

Dimers and circle patterns

Sanjay Ramassamy
École normale supérieure

Joint work with:

Richard Kenyon (Brown University)

Wai Yeung Lam (Brown University)

Marianna Russkikh (Université de Genève)

Journées de Combinatoire de Bordeaux
February 11 2019

Dimers and circle patterns



Part 1

Dimers and circle patterns

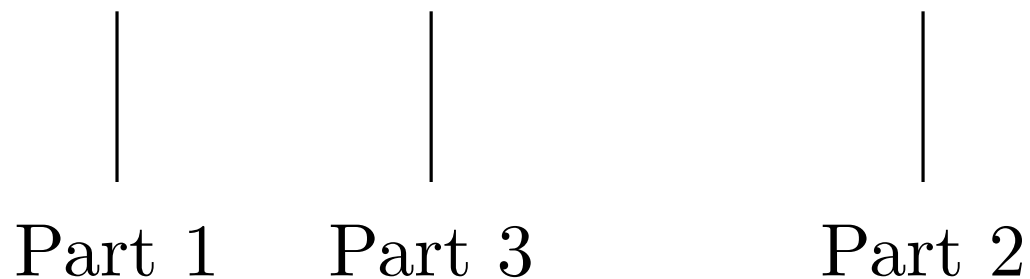
|

Part 1

|

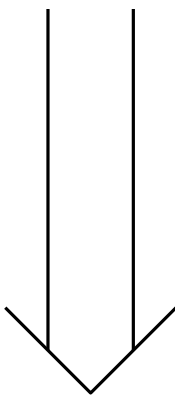
Part 2

Dimers and circle patterns



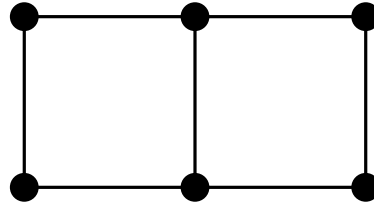
Dimers and circle patterns

Part 1 Part 3 Part 2

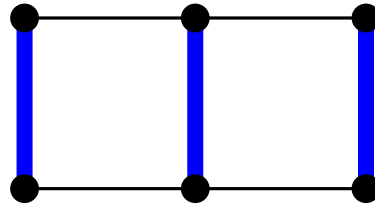


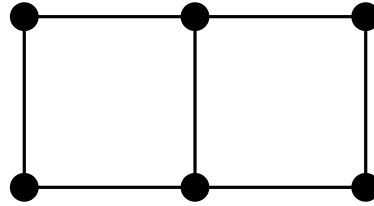
? — Part 4

1 The dimer model



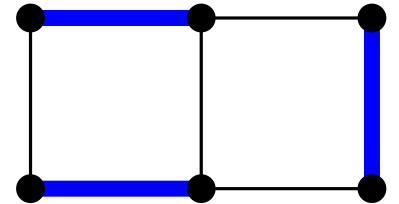
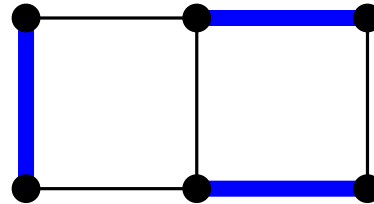
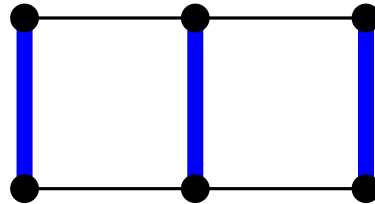
- *Dimer covering*: subset of edges such that each vertex is incident to exactly one edge.

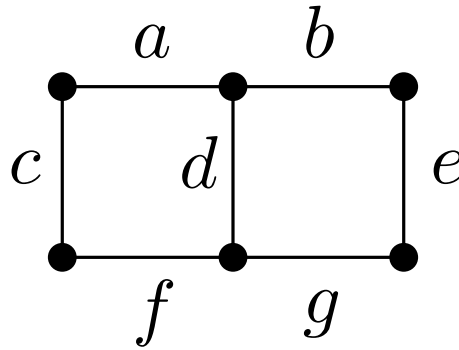




- *Dimer covering*: subset of edges such that each vertex is incident to exactly one edge.

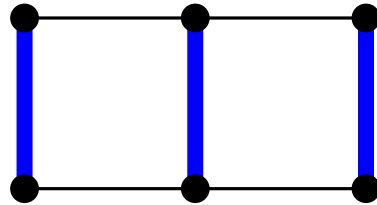
Dimer coverings:



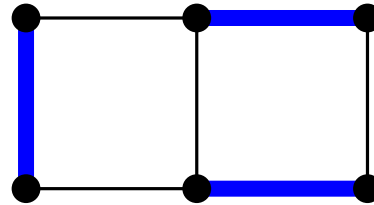


- *Dimer covering*: subset of edges such that each vertex is incident to exactly one edge.

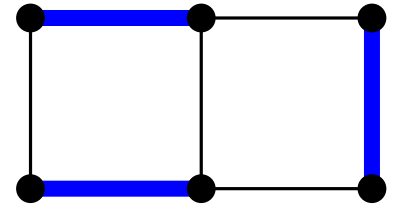
Dimer coverings:



cde



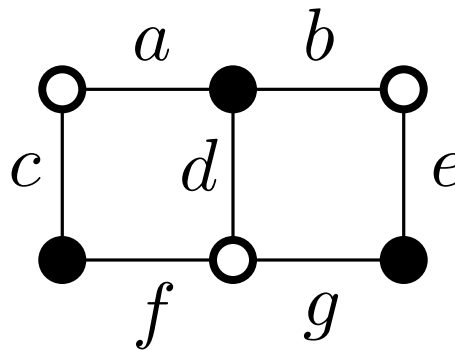
bcg



aef

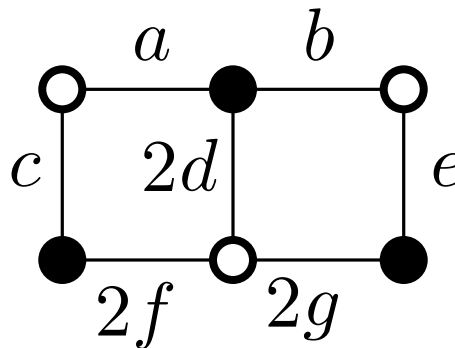
- *Boltzmann measure*: draw a dimer covering at random with probability proportional to its weight.

- Setting: *planar bipartite graphs* (vertices can be colored black and white such that each edge has two endpoints of different colors) with *positive edge weights*.



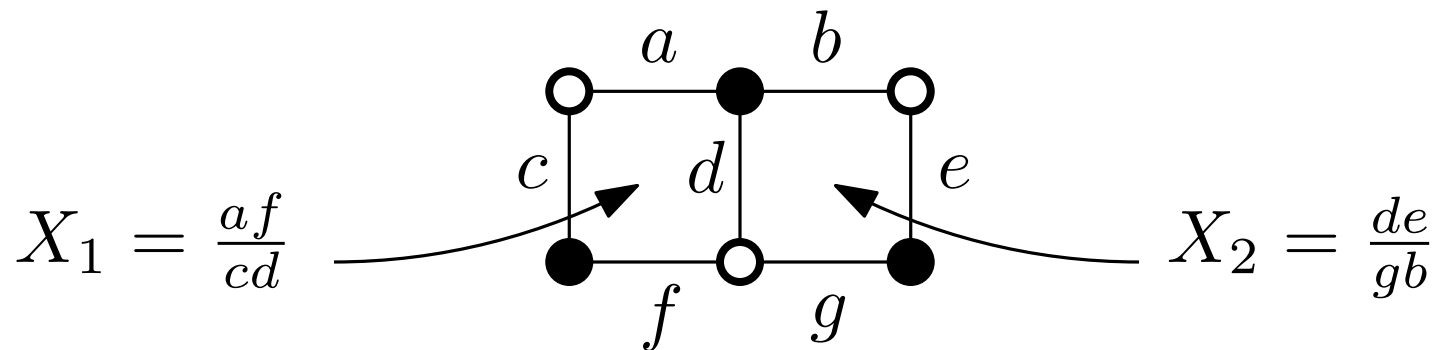
- Multiplying by $\lambda > 0$ the weight of every edge incident to a given vertex (*gauge transformation*) does not change the probability measure.
- Alternating products of edge weights around faces are coordinates on the space of edge weights modulo gauge.

- Setting: *planar bipartite graphs* (vertices can be colored black and white such that each edge has two endpoints of different colors) with *positive edge weights*.



