# Graph insertion operads

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Bordeaux

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# Table of contents

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# Associative Algebra vs Operad

| **Associative Algebra** $A$ | **Operad** $\mathcal{O}$ |
|---|---|
| extension cable | multi-socket |
|  |  |

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|  |  |
| product $\cdot$ | partial composition $\circ_*$ |
| words | trees |

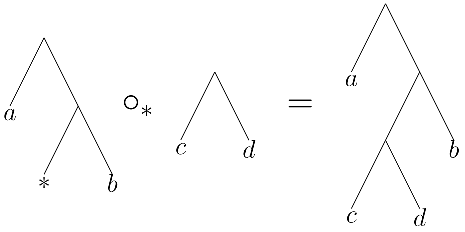
# Associative Algebra vs Operad

| **Associative Algebra** $A$ | **Operad** $\mathcal{O}$ |
| --- | --- |
| extension cable | multi-socket |
|  |  |
| product $\cdot$ | partial composition $\circ_*$ |
| words | trees |
| $x \in A$ | $x \in \mathcal{O}[V]$ |

And now some examples

# Example: Commutative magmatic

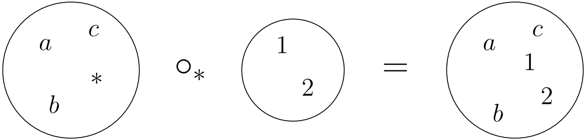
**ComMag**$[V]$ = non planar rooted binary trees with set of leaves $V$.




**ComMag**$[\{a, b, *\}]$ $\times$ **ComMag**$[\{c, d\}]$ $\rightarrow$ **ComMag**$[\{a, b, c, d\}]$

# Example: Commutative

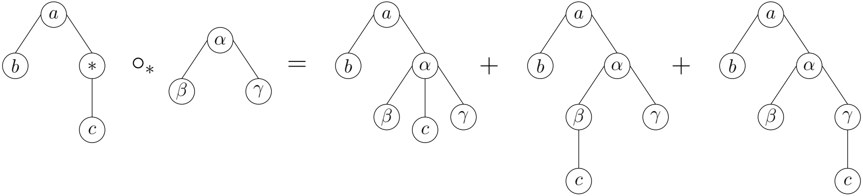
$$\mathbf{Com}[V] = \{V\}.$$




$$\mathbf{Com}[\{a,b,c,*\}] \times \mathbf{Com}[\{1,2\}] \to \mathbf{Com}[\{a,b,c,1,2\}]$$

# Example: Pre-Lie

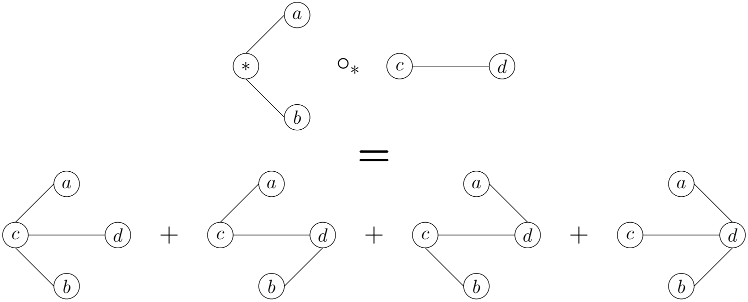
$\textbf{PLie}[V]$ = non planar rooted trees with set of vertices $V$.



$$\textbf{PLie}[\{a, b, c, *\}] \times \textbf{PLie}[\{\alpha, \beta, \gamma\}] \to \mathbb{K}\textbf{PLie}[\{a, b, c, \alpha, \beta, \gamma\}]$$

# Example: Kontsevich-Willwacher

$\mathbf{G}[V]$ = simple graphs with set of vertices $V$.



$$\mathbf{G}[\{a,b,*\}] \times \mathbf{G}[\{c,d\}] \to \mathbb{K}\mathbf{G}[\{a,b,c,d\}]$$

# Table of contents

# Species

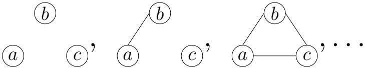
(Joyal) A *species* $P$ is given by the data of:

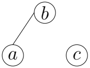
- for each finite set $I$, a vector space $\mathbb{K}P[I]$ of base $P[I]$,
- for each bijection $\sigma : I \to J$, a linear map

