The domino problem for surface groups

Sebastián **Barbieri Lemp** Joint work with Nathalie Aubrun and Etienne Moutot

Université de Bordeaux

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Classical domino problem

Consider a finite set τ of Wang tiles



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Consider a finite set τ of Wang tiles



Question:

is there a function $x:\mathbb{Z}^2\to\tau$ such that adjacent tiles share the same color?

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Question:

is there an algorithm which given a finite set of Wang tiles decides whether they tile the plane or not?

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Theorem (Berger 66') *No.*

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The domino problem is Undecidable.

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Domino problem

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Theorem (Kari 08')

. . .

The domino problem is undecidable in the binary hyperbolic tiling.

. . .

Let us consider the following ingredients:

- A directed, labeled (infinite) graph $\Gamma = (V, E, L)$.
- A finite set of colors \mathcal{A} .
- A finite list of **forbidden** colored labeled connected finite graphs \mathcal{F} .

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Domino problem for Γ :

Is there an algorithm which decides, given $(\mathcal{A}, \mathcal{F})$, whether there exists a coloring $x : V \to \mathcal{A}$ such that no graph from \mathcal{F} embeds?

The original domino problem:



Binary hyperbolic tiling:

. . .



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Binary hyperbolic tiling:



A particularly interesting case is when $\Gamma = (V, E, L)$ is the **Cayley** graph of a finitely generated group G given by the set of generators S.

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$$V = G$$
.
• $E = \{(g, gs) \mid g \in G, s \in S\}.$

•
$$L(g,gs) = s$$
.

Remark: the domino problem does not depend upon the set of generators S. These problems are all computationally (many-one) equivalent.

DP(G) is the domino problem of the group G.

Cayley graph of free group.



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Domino problem on groups.

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Domino conjecture

A finitely generated group has decidable domino problem if and only if it is virtually free.

Verified for polycyclic groups, Baumslag-Solitar groups, Branch groups.

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Theorem (Muller & Schupp '85)

A graph has decidable monadic second order logic (MSO) if and only if it has finite tree-width.

- Fact 1: A group is virtually free if and only if its Cayley graphs have finite tree-width.
- Fact 2: The domino problem can be expressed in MSO.

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If DC holds, then the domino problem contains all the complexity of MSO for finitely generated groups.

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¹https://math.stackexchange.com/questions/1834108/ cayley-graph-of-the-fundamental-group-of-the-2-torus => < >>

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The domino problem of the fundamental group of any closed orientable surface of positive genus is undecidable.

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Remark: we just need to show that the domino problem of

$$\pi_1\left(\textcircled{\bigcirc}\right)\cong\langle \mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\mid\mathsf{a}\mathsf{b}\mathsf{a}^{-1}\mathsf{b}^{-1}\mathsf{c}\mathsf{d}\mathsf{c}^{-1}\mathsf{d}^{-1}=1\rangle$$

is undecidable.

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Proof idea: use hyperbolicity.

• Step 1: show undecidability of DP for a class of graphs which embed nicely in the hyperbolic plane.

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- Step 4: profit.

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A (non-deterministic) **substitution** is a pair (\mathcal{A}, R) where \mathcal{A} is a finite alphabet and R is a set of pairs $(a \mapsto w) \in \mathcal{A} \times \mathcal{A}^*$.

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$$\mathcal{A} = \{0\}, R = \{(0 \mapsto 00)\}.$$

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$$\mathcal{A} = \{0, 1\}, R = \{(1 \mapsto 0), (0 \mapsto 01)\}.$$

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An infinite word $u = \ldots u_{-1}u_0u_1u_2 \cdots \in \mathcal{A}^{\mathbb{Z}}$ **produces** a word $v = \ldots v_{-1}v_0v_1v_2 \cdots \in \mathcal{A}^{\mathbb{Z}}$ if v can be obtained from u by applying a rule of R on each symbol.

That is, there exists a function $\Delta:\mathbb{Z}\to\mathbb{Z}$ such that :

$$(u_i \mapsto v_{\Delta(i)} \dots v_{\Delta(i+1)-1}) \in R$$
 for every $i \in \mathbb{Z}$

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- Join all consecutive symbols of *u_i* by edges from left to right.
- Join each symbol of u_i with the corresponding sequence of symbols it produces in u_{i+1} assigning labels from left to right.

Example 1: trivial substitution gives \mathbb{Z}^2 .

$$\mathcal{A} = \{0\} \ R = \{(0 \mapsto 0)\}.$$



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Example 2: Doubling substitution gives bin hyp tiling.

$$\mathcal{A} = \{0\} \ R = \{(0 \mapsto 00)\}.$$



Idea: take an orbit graph Γ .