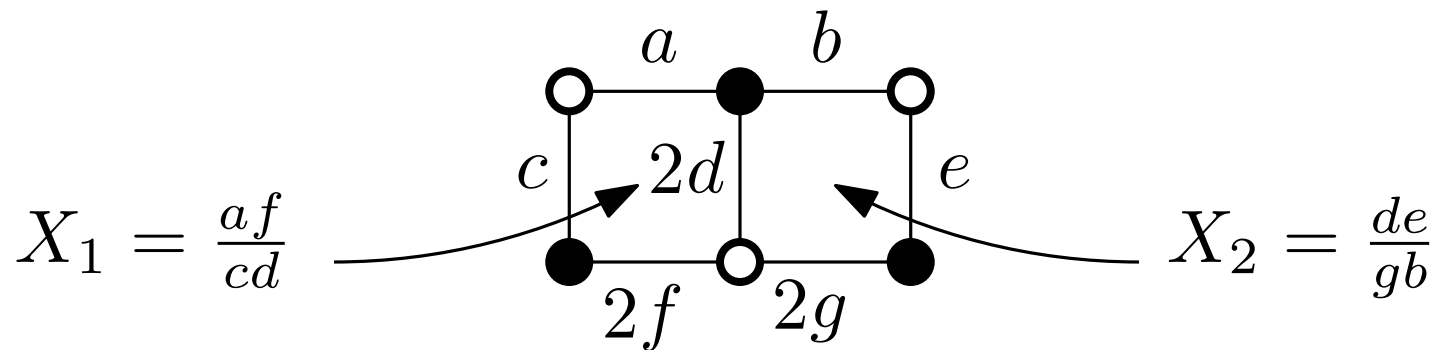
- Multiplying by $\lambda > 0$ the weight of every edge incident to a given vertex (*gauge transformation*) does not change the probability measure.
- Alternating products of edge weights around faces are coordinates on the space of edge weights modulo gauge.

- Setting: *planar bipartite graphs* (vertices can be colored black and white such that each edge has two endpoints of different colors) with *positive edge weights*.



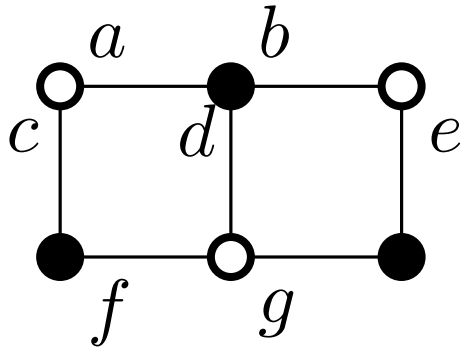
- Multiplying by $\lambda > 0$ the weight of every edge incident to a given vertex (*gauge transformation*) does not change the probability measure.
- Alternating products of edge weights around faces are coordinates on the space of edge weights modulo gauge.

- Setting: *planar bipartite graphs* (vertices can be colored black and white such that each edge has two endpoints of different colors) with *positive edge weights*.



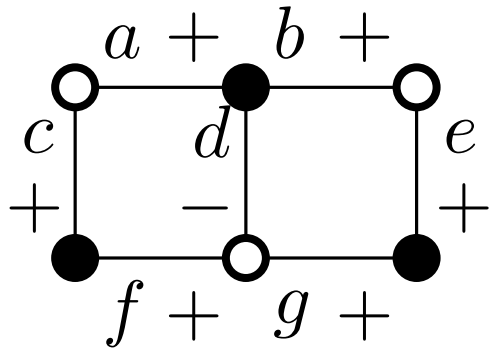
- Multiplying by $\lambda > 0$ the weight of every edge incident to a given vertex (*gauge transformation*) does not change the probability measure.
- Alternating products of edge weights around faces are coordinates on the space of edge weights modulo gauge.

The Kasteleyn matrix K



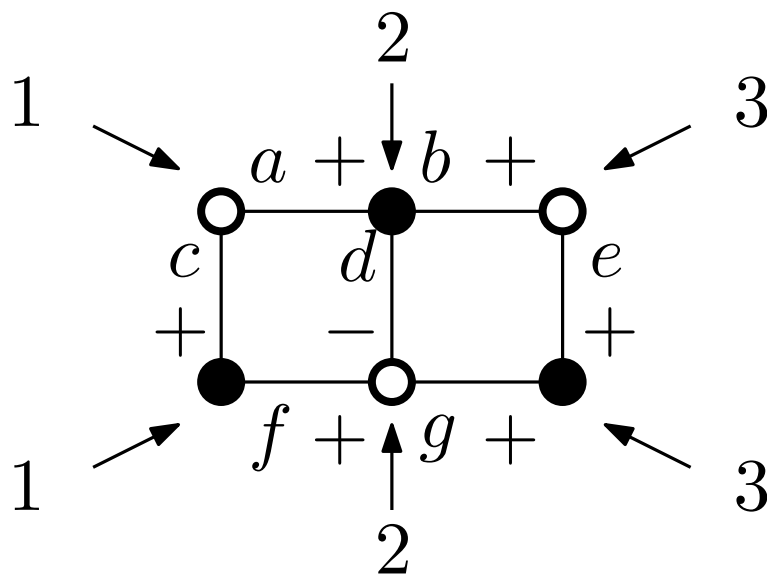
- *Kasteleyn signs*: assign a sign to each edge such that the number of minus signs around a face of degree 2 mod 4 (resp. 0 mod 4) is even (resp. odd).
- K : weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

The Kasteleyn matrix K



- *Kasteleyn signs*: assign a sign to each edge such that the number of minus signs around a face of degree 2 mod 4 (resp. 0 mod 4) is even (resp. odd).
- K : weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

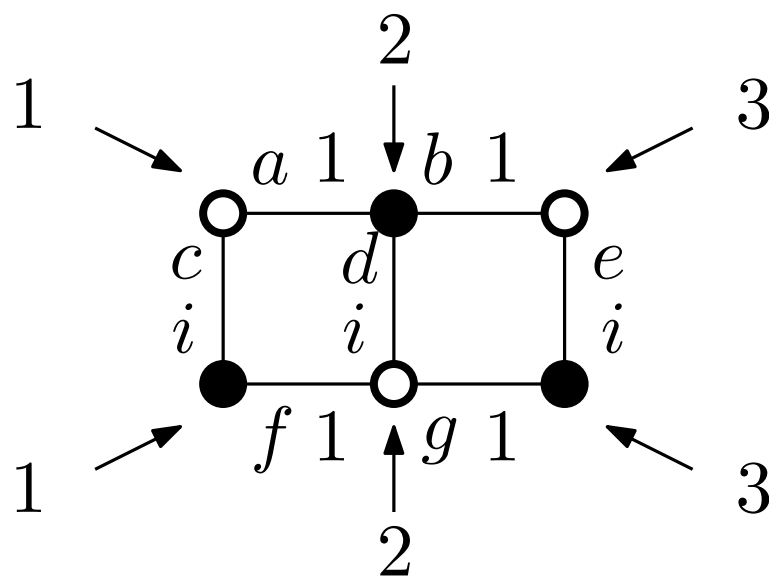
The Kasteleyn matrix K



$$\begin{matrix} \bullet_1 & \bullet_2 & \bullet_3 \\ \circ_1 & \circ_2 & \circ_3 \end{matrix} \begin{pmatrix} c & a & 0 \\ f & -d & g \\ 0 & b & e \end{pmatrix}$$

- *Kasteleyn signs*: assign a sign to each edge such that the number of minus signs around a face of degree 2 mod 4 (resp. 0 mod 4) is even (resp. odd).
- K : weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

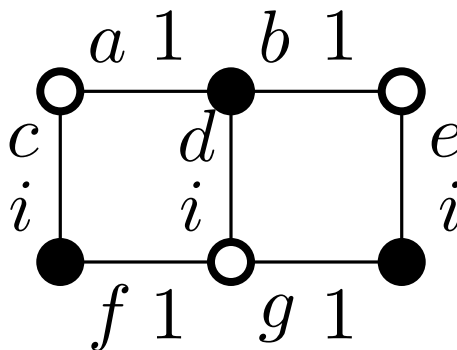
The Kasteleyn matrix K



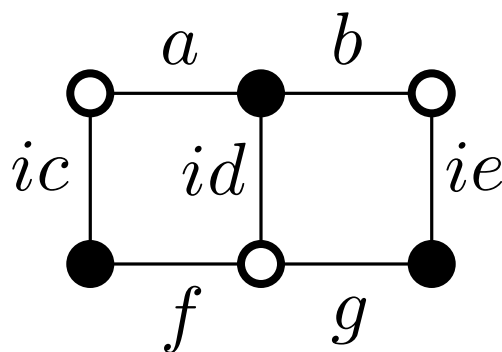
$$\begin{matrix} \bullet_1 & \bullet_2 & \bullet_3 \\ \circ_1 & \circ_2 & \circ_3 \end{matrix} \begin{pmatrix} ic & a & 0 \\ f & id & g \\ 0 & b & ie \end{pmatrix}$$

- *Complex Kasteleyn signs*: assign a unit complex number to each edge such that the alternating product of these numbers around a face of degree 2 mod 4 (resp. 0 mod 4) is 1 (resp. -1).
- K : weighted signed adjacency matrix with rows (resp. columns) indexed by white (resp. black) vertices.

- The partition function (sum of the weights of all dimer coverings) is $|\det K|$. The dimer correlations are given by minors of K^{-1} (Kasteleyn, Temperley-Fisher).
- Merge the complex Kasteleyn signs with the positive edge weights to get complex edge weights (entries of K).



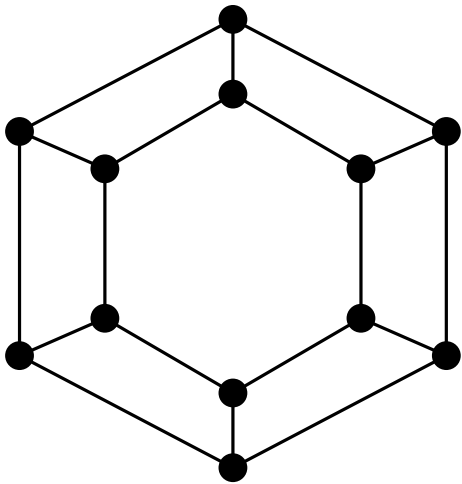
- The partition function (sum of the weights of all dimer coverings) is $|\det K|$. The dimer correlations are given by minors of K^{-1} (Kasteleyn, Temperley-Fisher).
- Merge the complex Kasteleyn signs with the positive edge weights to get complex edge weights (entries of K).



