Fast Synchronization of Random Automata

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Topic of the talk

Does the Černý conjecture hold with high probability?

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The Černý conjecture (1964)

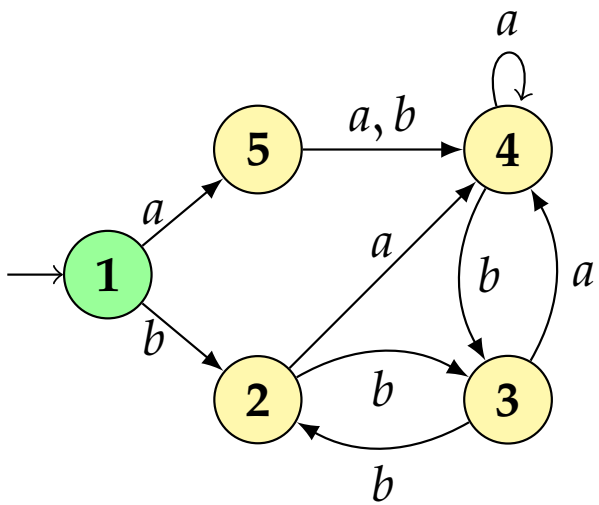
A synchronizing automaton with n states admits a synchronizing word of length at most $(n-1)^2$.

1. Automata & Synchronization

Deterministic and complete automata

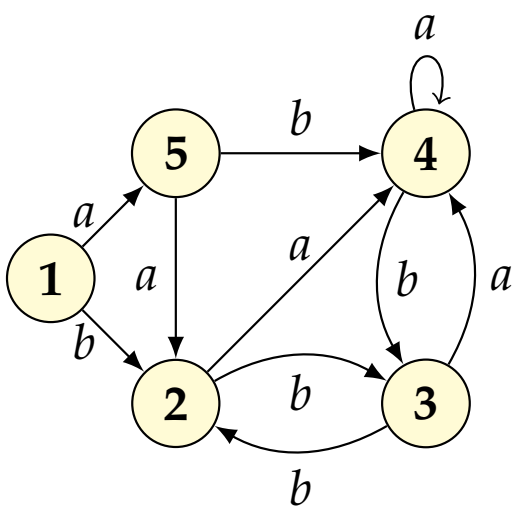
A (complete and deterministic) automaton is a directed graph s.t.:

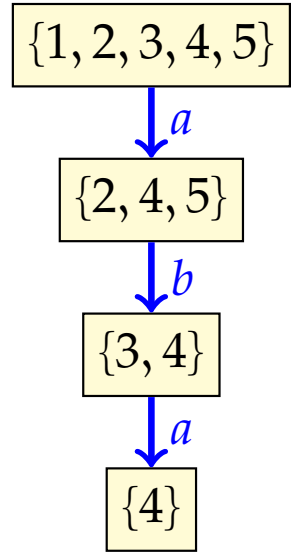
- vertices are called "states", and edges are called "transitions"
- For each state and for each letter a of a fixed alphabet A, there is exactly one outgoing transition labeled by a



Synchronizing automata

- An automaton is **synchronizing** when there exists a word that brings every state to one and the same state
- Such a word is a **synchronizing word**





- *aaaa* is a synchronizing word
- *aba* is a smaller synchronizing word

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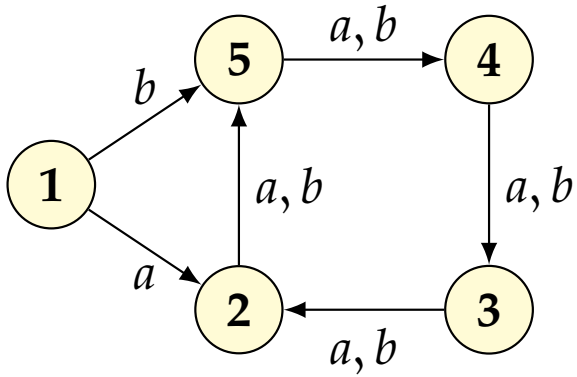


- This automaton is not synchronizing.

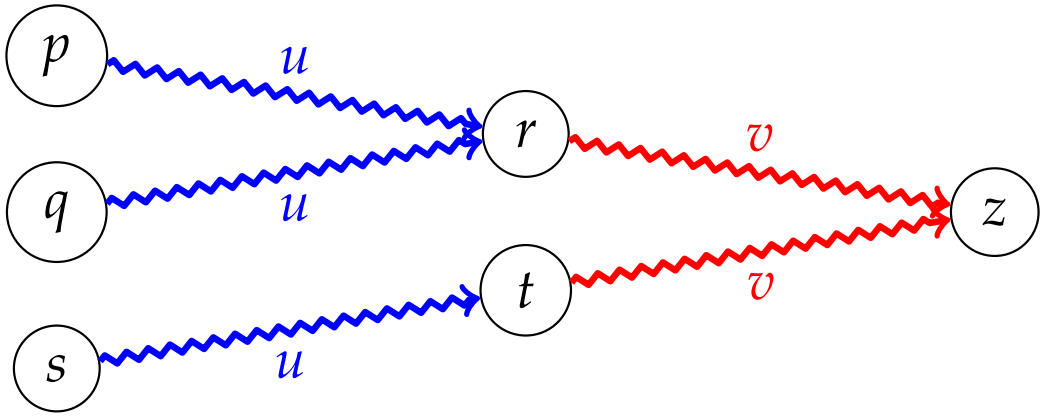
Pairwise Synchronization

- A pair of states (p,q) is synchronized when there exists a word u such that $p \cdot u = q \cdot u$.

