# The domino problem for surface groups 

Sebastián Barbieri Lemp<br>Joint work with Nathalie Aubrun and Etienne Moutot

Université de Bordeaux
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## Classical domino problem

Consider a finite set $\tau$ of Wang tiles


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## Question:

is there a function $x: \mathbb{Z}^{2} \rightarrow \tau$ such that adjacent tiles share the same color?


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## Theorem (Kari 08')

The domino problem is undecidable in the binary hyperbolic tiling.

## General setting

Let us consider the following ingredients:

- A directed, labeled (infinite) graph $\Gamma=(V, E, L)$.
- A finite set of colors $\mathcal{A}$.
- A finite list of forbidden colored labeled connected finite graphs $\mathcal{F}$.


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## Domino problem for $\Gamma$ :

Is there an algorithm which decides, given $(\mathcal{A}, \mathcal{F})$, whether there exists a coloring $x: V \rightarrow \mathcal{A}$ such that no graph from $\mathcal{F}$ embeds?

The original domino problem:


Binary hyperbolic tiling:


## Binary hyperbolic tiling:



## General setting: Cayley graphs

A particularly interesting case is when $\Gamma=(V, E, L)$ is the Cayley graph of a finitely generated group $G$ given by the set of generators $S$.

- $V=G$.
- $E=\{(g, g s) \mid g \in G, s \in S\}$.
- $L(g, g s)=s$.


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- $L(g, g s)=s$.

Remark: the domino problem does not depend upon the set of generators $S$. These problems are all computationally (many-one) equivalent.
$\operatorname{DP}(G)$ is the domino problem of the group $G$.

## Cayley graph of free group.



## Domino problem on groups.

## List of facts:

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## Domino conjecture

A finitely generated group has decidable domino problem if and only if it is virtually free.

Verified for polycyclic groups, Baumslag-Solitar groups, Branch groups.

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## Theorem (Muller \& Schupp '85)

A graph has decidable monadic second order logic (MSO) if and only if it has finite tree-width.

- Fact 1: A group is virtually free if and only if its Cayley graphs have finite tree-width.
- Fact 2: The domino problem can be expressed in MSO.


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If DC holds, then the domino problem contains all the complexity of MSO for finitely generated groups.

## Surface groups

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Remark: we just need to show that the domino problem of

$$
\pi_{1}(\circlearrowleft) \cong\left\langle a, b, c, d \mid a b a^{-1} b^{-1} c d c^{-1} d^{-1}=1\right\rangle
$$

is undecidable.

## How to prove it

Proof idea: use hyperbolicity.

- Step 1: show undecidability of DP for a class of graphs which embed nicely in the hyperbolic plane.


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- Step 3: reduce $\operatorname{DP}\left(\pi_{1}(\infty)\right)$ to $\operatorname{DP}(\Gamma)$.


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- Step 3: reduce $\operatorname{DP}\left(\pi_{1}(\infty)\right)$ to $\mathrm{DP}(\Gamma)$.
- Step 4: profit.


## How to prove it: nice class

A (non-deterministic) substitution is a pair $(\mathcal{A}, R)$ where $\mathcal{A}$ is a finite alphabet and $R$ is a set of pairs $(a \mapsto w) \in \mathcal{A} \times \mathcal{A}^{*}$.

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Example

- $\mathcal{A}=\{0\}, R=\{(0 \mapsto 00)\}$.
- $\mathcal{A}=\{0,1\}, R=\{(1 \mapsto 0),(0 \mapsto 01)\}$.


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\begin{aligned}
& \text { - } \mathcal{A}=\{0\}, R=\{(0 \mapsto 00)\} \\
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\end{aligned}
$$

An infinite word $u=\ldots u_{-1} u_{0} u_{1} u_{2} \cdots \in \mathcal{A}^{\mathbb{Z}}$ produces a word $v=\ldots v_{-1} v_{0} v_{1} v_{2} \cdots \in \mathcal{A}^{\mathbb{Z}}$ if $v$ can be obtained from $u$ by applying a rule of $R$ on each symbol.

That is, there exists a function $\Delta: \mathbb{Z} \rightarrow \mathbb{Z}$ such that :

$$
\left(u_{i} \mapsto v_{\Delta(i)} \ldots v_{\Delta(i+1)-1}\right) \in R \text { for every } i \in \mathbb{Z}
$$

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Let $\left\{u_{i}\right\}_{i \in \mathbb{Z}}$ be a sequence of bi-infinite words such that $u_{i}$ produces $u_{i+1}$ (with $\Delta_{i}$ ). We can associate an orbit graph.

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- Join all consecutive symbols of $u_{i}$ by edges from left to right.
- Join each symbol of $u_{i}$ with the corresponding sequence of symbols it produces in $u_{i+1}$ assigning labels from left to right.


