 Bonus

The nested block system

Idea

1. Take the *maximal* system of finite blocks

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Nested block system

One superblock 0000 Classification 00000 Bonus

The nested block system

Idea

1. Take the *maximal* system of finite blocks



Nested block system

One superblock 0000 Classification 00000 Bonus

The nested block system

Idea

1. Take the *maximal* system of finite blocks



Action on the maximal finite blocks...

Nested block system

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The nested block system

Idea

1. Take the *maximal* system of finite blocks



Nested block system

One superblock 0000 Classification 00000 Bonus

The nested block system

Idea

- 1. Take the *maximal* system of finite blocks
- 2. Take the minimal system of infinite blocks of the action of G on the maximal finite blocks



Nested block system

One superblock 0000 Classification 00000 Bonus

The nested block system

Idea

- 1. Take the *maximal* system of finite blocks
- 2. Take the *minimal* system of infinite blocks of the action of G on the maximal finite blocks \rightarrow finitely many superblocks



Nested block system

One superblock 0000 Classification 00000 Bonus

The nested block system

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- 2. Take the *minimal* system of infinite blocks of the action of G on the maximal finite blocks \rightarrow finitely many superblocks



Nested block system 00000000

One superblock $\bullet \circ \circ \circ$

Classificatio

Bonus



Nested block system 00000000

One superblock $\bullet \circ \circ \circ$

Classificatio 00000

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Bonus

One superblock: examples

Nested block system 00000000

One superblock $\bullet \circ \circ \circ$

Classificatio 00000 Bonus



Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classificatio 00000 Bonus



Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classificatio 00000 Bonus



Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classificatio 00000 Bonus



Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classificatio 00000 Bonus

One superblock: examples



 $G_{|B_0} = H_0$

Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classificatio 00000 Bonus

One superblock: examples



 $G_{|B_0} = H_0 , \quad \operatorname{Fix}(B_0)$

Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classificatio

Bonus

One superblock: examples



 $G_{|B_0} = H_0$, $\operatorname{Fix}(B_0)_{|B_1} = H_1$

Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus



 H_0 , H_1
Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus



Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus

One superblock: examples



 $H_0 \ , \ H_1 \ , \ H_2 \ , \ H_3$

Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus

One superblock: examples



 $H_0 , H_1 , H_2 , H_3 , H_4$

Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus



Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus



Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus



One superblock 0000



- " $H_0 \times \mathfrak{S}_{\infty}$ "
- $\rightarrow H_0$, Id , Id , Id , Id , Id . . .

Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus



- " $H_0 \times \mathfrak{S}_{\infty}$ " $\rightarrow H_0$, Id , Id , Id , Id , Id ...
- < " $H_0 \times \mathfrak{S}_{\infty}$ ", $H \wr \mathfrak{S}_{\infty} >$

Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus

One superblock: examples



• $< "H_0 \times \mathfrak{S}_\infty", H \wr \mathfrak{S}_\infty >$

 $H_0 \triangleright H$ w.l.o.g

Nested block system 00000000 One superblock $\bullet \circ \circ \circ$

Classification 00000 Bonus

One superblock: examples



Notation: $[H_0, H_\infty]$

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Classification 00000 Bonus

Subdirect product and synchronization

How to handle synchronizations between blocks ?

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Classificatio 00000 Bonus

Subdirect product and synchronization

How to handle synchronizations between blocks ?

Subdirect product of two groups, or actions

Subdirect product and synchronization

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Subdirect product of two groups, or actions

• Formalizes the *synchronization* between two actions

Subdirect product and synchronization

How to handle synchronizations between blocks ?

Subdirect product of two groups, or actions

- Formalizes the *synchronization* between two actions
- For instance, the actions on two different blocks

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Example

Subdirect product and synchronization

How to handle synchronizations between blocks ?

Subdirect product of two groups, or actions

- Formalizes the *synchronization* between two actions
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Example $< (1\ 2\ 3), (1\ 2), (4\ 5), (4\ 5\ 6) > \simeq \mathfrak{S}_3 \times \mathfrak{S}_3$

Subdirect product and synchronization

How to handle synchronizations between blocks ?