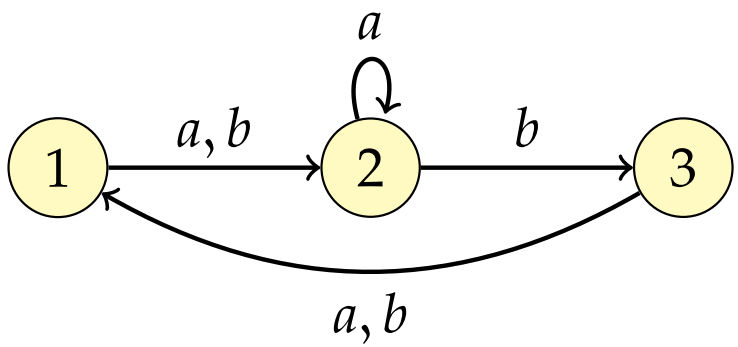
Lemma

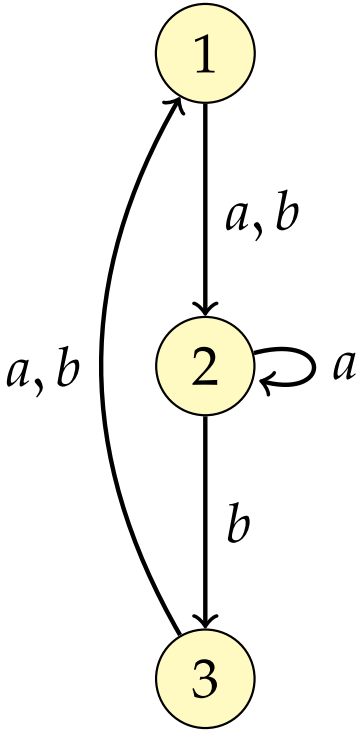
If every pair of states is synchronized, then the automaton is synchronizing.

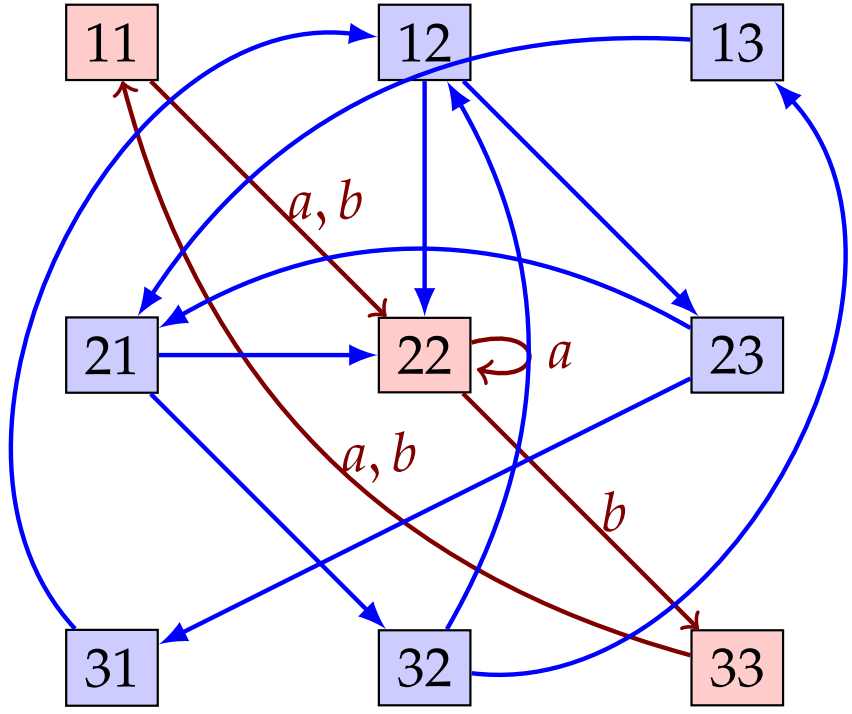
More precisely, is every pair is synchronized by a word of length at most ℓ, then the automaton is synchronized by a word of length at most $n(\ell - 1)$.



Checking Synchronization: Square of an Automaton







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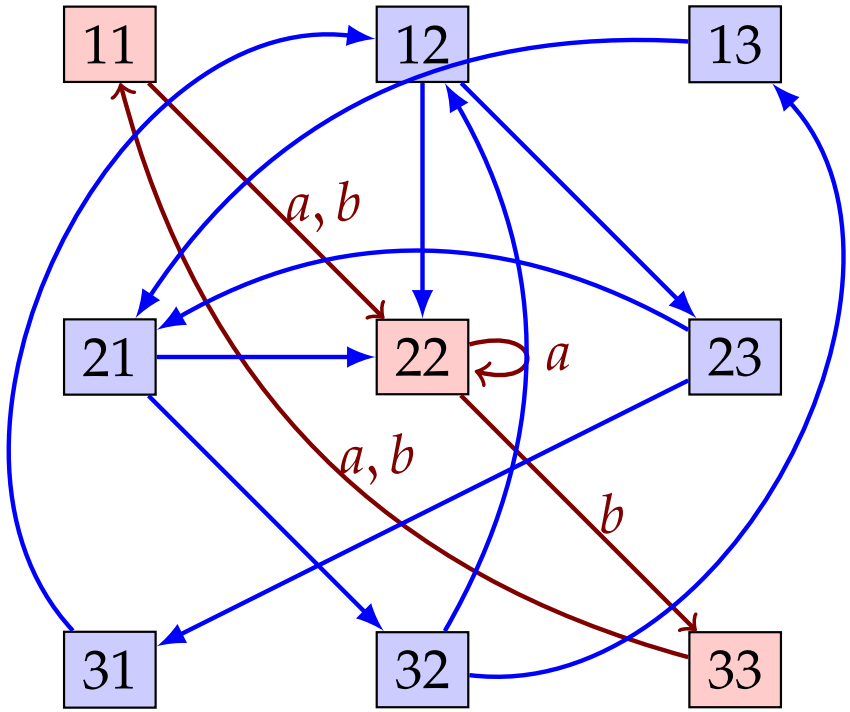