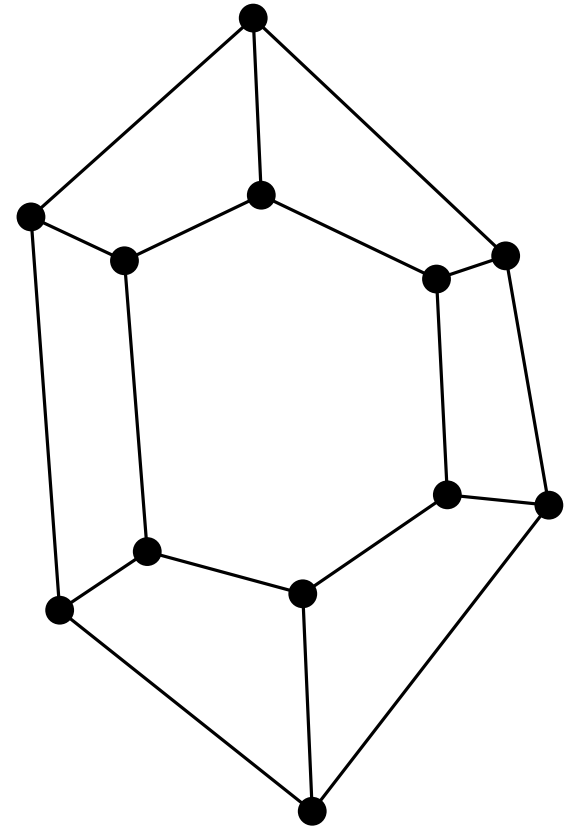
- The partition function (sum of the weights of all dimer coverings) is $|\det K|$. The dimer correlations are given by minors of K^{-1} (Kasteleyn, Temperley-Fisher).
- Merge the complex Kasteleyn signs with the positive edge weights to get complex edge weights (entries of K).
- The alternating product of complex edge weights is real positive (resp. real negative) around a face of degree 2 mod 4 (resp. 0 mod 4).

2 Circle patterns and circle centers

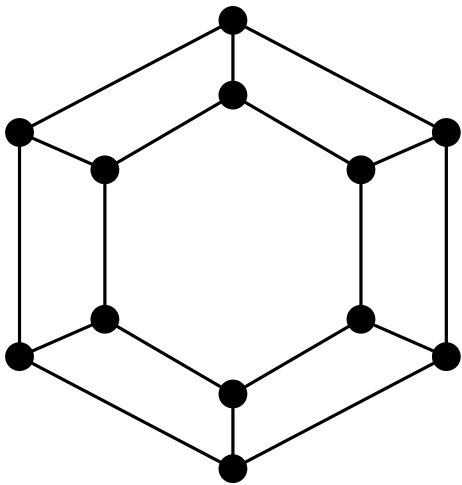
- *Circle pattern for G* : map from the vertex set of G to \mathbb{R}^2 sending all the vertices around any bounded face to concyclic points.



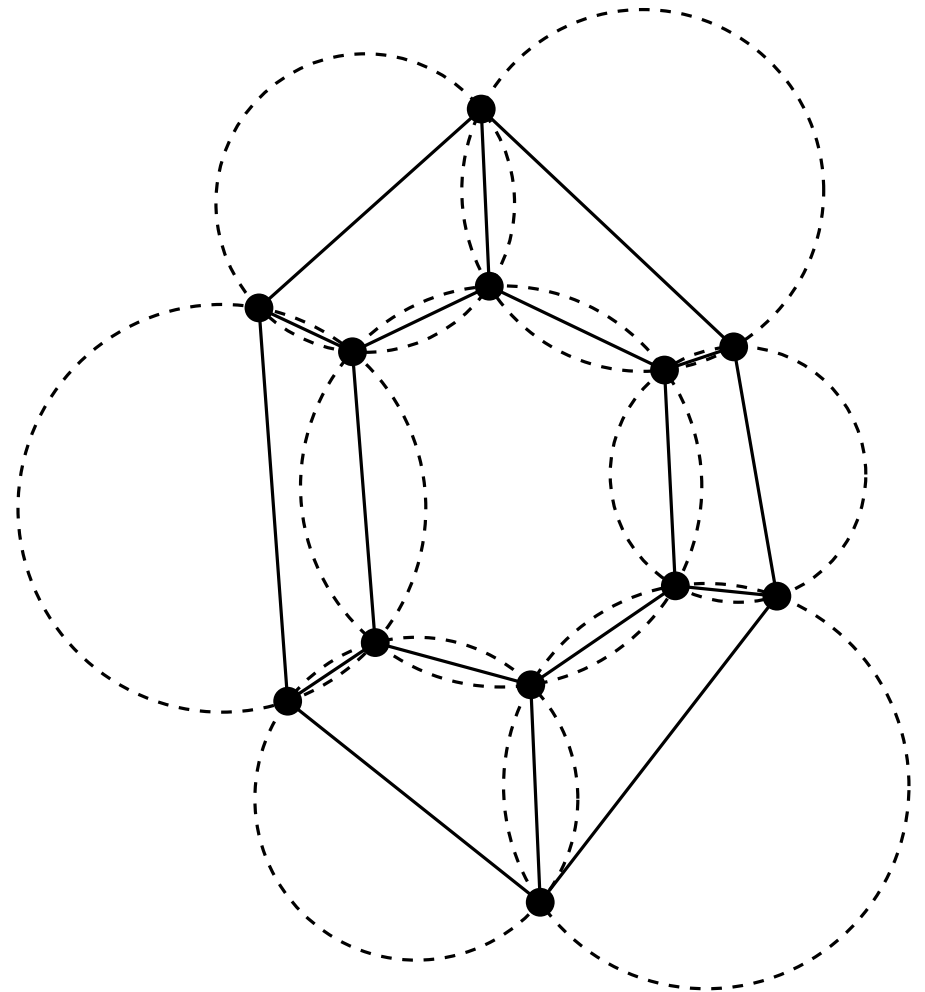
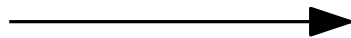
G planar



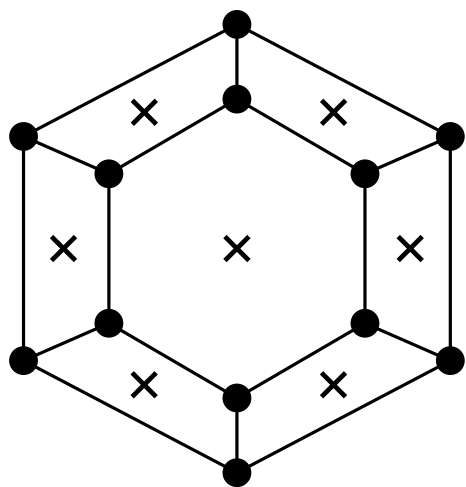
- *Circle pattern for G* : map from the vertex set of G to \mathbb{R}^2 sending all the vertices around any bounded face to concyclic points.



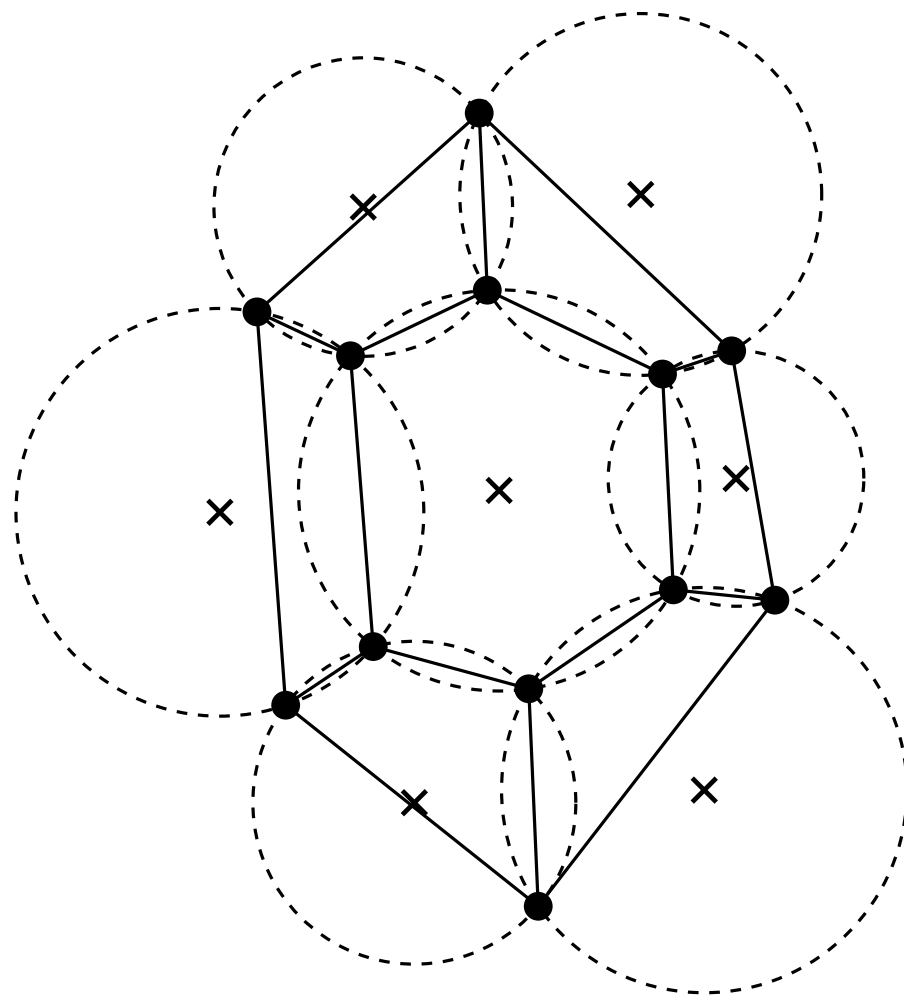
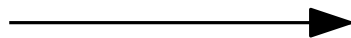
G planar