Subdirect product of two groups, or actions

- Formalizes the *synchronization* between two actions
- For instance, the actions on two different blocks

Example $< (1\ 2\ 3), (1\ 2), (4\ 5), (4\ 5\ 6) > \simeq \mathfrak{S}_3 \times \mathfrak{S}_3$ $< (1\ 2\ 3), (1\ 2)(4\ 5), (4\ 5\ 6) > \simeq$ subdirect product of $\mathfrak{S}_3 \times \mathfrak{S}_3$

Subdirect product and synchronization

How to handle synchronizations between blocks ?

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Example $< (1\ 2\ 3), (1\ 2), (4\ 5), (4\ 5\ 6) > \simeq \mathfrak{S}_3 \times \mathfrak{S}_3$ $< (1\ 2\ 3), (1\ 2)(4\ 5), (4\ 5\ 6) > \simeq \text{ subdirect product of } \mathfrak{S}_3 \times \mathfrak{S}_3$

Remark. The potential synchronizations of a group with another one are linked to its normal subgroups.

Nested block system 00000000 One superblock $\circ \circ \bullet \circ$

Classification 00000 Bonus

The tower determines $\operatorname{Stab}_G(\operatorname{blocks})$

Nested block system 00000000 One superblock $\circ \circ \bullet \circ$

Classification 00000 Bonus

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Nested block system 00000000 One superblock $\circ \circ \bullet \circ$

Classification 00000 Bonus

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Nested block system 00000000 One superblock $\circ \circ \bullet \circ$

Classification 00000 Bonus

The tower determines $\operatorname{Stab}_G(\operatorname{blocks})$



Nested block system

One superblock $000 \bullet$

Classification 00000 Bonus

One superblock: classification

• The tower determines G (uses the subdirect product and some technical lemmas)

Nested block system

One superblock $000 \bullet$

Classification 00000 Bonus

One superblock: classification

- The tower determines G (uses the subdirect product and some technical lemmas)
- Computer exploration on finitely many blocks

Nested block system

One superblock $\circ \circ \circ \bullet$

Classification 00000 Bonus

One superblock: classification

- The tower determines G (uses the subdirect product and some technical lemmas)
- Computer exploration on finitely many blocks
 - $\rightarrow\,$ Observation: always some $H_0,\ H,\ H,\ H,\ \dots,\ H,\ H_s$ Used the GAP database TransGrp 2.0.4 to browse transitive groups

Nested block system

Bonus

One superblock: classification

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 - \rightarrow Proof in the infinite case: always some $H_0, H, H, H \cdots$

Nested block system

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Classification

One superblock \Rightarrow $G = [H_0, H_\infty]$

Nested block system

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Classification

One superblock $\Rightarrow G = [H_0, H_\infty]$ $\mathbb{Q}[(X_o)_{o \in \operatorname{orb}(H)}]$

Nested block system

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Classification

One superblock \Rightarrow $G = [H_0, H_\infty]$

 $\mathcal{A}_G \simeq \mathbb{Q}[(X_o)_{o \in \operatorname{orb}(H)}]^{H_0}$

Nested block systen 00000000 One superblock 0000 Bonus

General case: minimal subgroup of finite index



Nested block systen 00000000 One superblock 0000 Classification •0000 Bonus

General case: minimal subgroup of finite index Normal subgroup K of G



Nested block systen 00000000 One superblock 0000 Classification •0000 Bonus

General case: minimal subgroup of finite index

Normal subgroup K of ${\cal G}$

• that fixes the kernel



Nested block systen 00000000 One superblock 0000 Classification •0000 Bonus

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Nested block systen 00000000 One superblock 0000 Classification •0000 Bonus

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Nested block system 00000000 One superblock 0000 Classification •0000 Bonus

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One superblock 0000 Bonus

General case: minimal subgroup of finite index

Normal subgroup K of G

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- of restrictions wreath products onto the superblocks



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 $\operatorname{Rev}(\mathbb{Q})$

General case: minimal subgroup of finite index

Normal subgroup K of G

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 $\operatorname{Aut}(\mathbb{Q})$

Nested block system 00000000 One superblock 0000 Classification 00000

Bonus

End of the proof of the conjectures

• In K, totally independent superblocks (and kernel)

Nested block system 00000000 One superblock

Classification 00000 Bonus

End of the proof of the conjectures

• In K, totally independent superblocks (and kernel) Consequence of the lack of finite index subgroups of \mathfrak{S}_{∞} , Aut(...) and wreath products