- Synchronizing iff there is a path from every (p, q) to a state (x, x)
- Checked in $\mathcal{O}(n^2)$
- Using the lemma: synchronizing word of length at most n^3

The Černý conjecture

Conjecture [Černý 64]

A synchronizing automaton with n states admits a synchronizing word of length at most $(n-1)^2$.

- $(n-1)^2$ is best possible
- n^3 is trivial
- First (general) known bound [Frankl] [Pin 83]: $\frac{1}{6}(n^3 - n)$.
- the conjecture holds for many families of automata
- [Szykuła 18] improve the coefficient of n^3 to $\frac{114}{685} = \frac{1}{6} - \frac{1}{4110}$

2. Settings

Deterministic and complete automata

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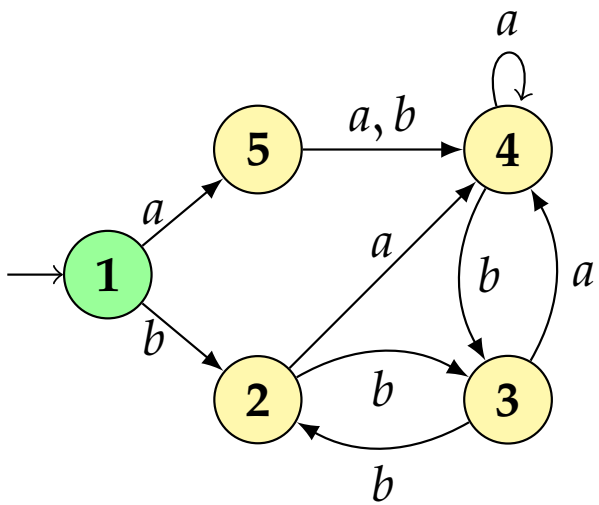


Random deterministic automata: some models

There are three main probabilistic models:

- **Uniform deterministic and complete automata:** the target of each transition is chosen uniformly at random, independently
- Uniform **accessible** deterministic and complete automata
- Uniform **minimal** automata

For these models, we can instead consider that each state is final with some fixed probability $p \in (0, 1)$, independently.

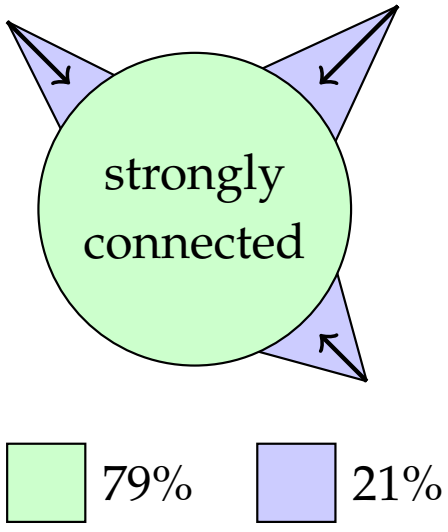
In this talk : our model is the first one, the simplest.

Random automata vs random digraphs

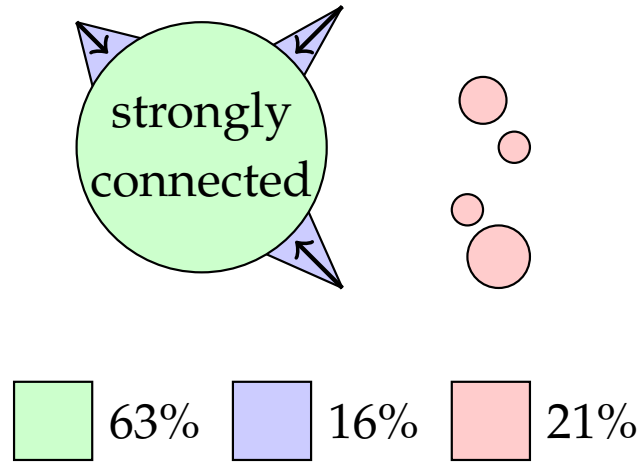
For $A = \{a, b\}$.

- **Random automata:** each state has 2 outgoing transitions
- **Random digraph (Erdős-Rényi):** each edge has probability $\frac{2}{n}$
- Let θ be the unique positive real solution of $1 - x = e^{-2x}$ ($\theta \approx 0.79$)

Random automaton



Random digraph [Karp 90]