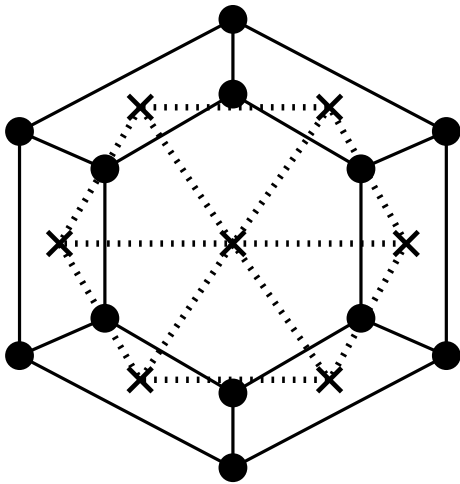
- *Circle pattern for G* : map from the vertex set of G to \mathbb{R}^2 sending all the vertices around any bounded face to concyclic points.



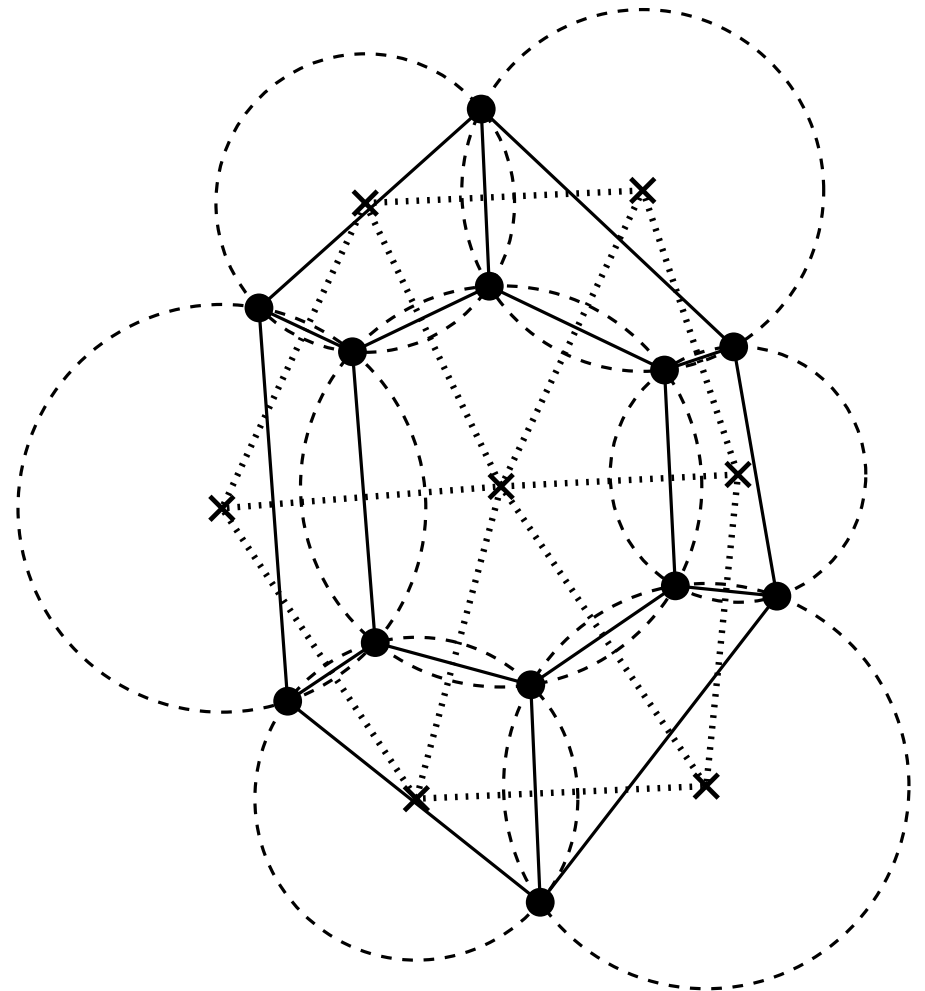
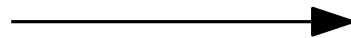
G planar



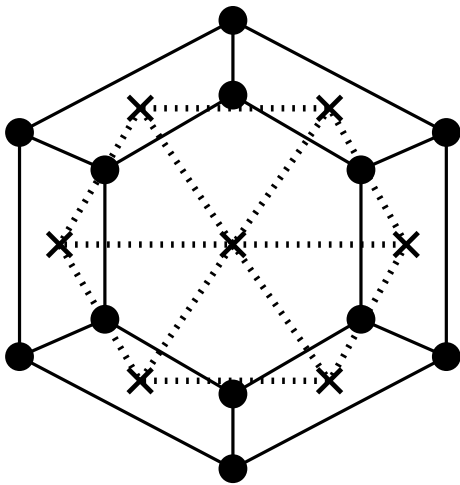
- *Circle pattern for G* : map from the vertex set of G to \mathbb{R}^2 sending all the vertices around any bounded face to concyclic points.



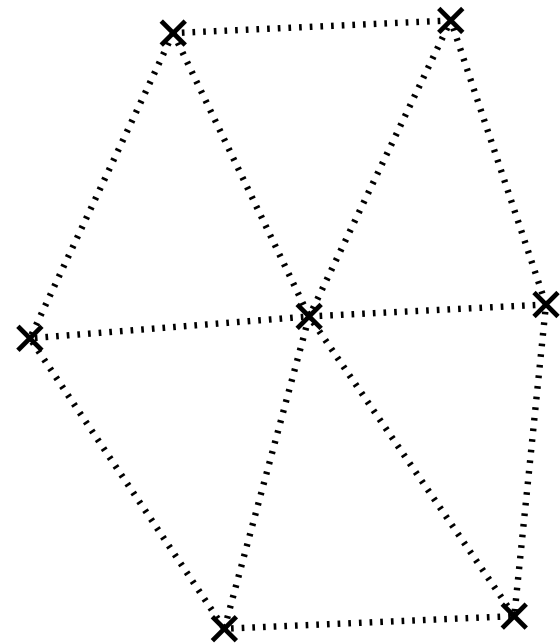
G planar



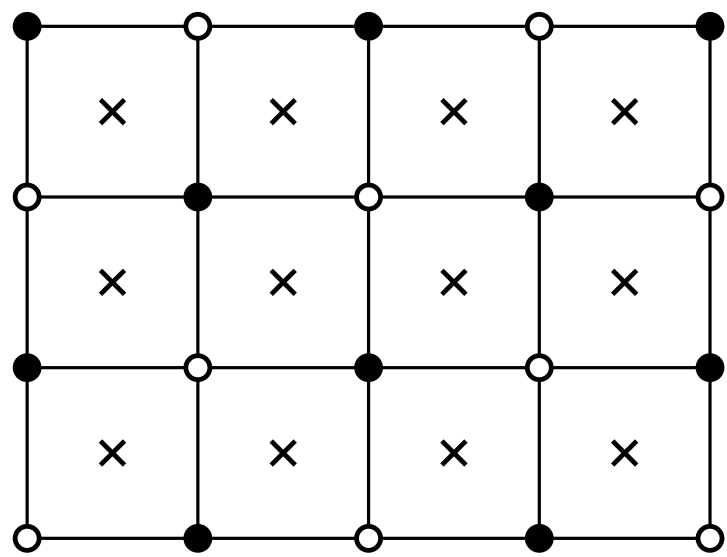
- *Circle centers for G* : drawing of the dual graph of G arising as centers of some circle pattern for G .



G planar

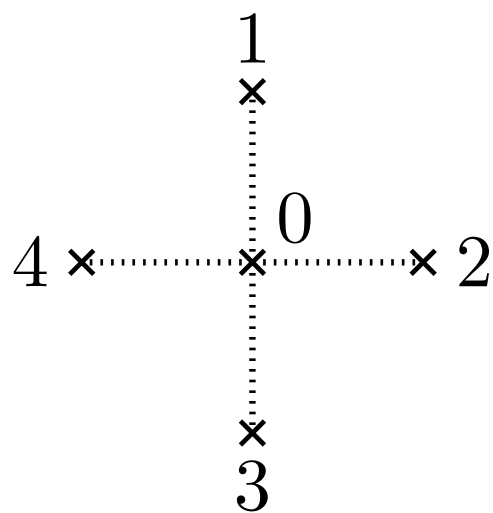


- Recover the circle pattern from the circle centers ?
How many circle patterns have the same centers ?
- Given a drawing of the dual graph of G , how to see if it corresponds to the centers of a circle pattern for G ?
- Answers in the case when G is bipartite.