Nested block system 00000000 One superblock

Classification 00000

Bonus

End of the proof of the conjectures

 In K, totally independent superblocks (and kernel)
 Consequence of the lack of finite index subgroups of 𝔅_∞, Aut(...) and wreath products

 \Rightarrow direct product of the restrictions $K^{(i)}$

Nested block system 00000000 One superblock

Classification 00000 Bonus

End of the proof of the conjectures

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 \Rightarrow direct product of the restrictions $K^{(i)}$ $\Rightarrow A = \Theta A$

 $\Rightarrow \mathcal{A}_K = \bigotimes_i \mathcal{A}_{K^{(i)}}$

Nested block system 00000000 Bonus

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Nested block system 00000000 One superblock

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free algebra finitely generated by the orbits of the $H^{(i)}$'s

Nested block system 00000000 One superblock

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Nested block system 00000000 One superblock

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• Fact: G acts by permutation on these generators

Nested block system 00000000 One superblock

Bonus

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Nested block system 00000000 One superblock

Classification 00000 Bonus

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Nested block system 00000000 One superblock

Bonus

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Which ends the proof of the conjectures!

Nested block system

One superblock 0000 Classification 00000

Bonus

Classification of P-oligomorphic groups

G_0 a finite permutation group



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Bonus

Classification of *P*-oligomorphic groups

G_0 a finite permutation group, \mathcal{B}_0 a block system.



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Bonus

Classification of P-oligomorphic groups

G_0 a finite permutation group, \mathcal{B}_0 a block system.

For each orbit of blocks

Nested block system 00000000 One superblock 0000 Classification 00000

Bonus

Classification of *P*-oligomorphic groups

 G_0 a finite permutation group, \mathcal{B}_0 a block system.

For each orbit of blocks, choose

1. One group of profile 1



Nested block system 00000000 One superblock 0000 Classification 00000

Bonus

Classification of P-oligomorphic groups

 G_0 a finite permutation group, \mathcal{B}_0 a block system.

- 1. One group of profile 1
 - Has to be \mathfrak{S}_{∞} if the blocks are singletons



Nested block system 00000000 One superblock

Classification 00000

Bonus

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Nested block system 00000000 One superblock 0000 Classification 00000

Bonus

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Nested block system 00000000 One superblock 0000 Classification 00000

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Nested block system 00000000 One superblock 0000 Classification 00000

Bonus

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For each orbit of blocks, choose

- 1. One group of profile 1
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2. One normal subgroup H of $H_0 = G_{0|B}$ for B in the orbit



Nested block system 00000000 One superblock

Classification 00000

Bonus

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Nested block system 00000000 One superblock 0000 $\begin{array}{c} {\rm Classification} \\ {\scriptstyle 000 \bullet 0} \end{array}$

Bonus

Consequences and perspectives

Finite encoding \rightarrow ground for algorithmics

Nested block system 00000000 One superblock

Classification 00000

Bonus

Consequences and perspectives

Finite encoding \rightarrow ground for algorithmics

• Implementation of *P*-oligomorphic groups (SageMath)

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Classification 00000

Bonus

Consequences and perspectives

Finite encoding \rightarrow ground for algorithmics

- Implementation of P-oligomorphic groups (SageMath)
- count *P*-oligomorphic groups, per growth rate of the profile

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 $5,\, 6,\, 14,\, 33,\, 32,\, 114,\, 47,\, 323,\, 260,\, 338,\, 50,\, 2108,\, 58,\, 430,\, \dots$

Bonus

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Still on the plate: explore higher growths

Bonus

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Still on the plate: explore higher growths

- Bounded: classified before
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- Subexponential: some properties remain true, to investigate
- Exponential: wilder primitive groups appear...

Bonus

Thank you for your attention !