Probabilistic Černý conjecture

Question 1

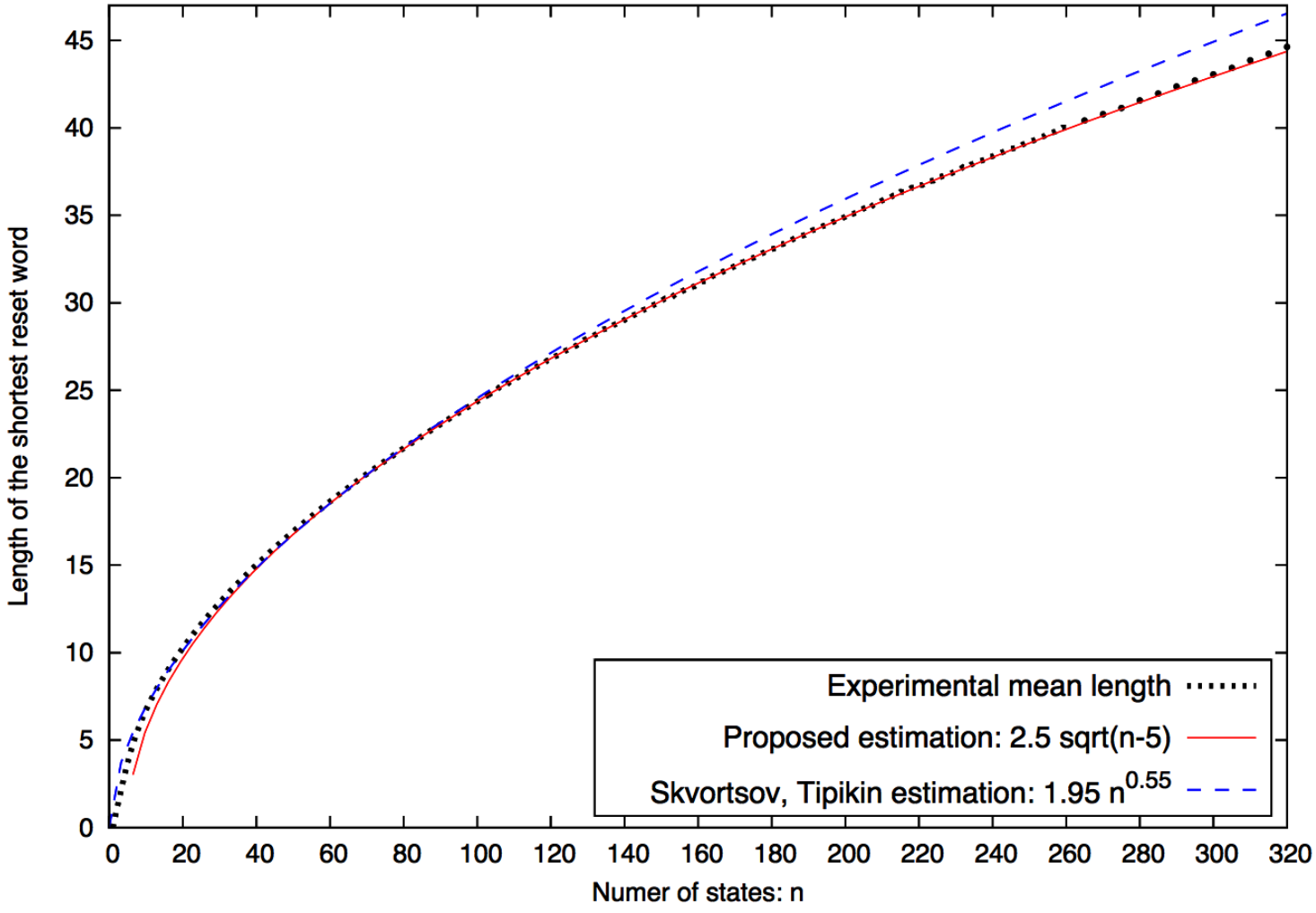
What is the probability that an automaton is synchronizing?

Question 2

Does the Černý conjecture hold with high probability?

Experiments [Kisielewicz, Kowalski and Szykuła 13]

:



The graphic comes from [Kisielewicz, Kowalski and Szykuła 13]

Answer to Q1: Berlinkov's theorem

Theorem [Berlinkov'16]

For alphabets with at least two letters, deterministic automata are synchronizing with high probability.

More precisely, a random automaton is not synchronizing with probability $\mathcal{O}(\frac{1}{n^{k/2}})$.

- For $k = 2$, the bound is tight: $\Theta(\frac{1}{n})$.

An answer to Q2: this talk

Theorem [N. RANDOM'16]

For alphabets with at least two letters, a random automaton admits a synchronizing word of length at most $\mathcal{O}(n \log^3 n)$ with high probability.

- The proof is independent of Berlinkov's proof
- It is more elementary
- The error term is not tight
- It provides information on the reset threshold

Corollary

For alphabets with at least two letters, the Černý conjecture holds with high probability.

An algebraic version

- in an automaton, the action of each letter a on the set of states is a mapping δ_a
- we are interested in the monoid generated by the δ_a's

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Theorem (Dixon 69)

Let σ and τ be two uniform random **permutations** of size n. With high probability, the group generated by σ and τ is either the symmetric group or the alternating group.

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Theorem (Dixon 69)