## Example 1: trivial substitution gives $\mathbb{Z}^{2}$.

$$
\mathcal{A}=\{0\} R=\{(0 \mapsto 0)\} .
$$



## Example 2: Doubling substitution gives bin hyp tiling.

$$
\mathcal{A}=\{0\} R=\{(0 \mapsto 00)\} .
$$



## Undecidability: reduce to example 2.

Idea: take an orbit graph $\Gamma$.


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In each vertex code a finite subgraph of the binary orbit graph + information on how to locally paste them together.


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Impose local consistency rules.

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- Suppose $\operatorname{DP}(\Gamma)$ is decidable.
- Use the previous tiling to encode the binary orbit graph.
- Let $(\mathcal{A}, \mathcal{F})$ be an alphabet and a set of forbidden patterns for the binary orbit graph. Use the encoding to simulate tilings in $\Gamma$.
- As $\operatorname{DP}(\Gamma)$ is decidable, we may use the associated algorithm to decide whether $(\mathcal{A}, \mathcal{F})$ admits a tiling of the binary orbit graph.
- contradiction $\checkmark$.


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## Warning

- We must check that the language of coded subgraphs is finite.
- We must check that the set of encodings is non-empty.


## Hyperbolic geometry to the rescue!

A substitution $(\mathcal{A}, R)$ has an expanding eigenvalue if there exists $\lambda>1$ and $v: \mathcal{A} \rightarrow \mathbb{R}^{+}$such that for every $\left(a \mapsto u_{1} \ldots u_{k}\right) \in R$ :

$$
\lambda v(a)=\left(v\left(u_{1}\right)+v\left(u_{2}\right)+\cdots+v\left(u_{k}\right)\right)
$$

## Example

$\mathcal{A}=\{0\} R=\{(0 \mapsto 00)\}$ admits the expanding eigenvalue $\lambda=2$.

$$
2 \lambda v(0)=(v(0)+v(0))
$$

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Remark: Tiling superpositions were introduced by D.B. Cohen and C. Goodman-Strauss to produce aperiodic tilings of surface groups.

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Theorem (Aubrun, B., Moutot)
For every orbit graph Г of a substitution with an expanding
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## Question

How does this relate to the fundamental group of $\square$
There is a "hidden" substitution in that group, namely $\mathcal{A}=\{\mathrm{a}, \mathrm{b}\}$ and

$$
\left\{\left(\mathrm{a} \mapsto \mathrm{ab}^{5} a b^{5} a b^{5} a b^{5} a b^{4}\right),\left(\mathrm{b} \mapsto a b^{5} a b^{5} a b^{5} a b^{5} a b^{5} a b^{4}\right) \cdot\right\}
$$

with $\lambda=17+12 \sqrt{2}$ and $v(\mathrm{~b}) / v(\mathrm{a})=\frac{1+\sqrt{2}}{2}$.


A way to look at this Cayley graph is as a translation surface obtained by pasting together octagons.




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- Encode substitution structure using a finite alphabet and local rules. $\checkmark$


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## Theorem (Aubrun, B., Moutot)

The domino problem is undecidable on the fundamental group of the closed orientable surface of genus 2.

## Word-hyperbolic groups

## Word-hyperbolic group

A finitely generated group is word-hyperbolic if the geodesic triangles of one of its Cayley graphs are $\delta$-slim for some $\delta>0$.


2
${ }^{2}$ From https://en.wikipedia.org/wiki/Hyperbolic_group

## Word-hyperbolic groups

Facts about word-hyperbolic groups:

- Virtually free groups $\checkmark$.
- Surface groups (genus $g \geq 2$ ) $\checkmark$.
- Nice computability properties: Finitely presented, decidable word problem, Dehn's algorithm works, language of shortlex geodesics is regular, etc.
- A random group is almost surely word-hyperbolic.


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Bottom line: testing ground for
Domino conjecture
A finitely generated group has decidable domino problem if and only if it is virtually free.

## word-hyperbolic groups

## Gromov's conjecture

The fundamental group of
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Facts:

- If a group $H$ embeds into a group $G$, then the domino problem of $G$ is computationally harder than the domino problem of $H$.
- If a word-hyperbolic group is not virtually free, it contains an embedded one-ended word-hyperbolic group.
- If GC holds, then every word-hyperbolic group which is not virtually free contains an embedded copy of the fundamental group of $\qquad$


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## Theorem

If GC holds, then the domino problem conjecture holds for every word-hyperbolic group.

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If GC holds, then the domino problem conjecture holds for every word-hyperbolic group.

- Fun fact: find a (non virt free) word-hyperbolic group with decidable domino problem and you shall attain fame and glory disprove Gromov's conjecture!
- Fun fact: Same can be shown with weaker versions of GC.


## Thank you for your attention!



The domino problem is undecidable on surface groups. https://arxiv.org/abs/1811.08420


[^0]:    ${ }^{1}$ https://math.stackexchange.com/questions/1834108/ cayley-graph-of-the-fundamental-group-of-the-2-torus