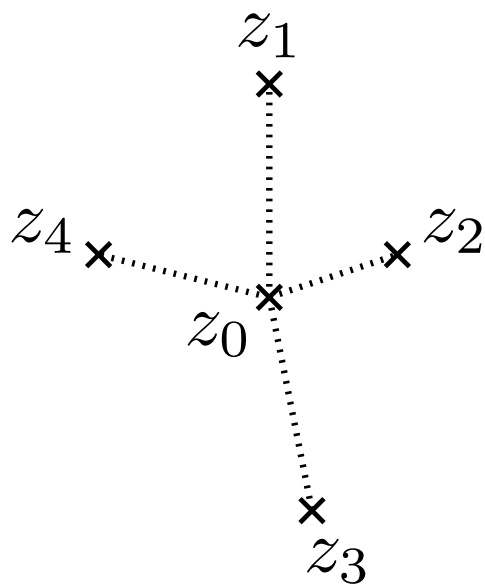


[Geogebra]

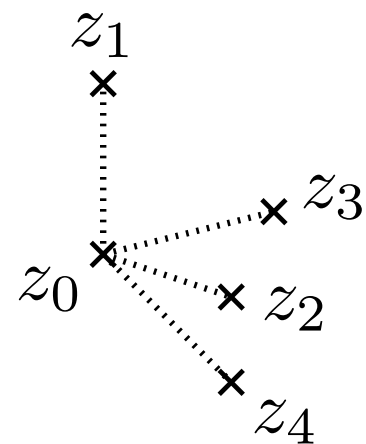
- From now on G is bipartite.
- 2-parameter family of patterns with the same centers.
- A drawing of the dual graph of G corresponds to circle centers for G if and only if around each dual vertex the sum of every other angle is $0 \bmod \pi$.
- An *embedding* of the dual graph of G corresponds to circle centers for G if and only if around each dual vertex the sum of every other angle is equal to π .



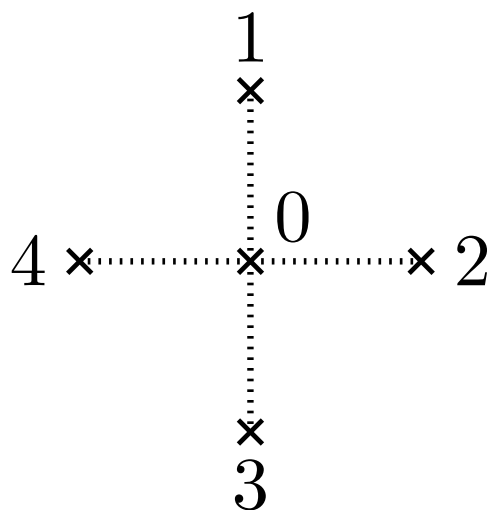
G



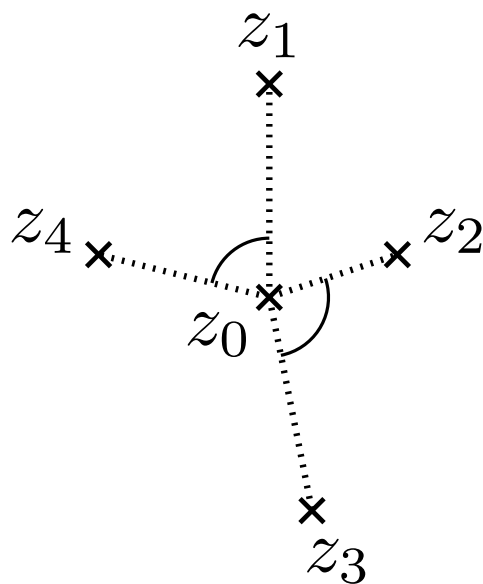
embedding



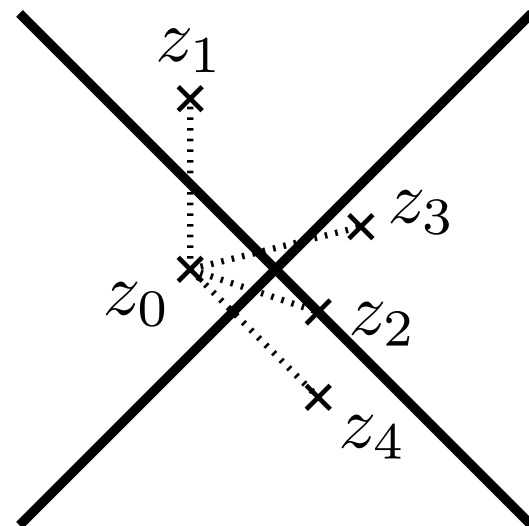
not embedding



G



embedding



not embedding

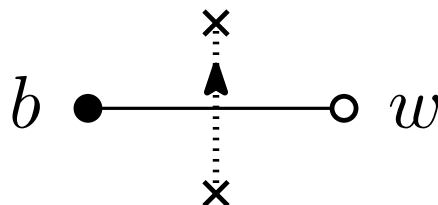


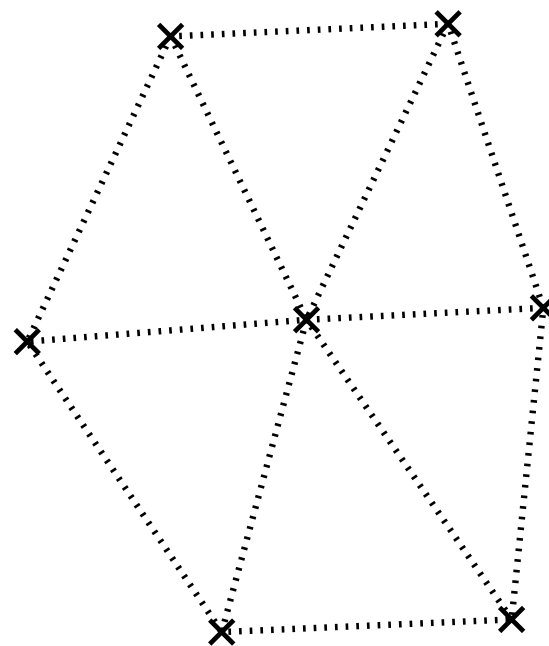
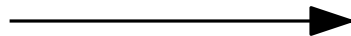
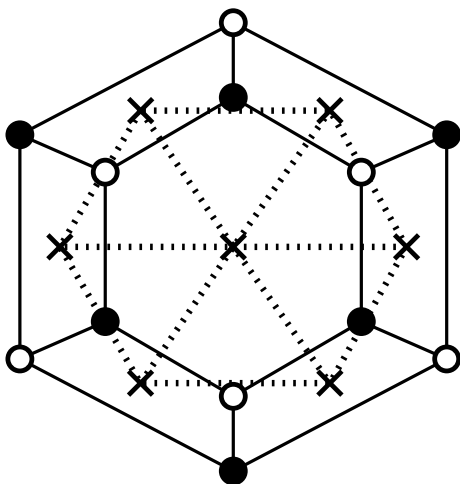
$$\arg \frac{z_4 - z_0}{z_1 - z_0} + \arg \frac{z_2 - z_0}{z_3 - z_0} = \pi \Leftrightarrow \frac{(z_2 - z_0)(z_4 - z_0)}{(z_1 - z_0)(z_3 - z_0)} \in \mathbb{R}_{<0}$$

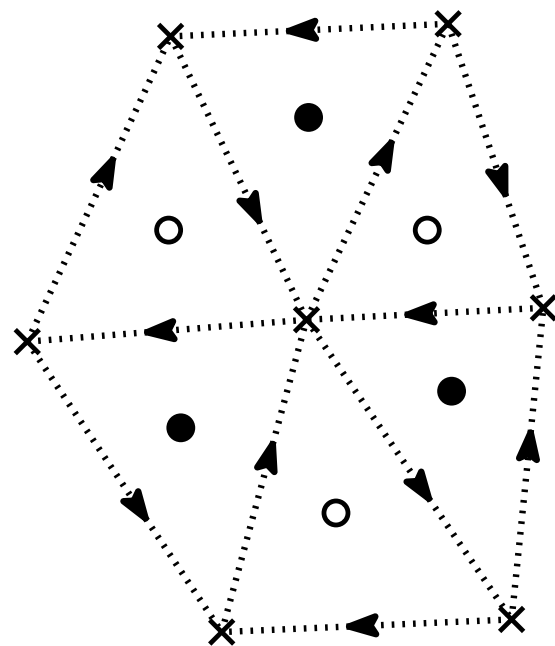
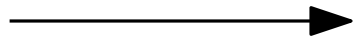
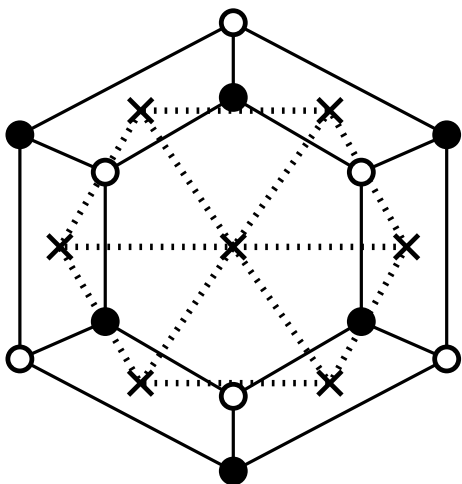
3 Dimer models and circle centers