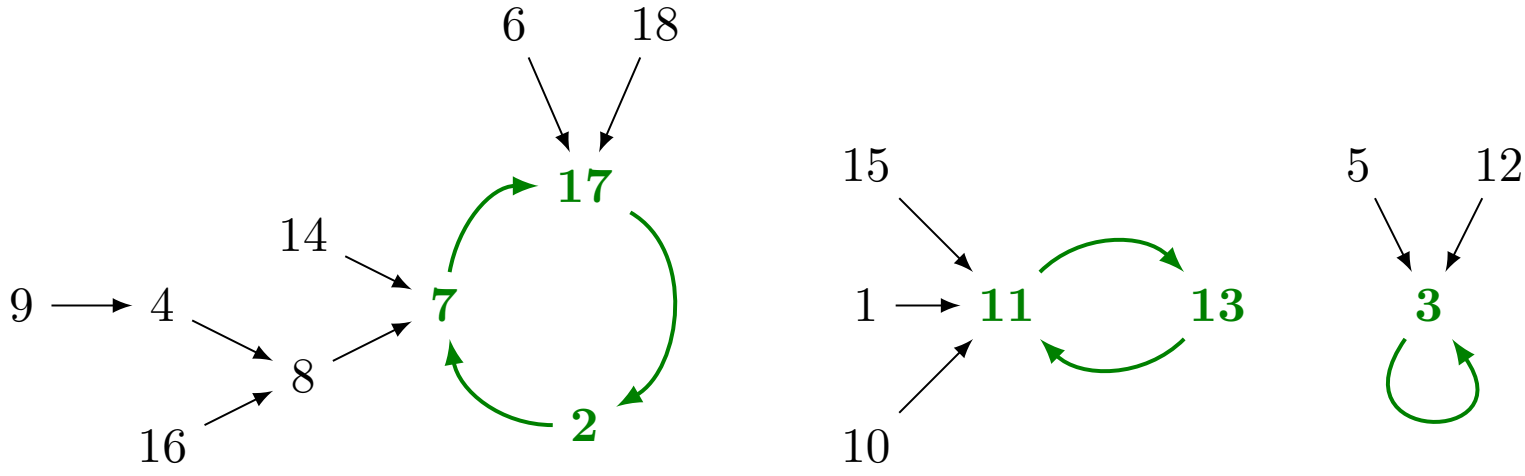
Let σ and τ be two uniform random **permutations** of size n. With high probability, the group generated by σ and τ is either the symmetric group or the alternating group.

- An automaton is **synchronized** $\Leftrightarrow$ the monoid **contains a constant map**

3. Proof sketch

Tuples of random mappings

- We consider **uniform random mapping** from $[n]$ to itself
- Its functional graph is a set of cycles of trees
- The mapping below has height 3, and it has 6 cyclic points



- A tuple of uniform random mappings defines a uniform random automaton (one mapping for the action of each letter)

A property of random p-mappings

Lemma

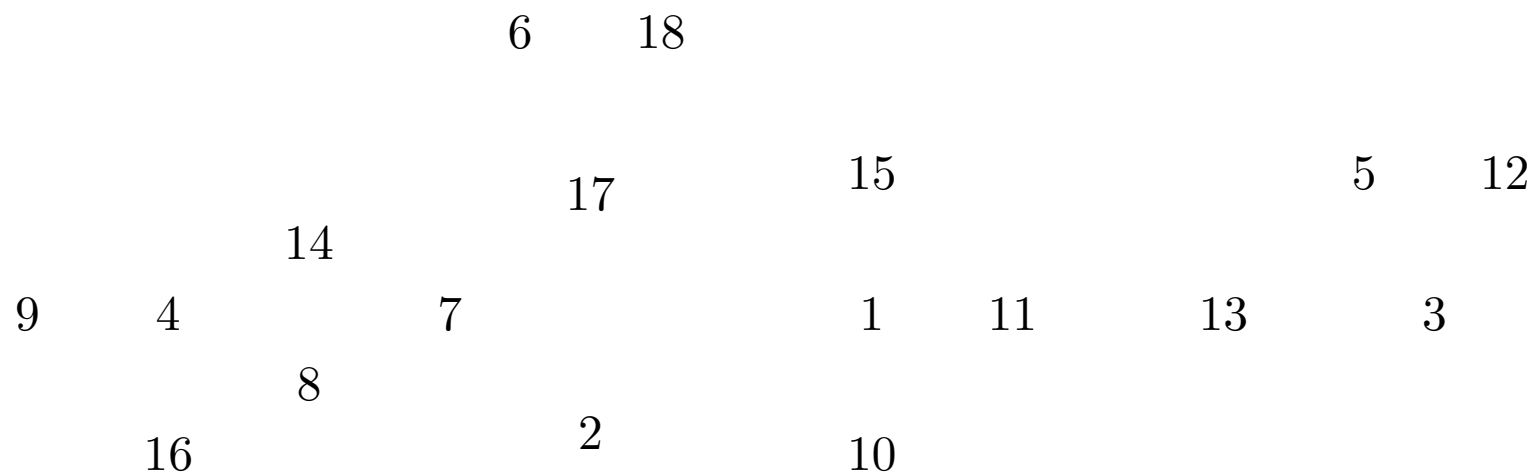
With high probability, a uniform random mapping has **height** at most $2\sqrt{n \log n}$ and has at most $2\sqrt{n \log n}$ **cyclic points**. It still holds for p-random mappings.

- **Proof:** Birthday paradox.

- random p-mapping: the image of each element is taken independently in $[n]$, following the probability distribution p.

First step: a-transitions

- **Main idea:**
  - Start from an automaton with no transitions
  - Add random transitions as needed
- First, we add all the a-transitions