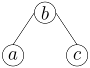
$$P[\sigma] : \mathbb{K}P[I] \to \mathbb{K}P[J],$$

such that

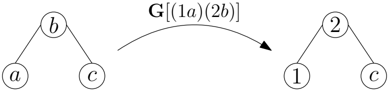
$$P[\tau \circ \sigma] = P[\tau] \circ P[\sigma] \text{ and } P[\mathrm{id}] = \mathrm{id}.$$

$$\mathbf{G}[\{a, b, c\}] = \Big\{$$  $$\Big\}$$


$$3$$ 

$$-$$ 

$$\in \mathbb{K}\mathbf{G}[\{a, b, c\}]$$



# Species

$$\begin{tikzcd} \mathbf{F} \arrow[r, hook] & \mathbf{G} \arrow[r, hook] & \mathbf{MG} \\ \mathbf{T} \arrow[r, hook] \arrow[u, hook] & \mathbf{G}_c \arrow[r, hook] \arrow[u, hook] & \mathbf{MG}_c \arrow[u, hook] \end{tikzcd}$$

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pointing: $\quad P^\bullet[V] = P[V] \times V$

$$\mathbf{T}^\bullet \hookrightarrow \mathbf{G}_c^\bullet \hookrightarrow \mathbf{MG}_c^\bullet$$

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\mathbf{F} \arrow[r, hook] & \mathbf{G} \arrow[r, hook] & \mathbf{MG} \\
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$$

pointing: $\quad P^\bullet[V] = P[V] \times V$

$$\mathbf{T}^\bullet \hookrightarrow \mathbf{G}_c^\bullet \hookrightarrow \mathbf{MG}_c^\bullet$$

$p \in P$ instead of "$V$ finite set and $p \in P[V]$"
$x \in p$ if $p \in P[\{x, \dots\}]$

# Partial composition

Let $P$ be a species. A *partial composition* over $P$ is given by:

for each disjoint pair of finite sets $V_1, V_2$

and for each element $* \in V_1$,

a map

$$\circ_* : P[V_1] \times P[V_2] \to \mathbb{K}P[V_1 \sqcup V_2 - \{*\}].$$

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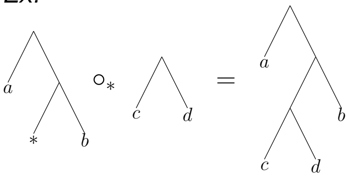
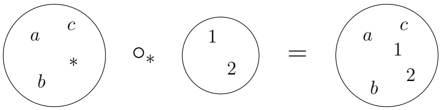


# Partial composition

A partial composition is *associative* if, for every $p, q, r \in P$:

for $*_1 \in p$ and $*_2 \in q$:

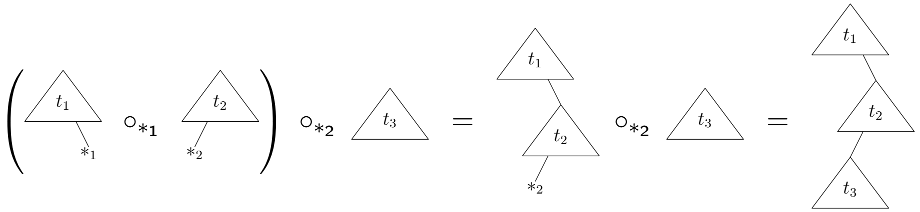
$$(p \circ_{*_1} q) \circ_{*_2} r = p \circ_{*_1} (q \circ_{*_2} r)$$

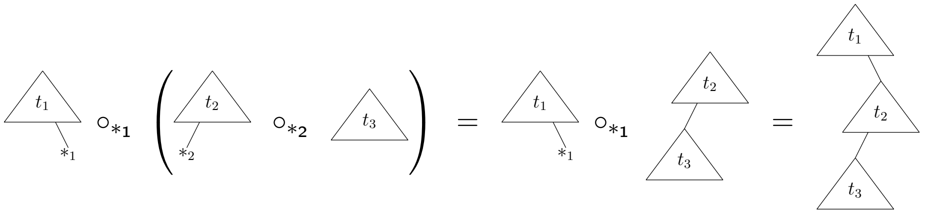
and for $*_1, *_2 \in p$:

$$(p \circ_{*_1} q) \circ_{*_2} r = (p \circ_{*_2} r) \circ_{*_1} q$$

# Partial composition

*Ex:* **ComMag**



# Partial composition

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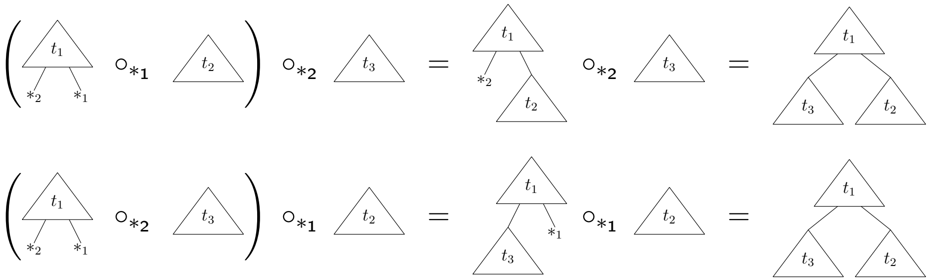

# Operad

## Definition

An *operad* is a species $\mathcal{O}$ with an associative partial composition $\circ$.