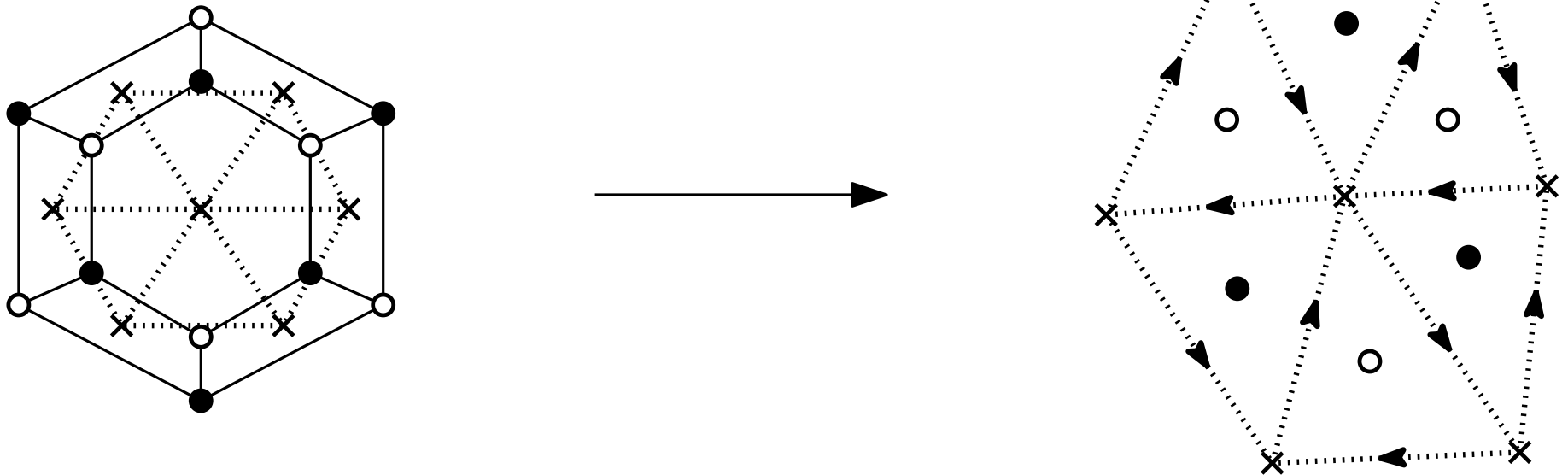
From circle centers to dimer weights

- Fix G a planar unweighted bipartite graph. Start with an embedding of the dual of G as circle centers.
- Construct complex edge weights for G associated to that embedding.
- For an edge in G between b and w , the weight is the vector (complex number) of its corresponding dual edge, oriented so that b lies to its left.

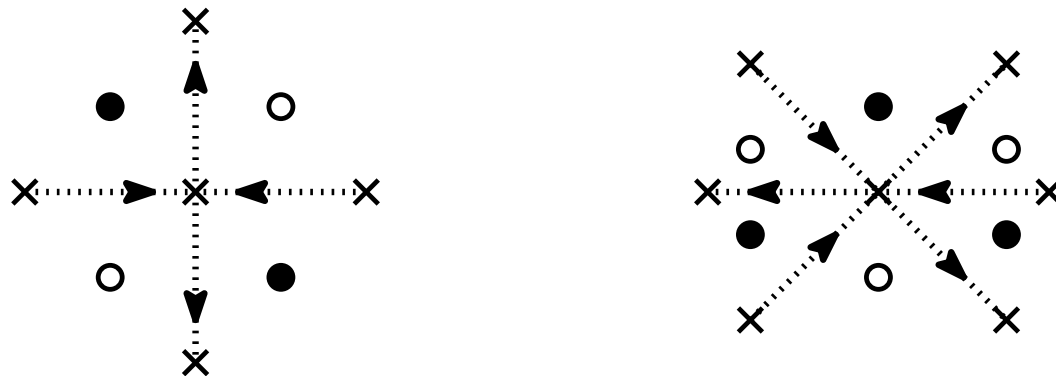


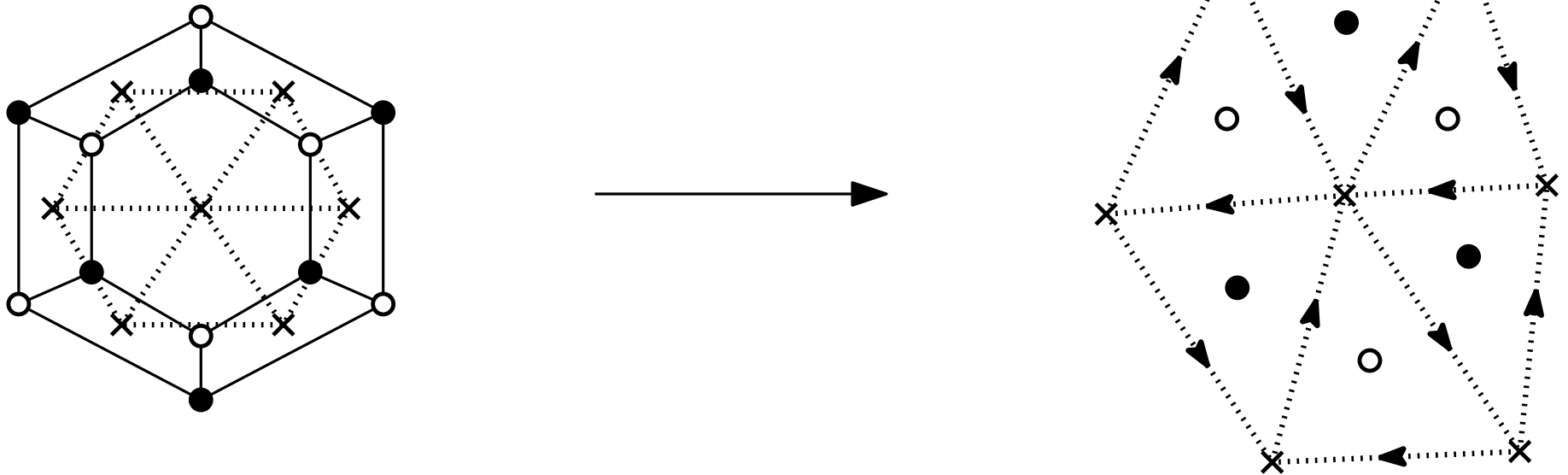






- The complex edge weights satisfy the Kasteleyn condition: the alternating product around of a face of degree $2 \bmod 4$ (resp. $0 \bmod 4$) is positive (resp. negative).





- The complex edge weights satisfy the Kasteleyn condition: the alternating product around of a face of degree $2 \bmod 4$ (resp. $0 \bmod 4$) is positive (resp. negative).
- Around every vertex, the sum of the complex edge weights is zero, i.e. the edge weights have zero divergence.

For a bipartite graph, the geometric local condition
“being centers of a circle pattern with embedded dual”
implies the local condition
“being Kasteleyn edge weights with zero divergence”
(Kenyon-Lam-R.-Russkikh, 2018)

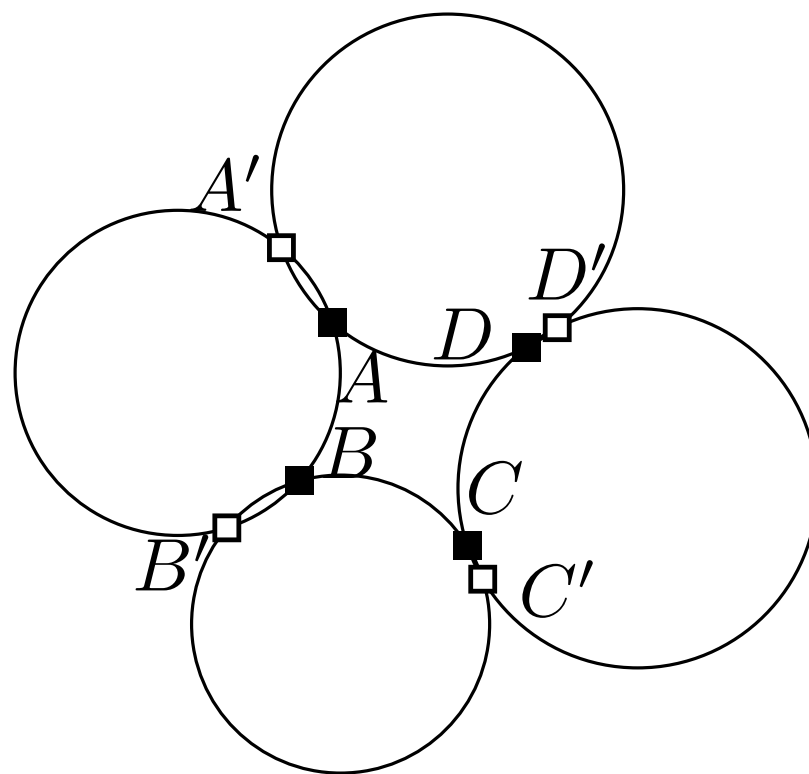
- The fact that circle center embeddings satisfy the Kasteleyn condition was also observed by Affolter (2018).
- Positive edge weights are obtained from circle centers as distances between adjacent centers.
- Going from circle centers to dimer edge weights is a local construction.

From dimer weights to circle centers

- Given a bipartite graph with positive edge weights, find gauge equivalent weights coming from circle centers.
- Possible for finite planar graphs and infinite planar graphs which are periodic in two directions (Kenyon-Lam-R.-Russkikh, 2018).
- The construction of a circle center embedding associated with a weighted planar graph G depends globally (not locally) on G .

