

On some arithmetic properties of Mahler functions

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Mahler functions

Definition

A ℓ -Mahler function is a series $f(x) = \sum_{k=0}^{+\infty} f_k x^k \in \mathbb{Q}[[x]]$ such that

$$a_n(x)f(x^{\ell^n}) + a_{n-1}(x)f(x^{\ell^{n-1}}) + \cdots + a_0(x)f(x) = 0$$

for some $\ell \geq 2$, $n \geq 1$ and $a_0(x), \dots, a_n(x) \in \mathbb{Q}(x)$ with $a_0(x)a_n(x) \neq 0$.

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- Renewal of interest : the generating series of an automatic sequence is a Mahler function.

Mahler functions of order 1

In the rest of the talk, we assume that

$$f(x) = p(x)f(x^\ell)$$

where $p(x) \in \mathbb{Z}[x]$ is a polynomial such that $p(0) = 1$.

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Up to a multiplicative constant, we have

$$f(x) = \sum_{n=0}^{\infty} f_n x^n = \prod_{i \geq 0} p(x^{\ell^i}) \in \mathbb{Z}[[x]].$$

Mahler functions of order 1

Examples

- $p(x) = 1 - x$, $\ell = 2$: $f(x) = \sum_{n \geq 0} (-1)^{t_n} x^n$ where $(t_n)_{n \geq 0}$ is the Thue-Morse sequence, *i.e.*, $t_n \in \{0, 1\}$ is the sum of the binary digits of n modulo 2.

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- $p(x) = 1 + x + \cdots + x^{m-1}$, ℓ, m prime, $\ell > m$: $f(x) = \sum_{n \geq 0} f_n x^n$ where $f_n = 1$ if the base ℓ expansion of n has only digits less than m and $f_n = 0$ otherwise.

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- $p(x) = 1 + x + x^2$, $\ell = 3$: $f(x) = \sum_{n \geq 0} f_n x^n$ where $(f_n)_{n \geq 0}$ is the Stern diatomic sequence, *i.e.*, f_n counts the way of writing n as a sum of powers of 2 using each power of 2 at most twice.

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Question (Vague version)

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More precisely, when is $f(x)$

- *rational ?*
- *algebraic ?*
- *automatic ?*

When is $f(x)$ algebraic?

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$$P(x, f(x)) = 0.$$

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Démonstration.

- $f(x) \in \mathbb{Z}[[x]]$ is the Taylor expansion at 0 of an analytic function on the unit disc $D(0, 1)$;
- Polyà-Carlson Theorem.



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Theorem (Duke and Nguyen)

Assume that ℓ is a prime and $p(x) = \phi_m(x)$ is the m -th cyclotomic polynomial. Then, $f(x)$ is rational if and only if ℓ divides m .

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Assume that ℓ is a prime. Then, the following conditions are equivalent :

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- 1 $f(x)$ is rational ;*
- 2 $p(x)$ is a product of cyclotomic polynomials of order divisible by ℓ ;*
- 3 $p(x)$ is monic and, for almost all roots of unity ζ of order prime to ℓ , $f(x)$ is bounded as z tends to ζ radially.*

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Idea of proof of radial boundedness \Rightarrow product of cyclotomic polynomials

Let ζ be a root of unity of sufficiently large order m prime to ℓ and let κ be the order of ℓ in $(\mathbb{Z}/m\mathbb{Z})^\times$.

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First step : *a priori* radial behaviour. We have

$$f(\zeta e^t) \sim m_\zeta(t) t^{-\frac{\log(\prod_{k=0}^{\kappa-1} p(\zeta^{\ell^k}))}{\log(\ell^\kappa)}} \text{ as } t \rightarrow 0^- \text{ where } m_\zeta(\ell^\kappa t) = m_\zeta(t).$$

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Second step : norm estimate. By assumption, $f(x)$ is bounded as $x \rightarrow \zeta$, so

$$\Re\left(-\frac{\log(\prod_{k=0}^{\kappa-1} p(\zeta^{\ell^k}))}{\log(\ell^\kappa)}\right) \geq 0 \Rightarrow \left|\prod_{k=0}^{\kappa-1} p(\zeta^{\ell^k})\right| \leq 1$$

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Third step : $p(\zeta)$ is a unit of $\mathbb{Z}[\zeta]$. We have

$$x := \prod_{\xi \text{ root of } 1 \text{ of order } m} p(\xi) = \prod_{i=1}^r \prod_{k=0}^{\kappa_i-1} p(\zeta_i^{\ell^k})$$

for some primitive m -th roots of unity ζ_1, \dots, ζ_r .

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But $x = N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta)) \in \mathbb{Z}$, so

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Fourth step : conclusion. For infinitely many roots of unity ζ , $p(\zeta)$ is a unit of $\mathbb{Z}[\zeta]$. A result due to Kaminski implies that $p(x)$ is a product of cyclotomic polynomials.