Context

- G permutation group of a countably infinite set E
- Profile φ_G : counts the orbits of subsets of E
- Hypothesis: $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron): $\varphi_G(n) \sim an^k$
- Conjecture (Macpherson): finite generation of the orbit algebra

Results

- Both conjectures hold !
- Classification of *P*-oligomorphic permutation groups
- The orbit algebra is an algebra of invariants (up to some 2-nilpotents)

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Example of a product in the cyclic group C_5

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Example of a product in the cyclic group \mathcal{C}_5



= 0

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Example of a product in the cyclic group \mathcal{C}_5



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The tower determines the group (0): "straight \mathfrak{S}_{∞} "

 ${\cal G}$ contains a set of "straight" swaps of blocks



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Subdirect product

Subdirect product of G_1 and G_2

- Formalizes the synchronization between G_1 and G_2
- Subgroup of $G_1 \times G_2$ (with canonical projections G_1 and G_2)
- $E = E_1 \sqcup E_2$ stable $\Rightarrow G$ subdirect product of $G_{|E_1}$ and $G_{|E_2}$

Synchronization in a subdirect product

Let $N_1 = \operatorname{Fix}_G(E_2)$ and $N_2 = \operatorname{Fix}_G(E_1)$.

$$\frac{G_1}{N_1} \simeq \frac{G}{N_1 \times N_2} \simeq \frac{G_2}{N_2}$$

A subdirect product with explicit N_i 's is explicit.

Remark. N_1 and N_2 are *normal* in G_1 and G_2 , so the possibilities of synchronization of a group is linked to its normal subgroups.

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Direct product in the case of finite blocks

"Speak, friend..."

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Direct product in the case of finite blocks

"Speak, friend..."

Example 3

 $C_3\times \mathfrak{S}_\infty$ acting on blocks of size 3

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

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$$\begin{array}{c} \mathrm{O}(\begin{array}{c} x \\ & \\ \end{array}) \\ \mathrm{O}(\begin{array}{c} x \\ & \\ \\ \end{array}) \end{array}$$

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Bonus

Direct product in the case of finite blocks

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Example 3

 $C_3\times \mathfrak{S}_\infty$ acting on blocks of size 3

 $G' = C_3 \ acting \ on \ (non \ empty) \ subsets$ $\mathbb{K}[x]^{G'} \longleftrightarrow$ Orbit algebra of $C_3 \times \mathfrak{S}_{\infty}$?

 $\mathcal{O}(\ x \underset{\bigotimes}{\bullet}).\mathcal{O}(\ x \underset{\bigotimes}{\bullet})$

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Bonus

Direct product in the case of finite blocks

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$$\mathcal{O}(x_{\textcircled{O}}) \mathcal{O}(x_{\textcircled{O}}) = \mathcal{O}(x_{\textcircled{O}}x_{\textcircled{O}})$$

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Bonus

Direct product in the case of finite blocks

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Example 3

 $C_3\times \mathfrak{S}_\infty$ acting on blocks of size 3

$$O(x_{0}) O(x_{0}) = O(x_{0} x_{0}) + O(x_{0} x_{0})$$

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 $\mathbb{K}[x]^{G'} \longleftrightarrow \text{Orbit algebra of } C_3 \times \mathfrak{S}_{\infty}$?

$$O(x_{0}) \cdot O(x_{0}) = O(x_{0} \cdot x_{0}) + O(x_{0} \cdot x_{0}) + O(x_{0} \cdot x_{0}) + O(x_{0} \cdot x_{0})$$
$$O(x_{0}) \cdot O(x_{0}) + O(x_{0} \cdot x_{0}) + O(x_{0} \cdot x_{0})$$

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Profile, conjectures	Nested block system	One superblock	Classification	Bonu
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The tower has shape H_0 , H, H, H, \cdots

Lemma to prove

G has tower H_0 H_1 H_2 $H_3 \Rightarrow H_1 = H_2$

Proof.

An element $s \in G$ stabilizing the blocks \leftrightarrow a quadruple

$$g \in H_1 \quad \to \quad \exists \ (1,g,h,k), \ h,k \in H_1.$$

Let σ be an element of G that permutes "straightforwardly" the first two blocks and fixes the other two.

Conjugation of x by σ in $G \to y = (g, 1, h, k)$ Then: $x^{-1}y = (g, g^{-1}, 1, 1)$

By arguing that the tower does not depend on the ordering of the blocks, g^{-1} and therefore g are in H_2 .

In the infinite case, apply to each restriction to four consecutive blocks of the fixator of the previous ones in G.

Profile, conjectures	Nested block system	One superblock	Classification	Bonus
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