*Ex:* $\mathbf{Com}, \mathbf{ComMag}, \mathbf{NAP}, \mathbf{PLie}, \mathbf{Dend}, \mathbf{Dup}, \mathbf{TSig}_{\mathbf{k}}^{||} \dots$

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What about graphs ?

In physics (Connes, Kreimer), other branch of mathematics (Kontsevich-Willwacher).

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$$\mathbf{PLie} \xrightarrow{\quad ? \quad} \mathbf{G}$$

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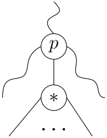
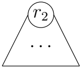
$$\mathbf{PLie} \xrightarrow{\quad ? \quad} \mathbf{MG}$$

# Table of contents

# Pre-Lie operad

As a species: $\mathbf{PLie} = \mathbf{T}^{\bullet}$. Partial composition, $t_1 \circ_* t_2 = ?$

$t_1 =$ 

, $t_2 =$ 

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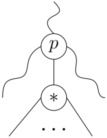
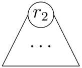
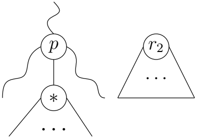
$t_1 =$ , $t_2 =$ 

1. take the disjoint union of $t_1$ and $t_2$



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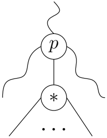
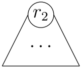
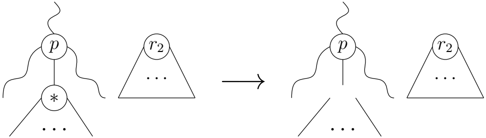
$t_1 =$  , $t_2 =$ 

1. take the disjoint union of $t_1$ and $t_2$
2. remove the vertex $*$



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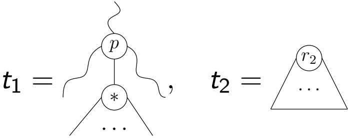
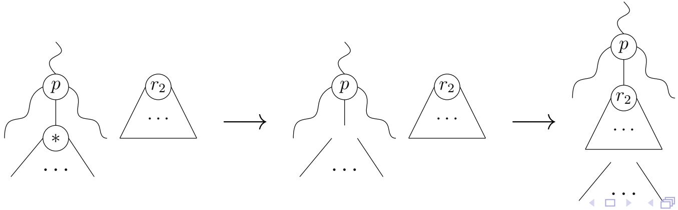



1. take the disjoint union of $t_1$ and $t_2$
2. remove the vertex $*$
3. if $* \neq r_1$, connect the parent of $*$ to $r_2$



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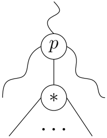
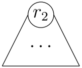
$t_1 =$  $,\quad t_2 =$ 

1. take the disjoint union of $t_1$ and $t_2$
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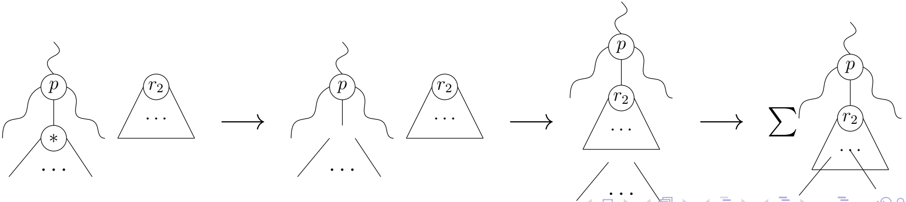
3. if $* \neq r_1$, connect the parent of $*$ to $r_2$
4. sum over all the ways to connect the children of $*$ to $t_2$



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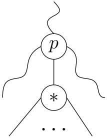
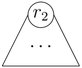
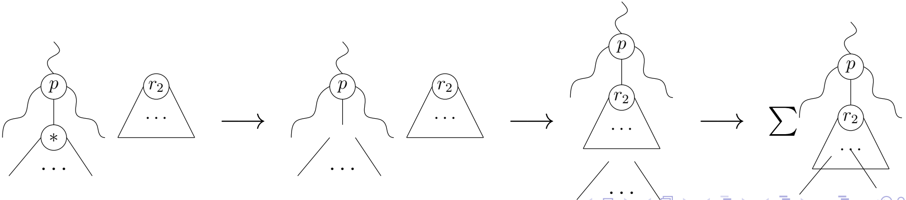
$t_1 =$ 