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Reminders on automatic sequences

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Example - Baum-Sweet 2-automaton

- $\ell = 2$, $Q = \{q_1, q_2, q_3\}$, $i = q_1$,
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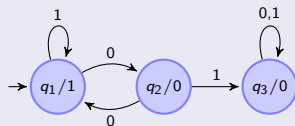
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Any ℓ -automaton generates a sequence $(u_n)_{n \in \mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$:

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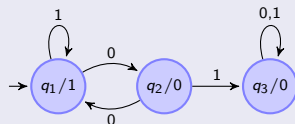
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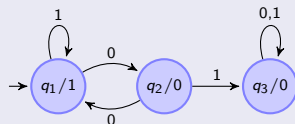
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$$(u_n)_{n \geq 0} = (1, 1, 0, 1, 1, 0, 0, 1, 0, \dots).$$

$$\text{For instance, } 5 = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \rightsquigarrow u_5 = 0.$$

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Automatic vs bounded

Remark

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Proposition (Becker)

$f(x)$ is automatic if and only if $(f_n)_{n \geq 0}$ is bounded.

When is $f(x)$ automatic?

The main result

Theorem (Checcoli-R)

- (i) If $\ell > \deg p(x)$, $f(x)$ is automatic if and only if the coefficients of $p(x)$ are in $\{0, \pm 1\}$.
- (ii) If $\ell = 2$ and $p(x) = x^2 + bx + 1$, $f(x)$ is automatic if and only if $b \in \{0, -1\}$.
- (iii) If $\ell \leq 3$ and $p(x) = x^3 + bx^2 + cx + 1$, $f(x)$ is automatic if and only if :
 - $\ell = 2$ and $b = c = 0$;
 - $\ell = 3$ and $(b, c) \in \{(0, 0), (-1, 0), (0, -1)\}$.

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Idea of proof

Let ζ be a root of unity of sufficiently large order m prime to ℓ and let κ be the order of ℓ in $(\mathbb{Z}/m\mathbb{Z})^\times$.

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Second step : norm estimate. By assumption, the coefficients of $f(x)$ are bounded, so $|tf(\zeta e^t)|$ is bounded as $t \rightarrow 0^-$. Therefore,

$$\Re\left(1 - \frac{\log(\prod_{k=0}^{\kappa-1} p(\zeta^{\ell^k}))}{\log(\ell^\kappa)}\right) \geq 0 \Rightarrow \left| \prod_{k=0}^{\kappa-1} p(\zeta^{\ell^k}) \right| \leq \ell^\kappa \Rightarrow |N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(p(\zeta))| \leq \ell^{[\mathbb{Q}(\zeta):\mathbb{Q}]}$$

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Third step : conclusion Applying this inequality to some explicit roots of unity ζ , we get polynomial inequalities for the coefficients of $p(x)$.

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... if $p(x)$ has higher degree

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We will see that the automaticity of $f(x)$ imposes restrictions on the order of vanishing of $p(x)$ at $x = 1$.

When is $f(x)$ automatic?

A necessary condition

Our norm estimates gives :

Theorem (Checcoli-R)

If $f(x)$ is automatic, then $v_{x-1}(p)$ can be explicitly bounded in terms of ℓ .

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Corollary (The case $\ell = 2$)

Assume that $\ell = 2$, that $f(x)$ is automatic and that one of the following conditions hold :

- 1** $p(j) \neq 0$;
- 2** $p(\zeta_5) \neq 0$, where ζ_5 is a primitive 5-th root of unity;
- 3** $p(\zeta_9) \neq 0$, where ζ_9 is a primitive 9-th root of unity.

Then, $v_{x-1}(p) = 0$ or 1.

Interpretation in terms of Newton polygons

Definition

The Newton polygon at $x = 1$ of

$$a_n(x)f(x^{\ell^n}) + a_{n-1}(x)f(x^{\ell^{n-1}}) + \cdots + a_0(x)f(x) = 0$$

is the convex hull in \mathbb{R}^2 of

$$\{(i, j) \mid i \in \mathbb{Z} \text{ and } j \geq v_{x-1}(a_i)\}.$$

This polygon is made of

- *two vertical half lines*
- *and of k vectors $(r_1, d_1), \dots, (r_k, d_k) \in \mathbb{N}^* \times \mathbb{Z}$ having pairwise distinct slopes, called the slopes L .*

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Question

Let $f(x)$ be a Mahler function of arbitrary order, and consider its minimal equation

$$a_n(x)f(x^{\ell^n}) + a_{n-1}(x)f(x^{\ell^{n-1}}) + \cdots + a_0(x)f(x) = 0.$$

If $f(x)$ is automatic, what can be said on the slopes of this minimal operator?