First step: *a*-transitions

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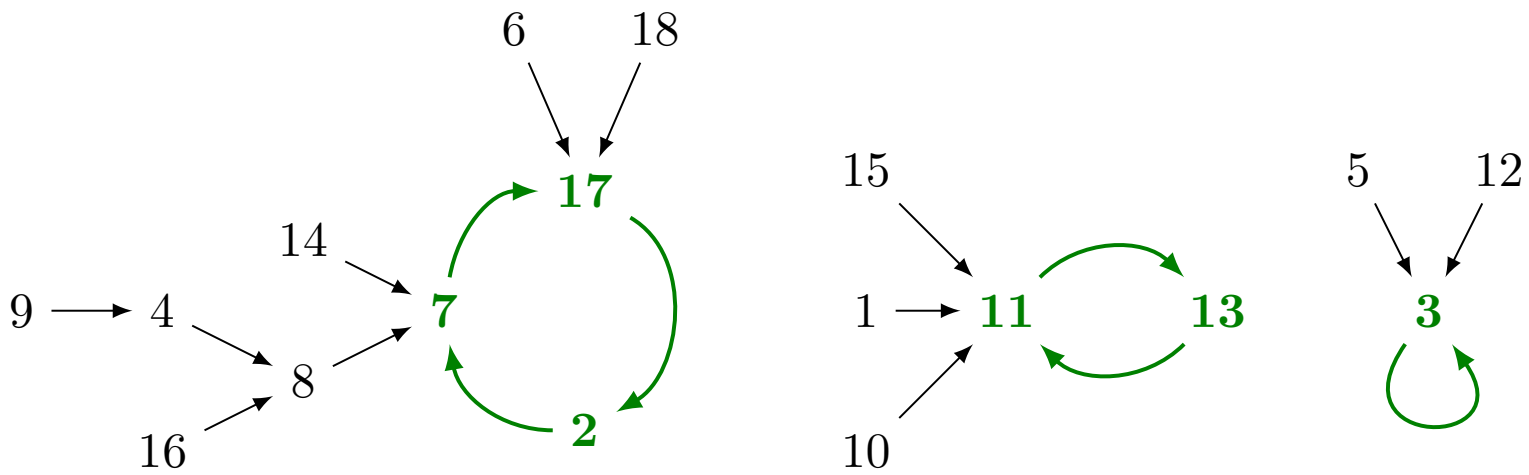
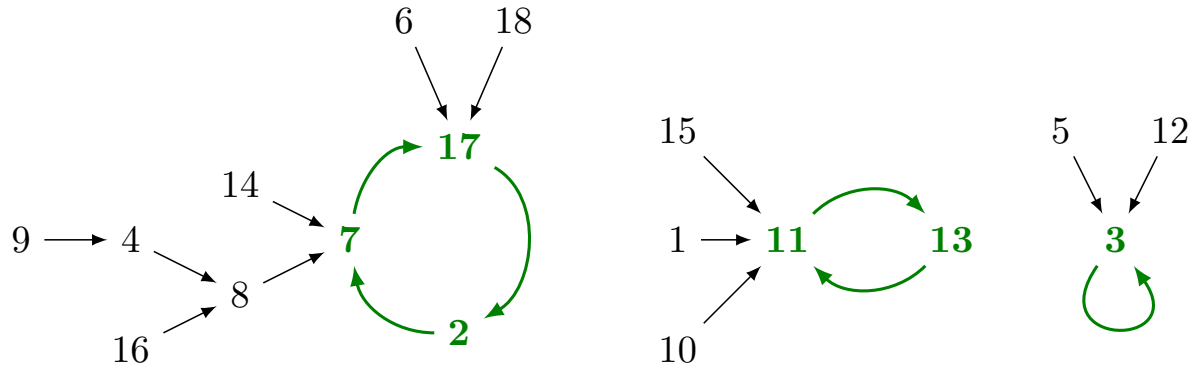
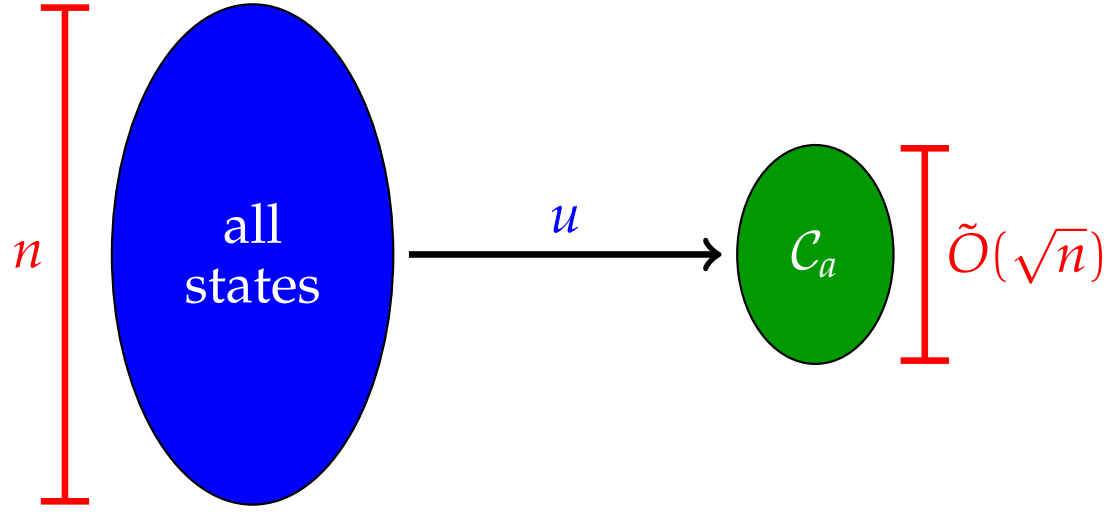