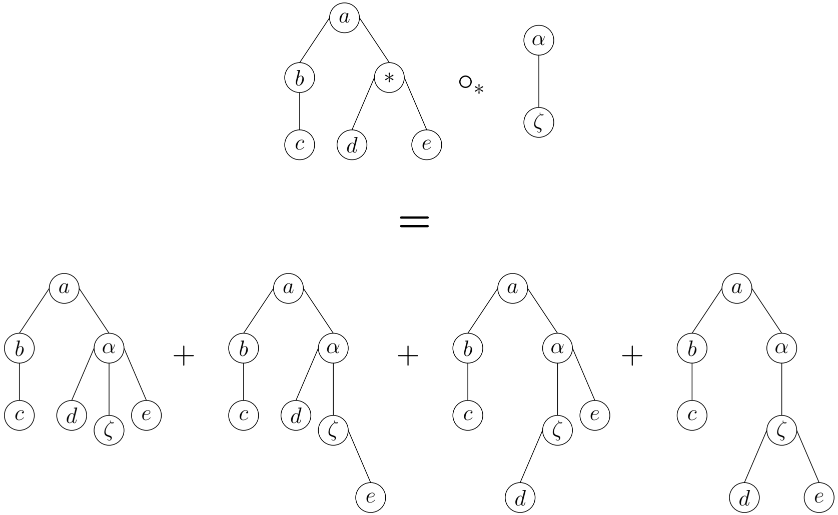
, $t_2 =$ 


1. take the disjoint union of $t_1$ and $t_2$
2. remove the vertex $*$
3. if $* \neq r_1$, connect the parent of $*$ to $r_2$
4. sum over all the ways to connect the children of $*$ to $t_2$
5. the new root is $r_1$ is $* \neq r_1$ and $r_2$ else.



# Pre-Lie operad

*Ex:*



# Canonical multigraph operad

As a species: **MG**. Partial composition, $g_1 \circ_* g_2 = ?$

1. take the disjoint union of $g_1$ and $g_2$
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3. sum over all the ways to connect the neighbours of $*$ to $g_2$.

# Canonical multigraph operad

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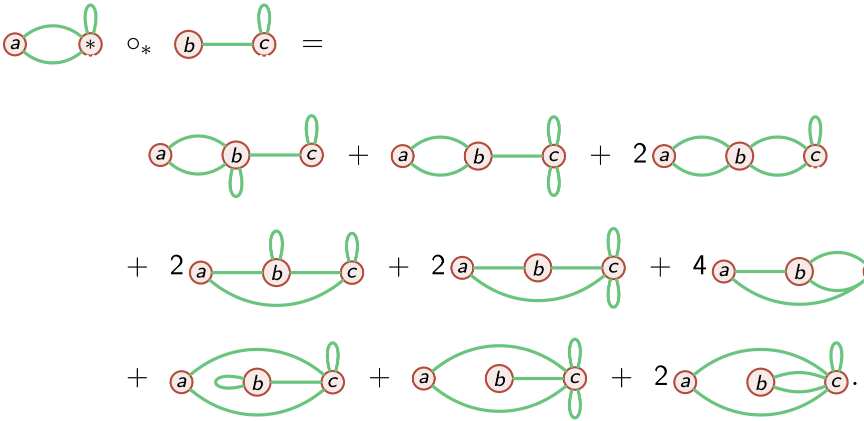
*Remark:*

$$\begin{tikzcd}
\mathbf{F} \arrow[r, hook] & \mathbf{G} \arrow[r, hook] & \mathbf{MG} \\
\mathbf{T} \arrow[u, hook] \arrow[r, hook] & \mathbf{G}_c \arrow[u, hook] \arrow[r, hook] & \mathbf{MG}_c \arrow[u, hook]
\end{tikzcd}$$

On **G**: Kontsevich-Willwacher.

*Ex:*



# Table of contents

As species: $\mathbf{T}^{\bullet} \hookrightarrow \mathbf{MG}_c^{\bullet}$

As operads: $\mathbf{PLie} \hookrightarrow \color{red}{\mathbf{MG}_c^{\bullet}}$ ?