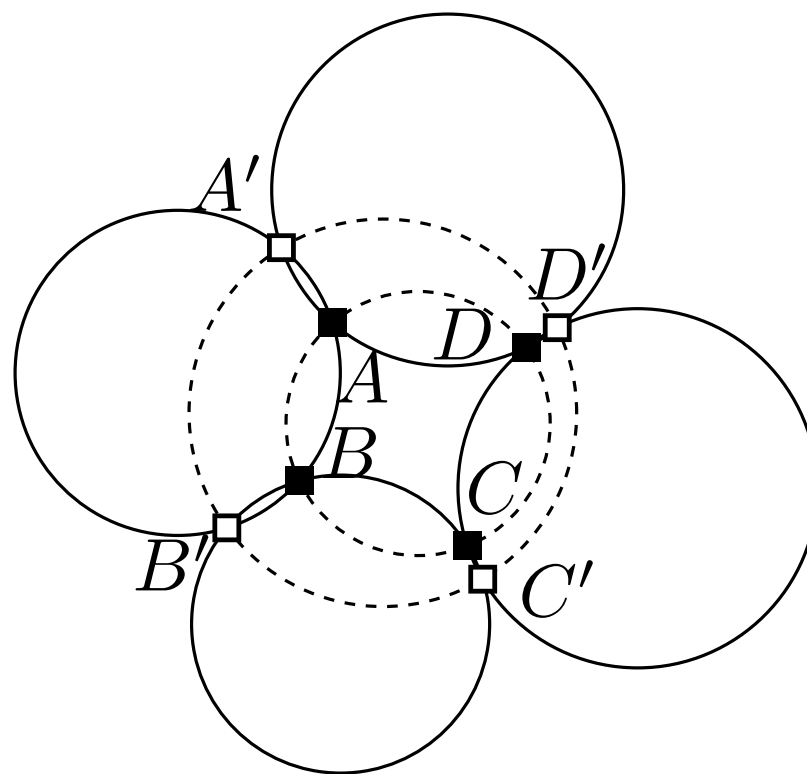
4 Motivation and perspectives

Miquel's theorem



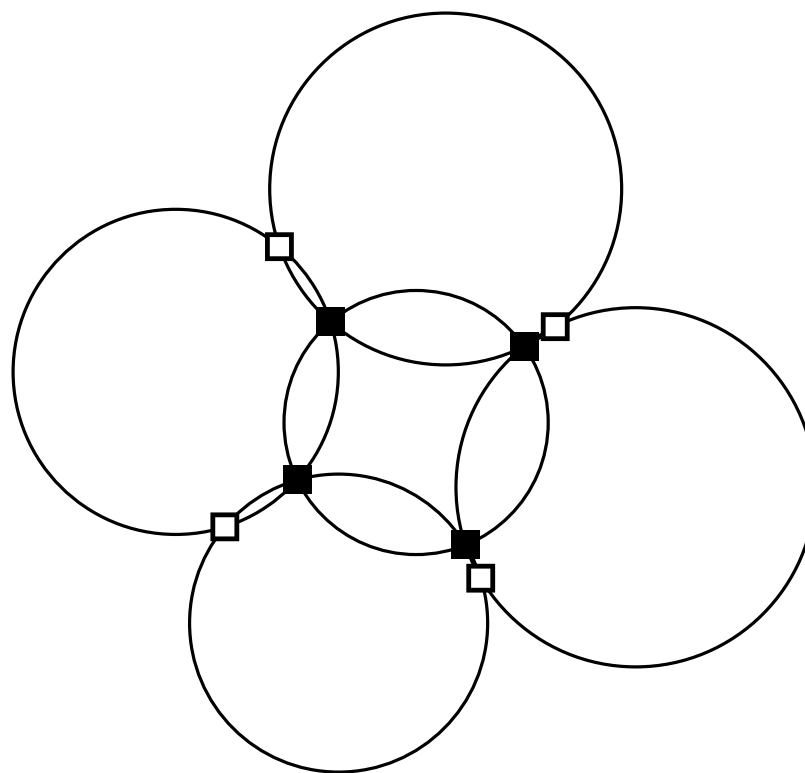
Theorem (Miquel, 1838). *In this setting,
 A, B, C, D concyclic $\Leftrightarrow A', B', C', D'$ concyclic.*

Miquel's theorem



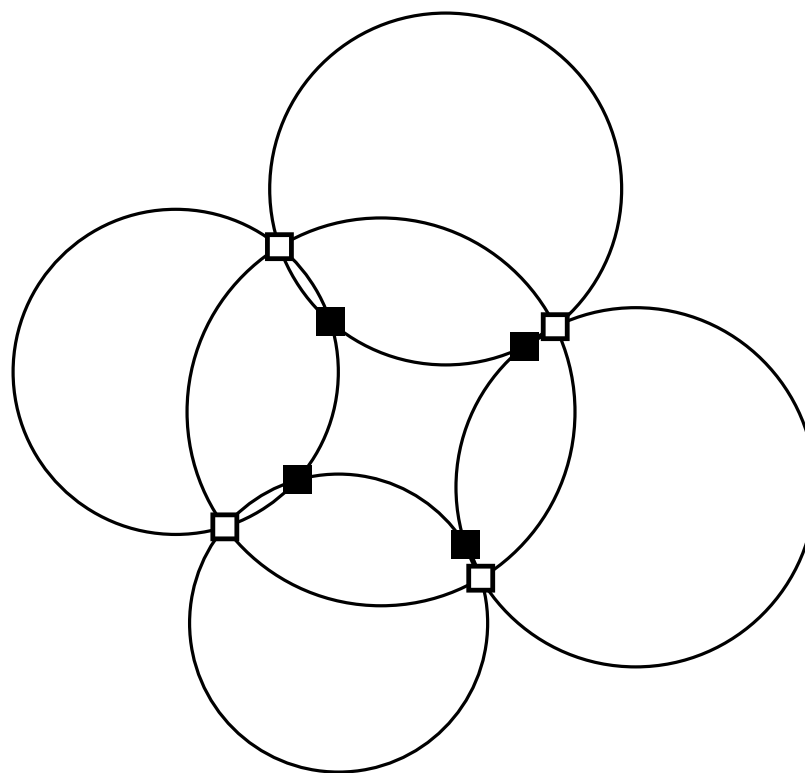
Theorem (Miquel, 1838). *In this setting,
 A, B, C, D concyclic $\Leftrightarrow A', B', C', D'$ concyclic.*

Miquel's theorem



Theorem (Miquel, 1838). *In this setting,
 A, B, C, D concyclic $\Leftrightarrow A', B', C', D'$ concyclic.*

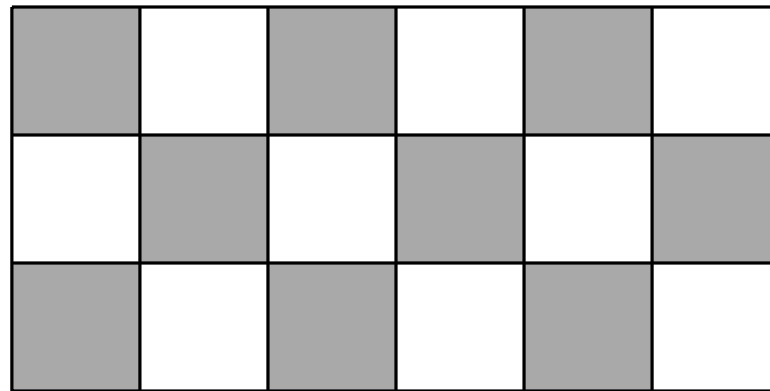
Miquel's theorem



Theorem (Miquel, 1838). *In this setting,
 A, B, C, D concyclic $\Leftrightarrow A', B', C', D'$ concyclic.*

Miquel dynamics

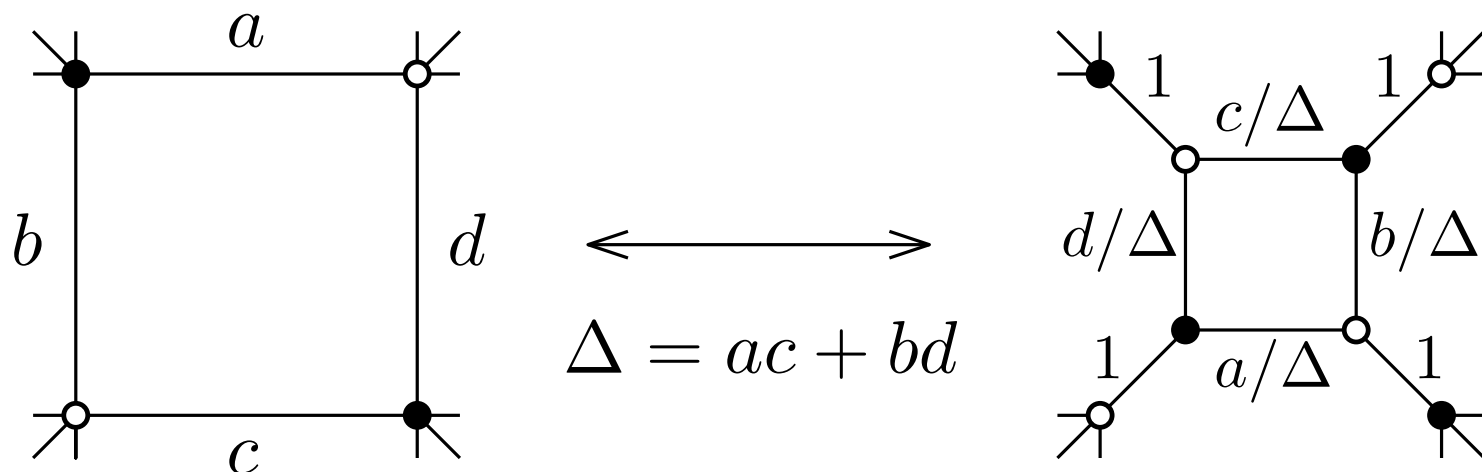
- Discrete-time dynamics on the space of square-grid circle patterns: alternate Miquel moves on all the white faces then on all the black faces.
- Invented by Kenyon, first properties studied in [R. 2018] and [Glutsyuk-R. 2018].