Shrinking to a-cyclic points

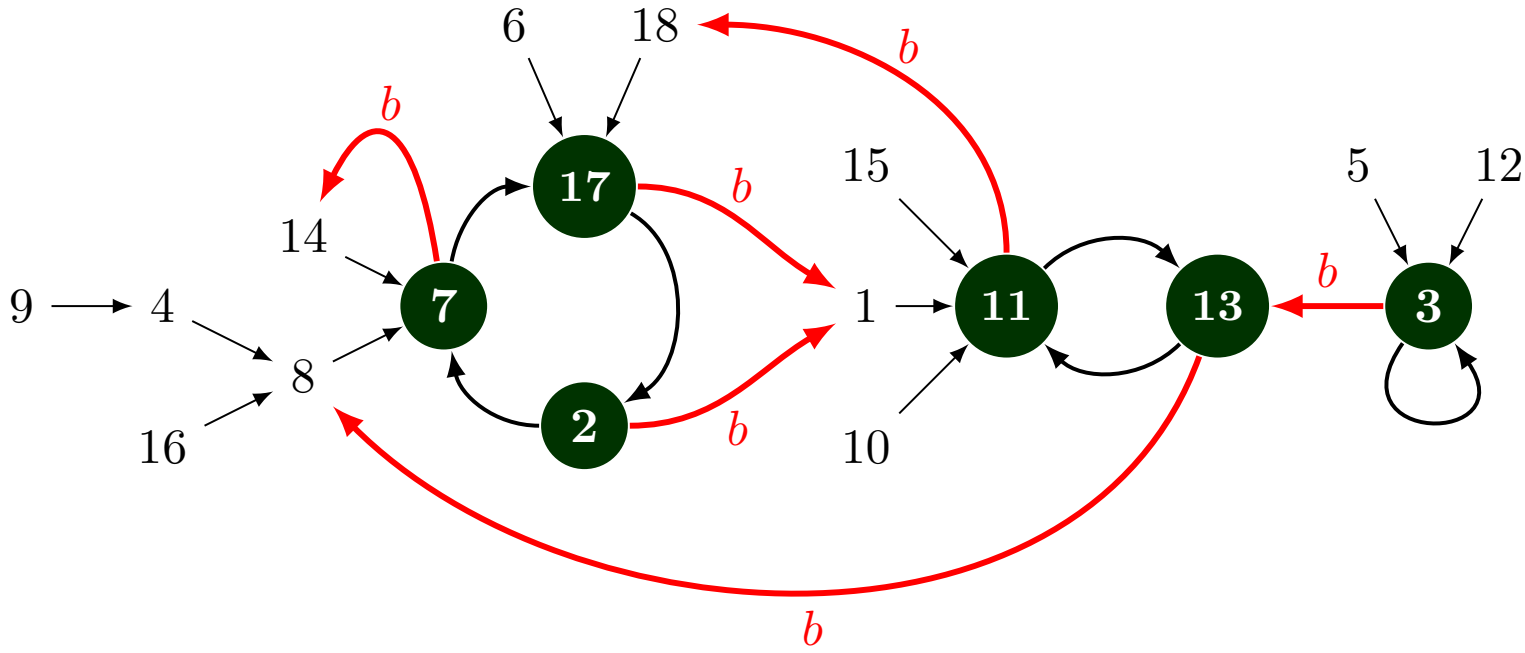


- $\mathcal{C}_a$ is the set of a-cyclic points
- W.h.p., $|\mathcal{C}_a| \leq 2\sqrt{n \log n}$ and $u = a^{2\sqrt{n \log n}}$ maps $[n]$ to $\mathcal{C}_a$



Shrinking $\mathcal{C}_a$

- We fix the a-transitions
- We now generate b-transitions starting from the states of $\mathcal{C}_a$

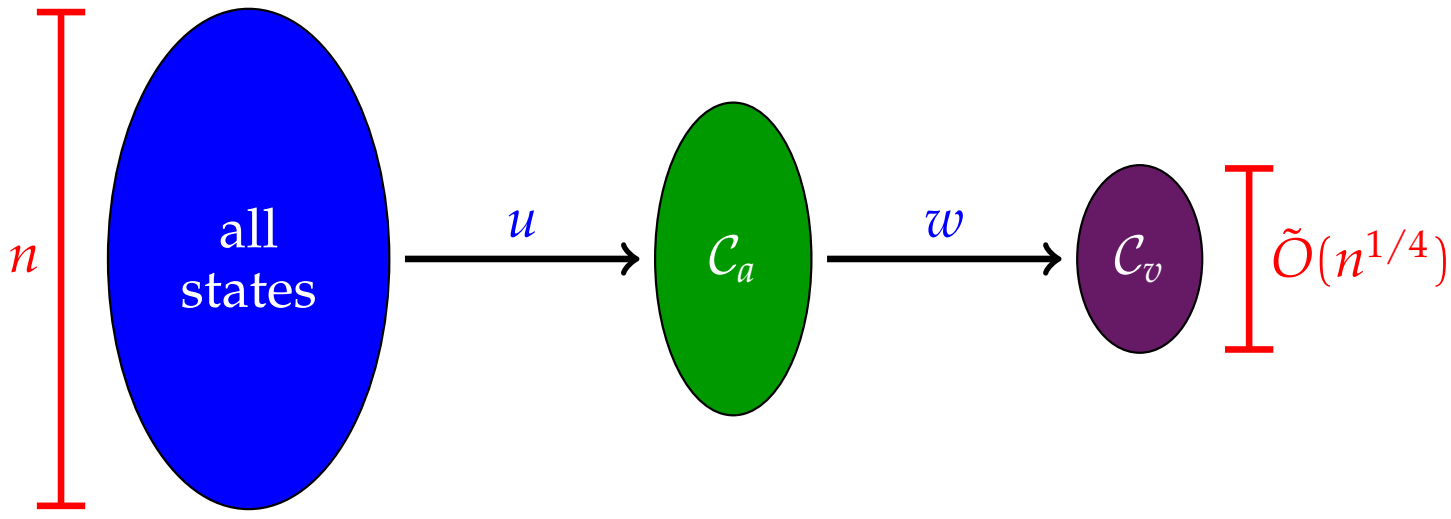


- We consider the map $\delta_v : \mathcal{C}_a \to \mathcal{C}_a$ defined with $v = bu$

$$\mathbb{P}\left(f(p) = q\right) = \frac{\text{number of preimages of } q \text{ by } u}{n}$$