# The $\phi$ morphism

$$\begin{aligned} \phi : \mathbf{MG}_c &\hookrightarrow \mathbf{MG}_c^{\bullet} \\ g &\mapsto \sum_{r \in g} (g, r), \end{aligned}$$

**Proposition**

The species morphism $\phi$ induces an operad morphism $\mathbf{T} \hookrightarrow \mathbf{PLie}$.

$$
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$$
\begin{tikzcd}
\mathbf{T} \arrow[r, hook, "\phi"] \arrow[d, hook] & \mathbf{PLie} \arrow[d, hook] \\
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# The $\phi$ morphism

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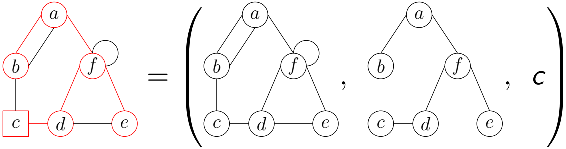
$$\begin{tikzcd} \mathbf{T} \arrow[r, "\sim"] \arrow[d, hook] & \phi(\mathbf{T}) \arrow[r, hook] \arrow[d, hook] & \mathbf{PLie} \arrow[d, hook] \\ \mathbf{MG}_c \arrow[r, "\sim"] & \color{green}{\phi(\mathbf{MG}_c)} \arrow[r, hook, green] & \color{red}{\mathbf{MG}_c^\bullet} \end{tikzcd}$$

$$\cancel{\mathbf{MG}_c^\bullet}$$

$\rightarrow$ **ST**: spanning tree species.

$$\mathbf{ST}[V] = \{(g, t, v) \mid g \in \mathbf{MG}_c[V], t \text{ spanning tree of } g, v \in V\}$$

*Ex:*



$\in \mathbf{ST}[\{a, b, c, d, e, f\}]$

# Spanning tree operad

## Definition (Hadamart product)

The Hadamard product $P \times Q$ of two species is defined by
$P \times Q[V] = P[V] \times Q[V]$.
If $P$ and $Q$ are operads then $P \times Q$ is also an operad with
$(p_1, q_1) \circ_* (p_2, q_2) = (p_1 \circ_* p_2, q_1 \circ_* q_2)$.

# Spanning tree operad

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$$\mathbf{ST} \xrightarrow{\sim} \mathbf{MG} \times \mathbf{PLie}$$

$$(g, t, v) \mapsto (g \setminus t, (t, v))$$

$\Rightarrow$ operad structure on **ST**

$$\begin{tikzcd}
\mathsf{T} \arrow[r, "\sim"] \arrow[d, hook] & \phi(\mathsf{T}) \arrow[r, hook] \arrow[d, hook] & \mathsf{PLie} \arrow[d, hook] \\
\mathsf{MG}_c \arrow[r, "\sim"] & \phi(\mathsf{MG}_c) \arrow[r, hook] & \mathsf{ST}
\end{tikzcd}$$

# The sub-operad $\mathcal{O}_1$

$$\mathcal{O}_1[V] = \{\sum_{v \in V} (g, t_v, v) \,|\, g \in MG_c[V], t_v \text{ spanning tree of } g\} \subset \mathbb{K}\mathbf{ST}[V]$$

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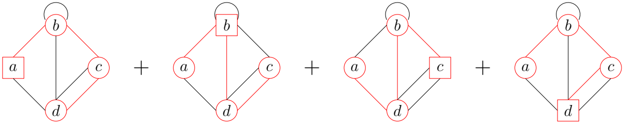
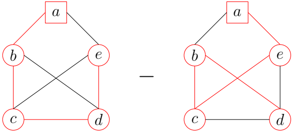



$\in \mathcal{O}_1[\{a, b, c, d\}]$




$\in \mathbb{K}\mathcal{O}_1[\{a, b, c, d, e\}]$

## The sub-operad $\mathcal{O}_1$

**Lemma**

*$\mathcal{O}_1$ is a sub-operad of* $\mathbf{ST}$ *(i.e $\mathcal{O}_1 \circ_* \mathcal{O}_1 \subset \mathcal{O}_1$).*

$$\mathcal{O}_1 \cap \mathbf{PLie} = \phi(T)$$

# The sub-operad $\mathcal{O}_1$

**Lemma**

*$\mathcal{O}_1$ is a sub-operad of* $\mathbf{ST}$ *(i.e $\mathcal{O}_1 \circ_* \mathcal{O}_1 \subset \mathcal{O}_1$).*

$$\mathcal{O}_1 \cap \mathbf{PLie} = \phi(T)$$

but $\mathcal{O}_1 \neq \phi(\mathbf{MG})$.