In each vertex code a finite subgraph of the binary orbit graph + information on how to locally paste them together.



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Impose local consistency rules.

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- Suppose DP(Γ) is decidable.
- Use the previous tiling to encode the binary orbit graph.
- Let (A, F) be an alphabet and a set of forbidden patterns for the binary orbit graph. Use the encoding to simulate tilings in Γ.
- As DP(Γ) is decidable, we may use the associated algorithm to decide whether (A, F) admits a tiling of the binary orbit graph.
- contradiction √.

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Warning

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Warning

- We must check that the language of coded subgraphs is finite.
- We must check that the set of encodings is **non-empty**.

A substitution (\mathcal{A}, R) has an **expanding eigenvalue** if there exists $\lambda > 1$ and $v : \mathcal{A} \to \mathbb{R}^+$ such that for every $(a \mapsto u_1 \dots u_k) \in R$:

$$\lambda v(a) = (v(u_1) + v(u_2) + \cdots + v(u_k))$$

Example

 $\mathcal{A} = \{0\} \ R = \{(0 \mapsto 00)\}$ admits the expanding eigenvalue $\lambda = 2$.

 $2\lambda v(0) = (v(0) + v(0))$

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To every orbit of a substitution with an expanding eigenvalue we can associate canonically a tiling of \mathbb{H}^2 .



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Finitely many (coded) ways to intersect \implies finite alphabet There is an encoding \implies non-emptiness

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Finitely many (coded) ways to intersect \implies finite alphabet There is an encoding \implies non-emptiness

Remark: Tiling superpositions were introduced by D.B. Cohen and C. Goodman-Strauss to produce aperiodic tilings of surface groups.

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For every orbit graph Γ of a substitution with an expanding eigenvalue $DP(\Gamma)$ is undecidable.

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There is a "hidden" substitution in that group, namely $\mathcal{A} = \{\mathtt{a}, \mathtt{b}\}$ and

$$\left\{\left(a\mapsto ab^5ab^5ab^5ab^5ab^4\right),\left(b\mapsto ab^5ab^5ab^5ab^5ab^5ab^4\right).\right\}$$

with $\lambda = 17 + 12\sqrt{2}$ and $v(b)/v(a) = \frac{1+\sqrt{2}}{2}$.

Skip Proof

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A way to look at this Cayley graph is as a translation surface obtained by pasting together octagons.







Surface group: vertex with ancestor

If a vertex is **connected** by a generator with the previous ring then the sequences of vertices in the next level it sees follows the following pattern:



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If a vertex is **not connected** by a generator with the previous ring then the sequences of vertices in the next level it sees follows the following pattern:



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Theorem (Aubrun, B., Moutot)

The domino problem is undecidable on the fundamental group of the closed orientable surface of genus 2.

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Word-hyperbolic group

A finitely generated group is **word-hyperbolic** if the geodesic triangles of one of its Cayley graphs are δ -slim for some $\delta > 0$.



²From https://en.wikipedia.org/wiki/Hyperbolic_group

Facts about word-hyperbolic groups:

- Virtually free groups \checkmark .
- Surface groups (genus $g \ge 2$) \checkmark .
- Nice computability properties: Finitely presented, decidable word problem, Dehn's algorithm works, language of shortlex geodesics is regular, etc.
- A random group is almost surely word-hyperbolic.

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Bottom line: testing ground for

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The fundamental group of embeds into any one-ended word-hyperbolic group.

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Facts:

- If a group *H* embeds into a group *G*, then the domino problem of *G* is computationally harder than the domino problem of *H*.
- If a word-hyperbolic group is not virtually free, it contains an embedded one-ended word-hyperbolic group.
- If GC holds, then every word-hyperbolic group which is not virtually free contains an embedded copy of the fundamental group of

The fundamental group of embeds into any one-ended word-hyperbolic group.

Theorem

If GC holds, then the domino problem conjecture holds for every word-hyperbolic group.

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Theorem

If GC holds, then the domino problem conjecture holds for every word-hyperbolic group.

• Fun fact: find a (non virt free) word-hyperbolic group with decidable domino problem and you shall attain fame and glory disprove Gromov's conjecture!

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If GC holds, then the domino problem conjecture holds for every word-hyperbolic group.

- Fun fact: find a (non virt free) word-hyperbolic group with decidable domino problem and you shall attain fame and glory disprove Gromov's conjecture!
- Fun fact: Same can be shown with weaker versions of GC.

Thank you for your attention!



The domino problem is undecidable on surface groups. https://arxiv.org/abs/1811.08420

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