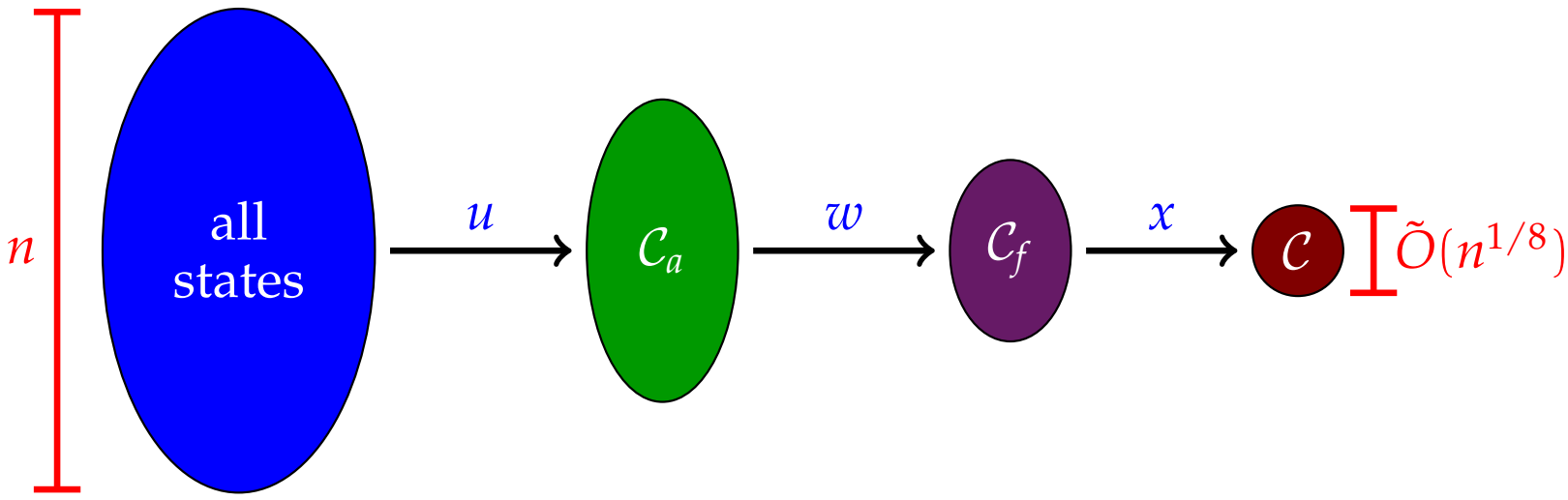
Shrinking $\mathcal{C}_a$

- δ_v is a **random p-mapping** on $\mathcal{C}_a$
- Let $w = v^{\beta_n}$, with $\beta_n = 3n^{1/4}(\log n)^{3/4}$
- W.h.p $|\mathcal{C}_w| \leq \beta_n$ and δ_w maps $\mathcal{C}_a$ to $\mathcal{C}_w$



Shrinking once more

- Using a third letter c, we can do the same trick once again
- If $A = \{a, b\}$, with some care, we can use $c = bb$

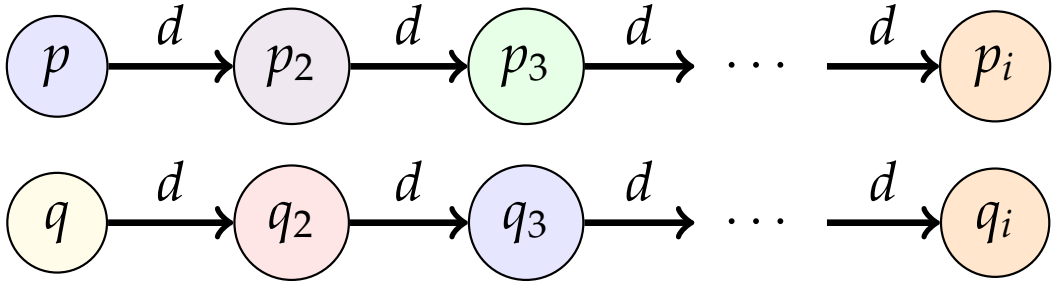


- W.h.p. the word $s = uwx$ has length at most $n^{7/8} \log^3 n$ and maps the set of states to a set C of size $\tilde{O}(n^{1/8})$.

Synchronizing $\mathcal{C}$

- We synchronize every pair (p, q) of states of $\mathcal{C}$

Repeatedly draw d-transitions starting from p and q:



- if $\delta_s(p_i) = \delta_s(q_i)$ the synchronization is a **success**
- if p_i or q_i already have a d-transition or if the sequence is too long, it is a **failure**
- **Proposition:** w.h.p. this synchronization process is a **success** for every pair of states of $\mathcal{C}$.

Proof sketch

- Shrink the set of states three times to a set $\mathcal{C}$ of size $\approx n^{1/8}$
- Synchronize pairs of states of $\mathcal{C}$ using words of the form $d^i \cdot s$
- **(technical)** adapt the proof to alphabets with two letters

Theorem (N. 16)

For alphabets with at least two letters, a random automaton admits a synchronizing word of length at most $\mathcal{O}(n \log^3 n)$ with high probability.

Further directions

- The expected reset threshold in $\Theta(\sqrt{n})$
- The error term in $\Theta(1/n)$

- Proof of the **Černý conjecture** ...

Thanks!