$\phi(\mathbf{MG})$ $\qquad$ $\mathcal{O}_1$

$\sum_{v \in g}(g, v)$ $\qquad$ $\sum_{v \in g}(g, t_v, v)$

How to choose a spanning tree for each vertex ?

$\phi(\mathbf{MG})$ $\qquad$ $\mathcal{O}_1$

$\sum_{v\in g}(g,v) \quad \sum_{v\in g}(g,t_v,v)$

How to choose a spanning tree for each vertex ?

We don't need to choose!

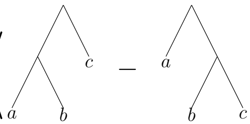
# Ideal

## Definition

An ideal of an operad $\mathcal{O}$ is a sub-species $\mathcal{I}$ such that $\mathcal{O} \circ_* \mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I} \circ_* \mathcal{O} \subseteq \mathcal{I}$.
The quotient species $\mathcal{O}/\mathcal{I}$ defined by $\mathbb{K}\mathcal{O}/\mathcal{I}[V] = \mathbb{K}\mathcal{O}[V]/\mathbb{K}\mathcal{I}[V]$ then has a natural operad structure.

*Ex:*

$\mathcal{O} = \mathbf{ComMag}$

$\mathcal{I} = \Big\langle$  $\Big\rangle$



$\mathbf{ComMag}/\mathcal{I} \cong \mathbf{Com}$

# The ideal $\mathcal{I}_1$

$$\mathcal{I}_1[V] = \{(g, t_1, v) - (g, t_2, v) \,|\, g \in \mathbf{MG}_c[V], \\ t_1,\ t_2 \text{ spanning trees of } g, v \in V\} \\ \subset \mathbb{K}\mathbf{ST}[V]$$

**Lemma**

*$\mathcal{I}_1$ is an ideal of $\mathcal{O}_1$.*

$$\mathcal{O}_1/\mathcal{I}_1 \cong \phi(\mathbf{MG}_c)$$

# Complete diagram

$$
\begin{tikzcd}
\mathsf{T} \arrow[r, "\sim"] \arrow[d, hook] & \mathsf{PLie} \cap \mathcal{O}_1/\mathcal{I}_1 \arrow[r, equal] \arrow[d, hook] & \mathsf{PLie} \cap \mathcal{O}_1 \arrow[r, hook] \arrow[d, hook] & \mathsf{PLie} \arrow[d, hook] \\
\mathsf{MG}_c \arrow[r, "\sim"] & \mathcal{O}_1/\mathcal{I}_1 & \mathcal{O}_1 \arrow[l, two heads] \arrow[r, hook] & \mathsf{ST}
\end{tikzcd}
$$

# Complete diagram

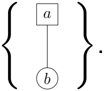
$$
\begin{tikzcd}
\mathbf{T} \arrow[r, "\sim"] \arrow[d, hook] & \mathbf{PLie} \cap \mathcal{O}_1/\mathcal{I}_1 \arrow[r, equal] \arrow[d, hook] & \mathbf{PLie} \cap \mathcal{O}_1 \arrow[r, hook] \arrow[d, hook] & \mathbf{PLie} \arrow[d, hook] \\
\mathbf{G}_c \arrow[r, "\sim"] \arrow[d, hook] & \mathbf{G}_c \cap \mathcal{O}_1/\mathcal{I}_1 \arrow[d, hook] & \mathbf{G}_c \cap \mathcal{O}_1 \arrow[l, two heads] \arrow[r, hook] \arrow[d, hook] & \mathbf{G}_c \cap \mathbf{ST} \arrow[d, hook] \\
\mathbf{MG}_c \arrow[r, "\sim"] & \mathcal{O}_1/\mathcal{I}_1 & \mathcal{O}_1 \arrow[l, two heads] \arrow[r, hook] & \mathbf{ST}
\end{tikzcd}
$$

# Table of contents

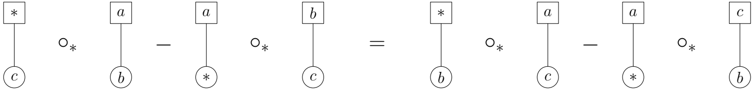
Description of an operad by its generators and relations.

*Ex:* **PLie**

generators:



relations:



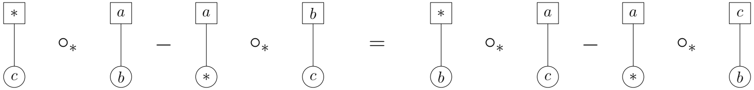
# Generators and relations

Description of an operad by its generators and relations.

*Ex:* **PLie**

generators: $\left\{ \right.$  $\left. \right\}$.

relations:




What about **ST** ? **ST** $\cong$ **MG** $\times$ **PLie**

$\rightarrow$ what about **MG** ?

# Generators and relations

$\mathbf{MG}$: too involved.

$\mathbf{MG}_c$ ?

$\mathbf{MG}$: too involved.

$\mathbf{MG}_c$ ? too involved.

$\mathbf{G}_c$ ?

**MG**: too involved.

$\mathbf{MG}_c$ ? too involved.

$\mathbf{G}_c$ ? too involved.

. . . **T** ?

$\mathbf{MG}$: too involved.

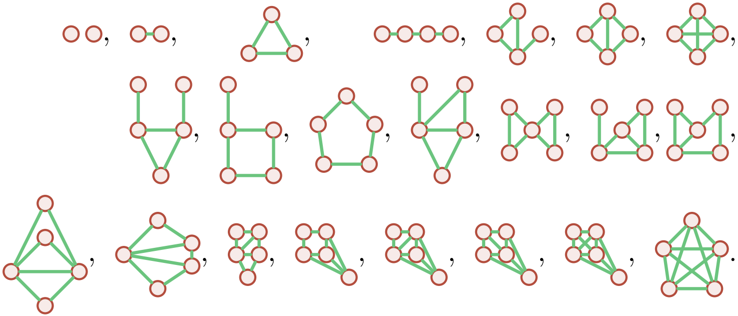
$\mathbf{MG}_c$ ? too involved.

$\mathbf{G}_c$ ? too involved.

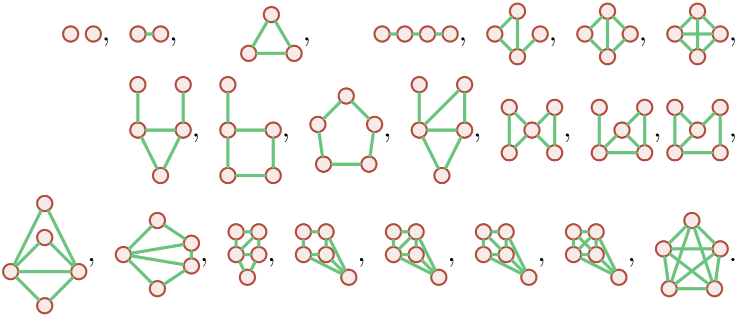
$\dots\mathbf{T}$ ? still too involved.

$\rightarrow$ finitely generated sub-operads.

# Graph generators



# Graph generators



## Lemma

*Any graph with $n$ vertices and at least $\binom{n-1}{2}+1$ edges must be a generator.*

# Graph generators

**Lemma**

*Any graph with n vertices and at least $\binom{n-1}{2}+1$ edges must be a generator.*

*Proof:* $v(g)$ = number of vertices, $e(g)$ number of edges.

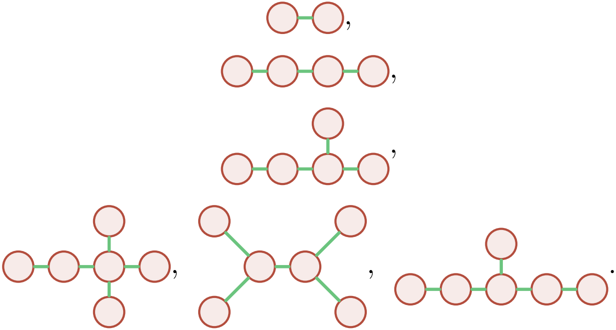
$g_1 \circ_* g_2 = g + \cdots \Rightarrow e(g) = e(g_1) + e(g_2)$

maximum when $e(g_i) = \binom{v(g_i)}{2}$

$\binom{v(g_1)}{2} + \binom{v(g_2)}{2} = \binom{v(g_1)}{2} + \binom{v(g)+1-v(g_1)}{2}$

maximum when $v(g) = 2$ or $v(g) = n-1$ and equal to $\binom{n-1}{2}$.

# Tree generators



# finitely generated sub-operads

- $\{ a \ \ b \}$

relation:

$$a \ \ * \ \circ_* \ b \ \ c \ = \ * \ \ c \ \circ_* \ a \ \ b.$$

- $\left\{ \textcircled{a}\ \textcircled{b} \right\} \cong \textbf{Com}$

relation:

$$\textcircled{a}\ \textcircled{*} \circ_* \textcircled{b}\ \textcircled{c} = \textcircled{*}\ \textcircled{c} \circ_* \textcircled{a}\ \textcircled{b}.$$

- $\{ (a) \ (b) \} \cong \mathbf{Com}$

relation:

$$(a) \ (*) \circ_* (b) \ (c) = (*) \ (c) \circ_* (a) \ (b).$$

- $\{ (a)\!-\!(b) \}$

no relations.

# finitely generated sub-operads

- $\{a \quad b\} \cong$ **Com**

relation:

$$a \quad * \quad \circ_* \quad b \quad c \quad = \quad * \quad c \quad \circ_* \quad a \quad b.$$

- $\{a - b\} \cong$ **ComMag**

no relations.

- **SP**

generators: $\{ a \;\; b,\; a - b \}$

relations:

$$a \;\; * \circ_* b \;\; c \;=\; * \;\; c \circ_* a \;\; b,$$

$$a - * \circ_* b \;\; c \;=\; c \;\; * \circ_* a - b \;+\; b \;\; * \circ_* a - c.$$

- **SP** Koszul.

generators: $\{ a \;\; b ,\; a - b \}$

relations:

$$a \;\; * \circ_* b \;\; c = * \;\; c \circ_* a \;\; b,$$

$$a - * \circ_* b \;\; c = c \;\; * \circ_* a - b + b \;\; * \circ_* a - c.$$

# finitely generated sub-operads

- **SP** Koszul.

generators: $\{a\ \ b,\ a — b\}$

relations:

$$a\ \ * \circ_* b\ \ c = *\ \ c \circ_* a\ \ b,$$

$$a — * \circ_* b\ \ c = c\ \ * \circ_* a — b + b\ \ * \circ_* a — c.$$

- **SP**$^!$ generators: $\{a\ \ b^\vee,\ a — b^\vee\}$

relations:

$$a — *^\vee \circ_* b — c^\vee = 0,$$

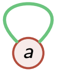
$$a\ \ *^\vee \circ_* b — c^\vee + c — *^\vee \circ_* a\ \ b^\vee + b — *^\vee \circ_* a\ \ c^\vee = 0,$$

$$a\ \ *^\vee \circ_* b\ \ c^\vee + c\ \ *^\vee \circ_* a\ \ b^\vee + b\ \ *^\vee \circ_* c\ \ a^\vee = 0.$$

# finitely generated sub-operads

- **LP**

generators:

$$\left\{ \right.$$  $$, \; a \quad b \left. \right\}.$$

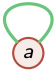
relations:

$$a \quad * \quad \circ_* \quad b \quad c \quad = \quad * \quad c \quad \circ_* \quad a \quad b.$$

$$\ldots \; ?$$

- **LP**

generators:

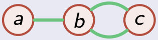
$\Big\{$  , $a$ $b$ $\Big\}$.

relations:

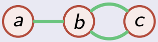
$$a \quad * \quad \circ_* \quad b \quad c \quad = \quad * \quad c \quad \circ_* \quad a \quad b.$$

$$\dots ?$$

**Lemma**

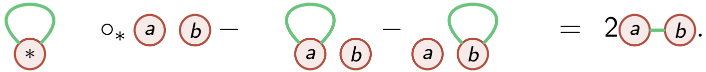
*We have* **SP** $\subset$ **LP** *but*  $\notin$ **LP**.

## Lemma

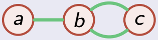
*We have* **SP** $\subset$ **LP** *but*  $\notin$ **LP**.

*Proof:*

**SP** $\subset$ **LP**:

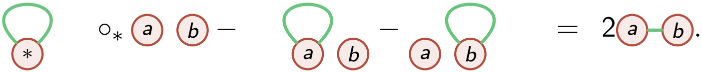


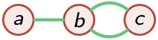
## Lemma

*We have* **SP** $\subset$ **LP** *but*  $\notin$ **LP**.

*Proof:*

**SP** $\subset$ **LP**:



 $\notin$ **LP**: Computer algebra

# Conclusion

Extensions and links of **PLie** and Kontsevich-Willwacher operads.

Study of finite case.

# Conclusion

Extensions and links of **PLie** and Kontsevich-Willwacher operads.

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General method to compute graph operads.

# Conclusion

Extensions and links of **PLie** and Kontsevich-Willwacher operads.

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Perspectives

generators and relations of **ST** and co ?

description of **LP** ?

$\mathbf{SP}^!$ link with permutations ?

$\mathbf{ComMag} \hookrightarrow \mathbf{T} \hookrightarrow \mathbf{PLie}$ further factorization ?

Thank you for your attention.

arXiv:1912.06563