[Mathematica]

Dimer local move

- Urban renewal:



Theorem (Affolter 2018, Kenyon-Lam-R.-Russkikh). *The Miquel move for circle patterns commutes with the urban renewal for dimer models.*

- The identification with the Goncharov-Kenyon dimer dynamics yields the integrability of Miquel dynamics.

Embeddings in statistical mechanics

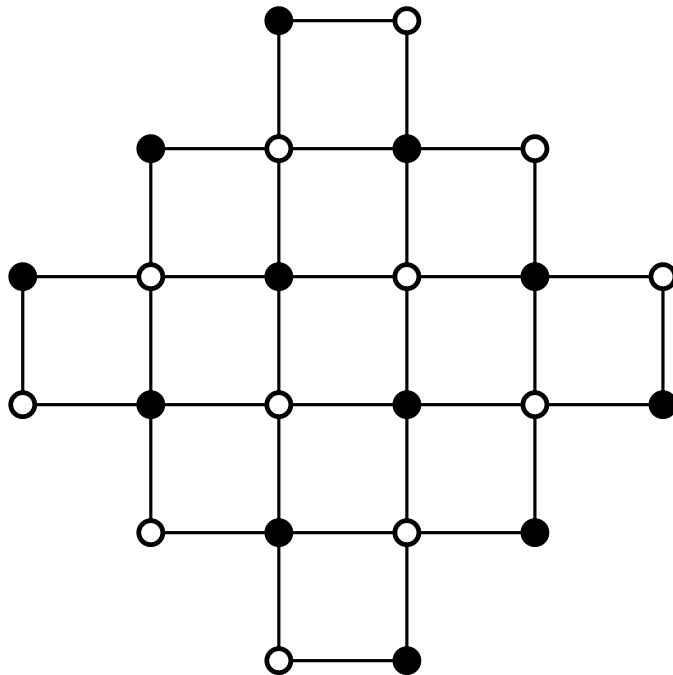
- Consider an infinite planar graph periodic in two directions on which we study a statistical mechanical model (random walk, dimers, Ising,...).
- Find an embedding of it such that universal conformally invariant objects appear in the scaling limit.
- Same issue for formulating the convergence to Liouville quantum gravity of random planar maps decorated with some statistical mechanical model.

Embeddings in statistical mechanics

Theorem (Kenyon-Lam-R.-Russkikh, 2018). *Circle center embeddings for dimers generalize the Tutte embedding adapted to spanning trees and the s -embeddings adapted to the Ising model.*

What to expect ?

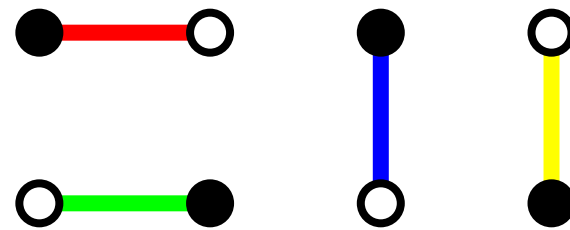
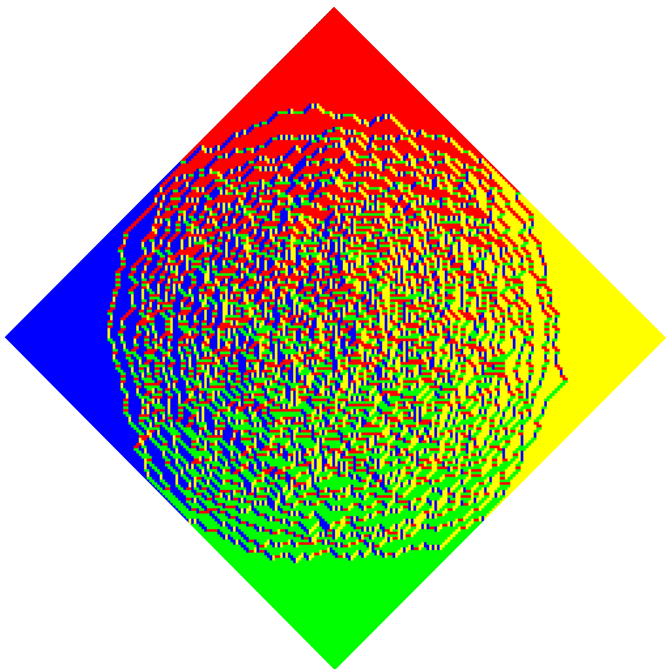
- Local formulas for the correlations and the free energy (Kenyon, Boutillier, de Tilière, Raschel).
- Limit shape results.



Aztec diamond of size 3

What to expect ?

- Local formulas for the correlations and the free energy (Kenyon, Boutillier, de Tilière, Raschel).
- Limit shape results.



picture by Cris Moore

tuvalu.santafe.edu/~moore/aztec256.gif

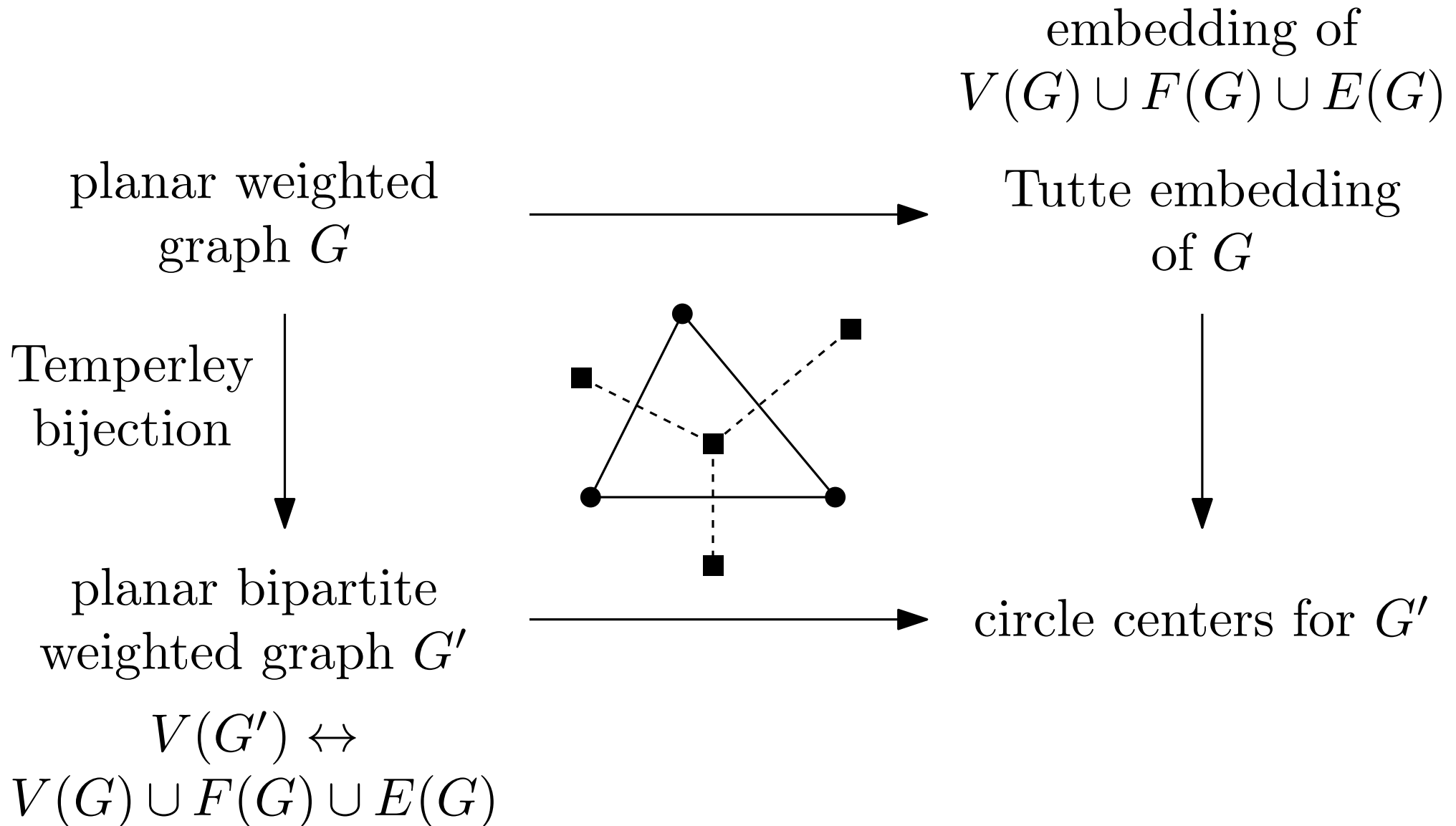
What to expect ?

- Local formulas for the correlations and the free energy (Kenyon, Boutillier, de Tilière, Raschel).
- Limit shape results.
- Work in progress joint with Dmitry Chelkak: Aztec diamond via circle patterns.

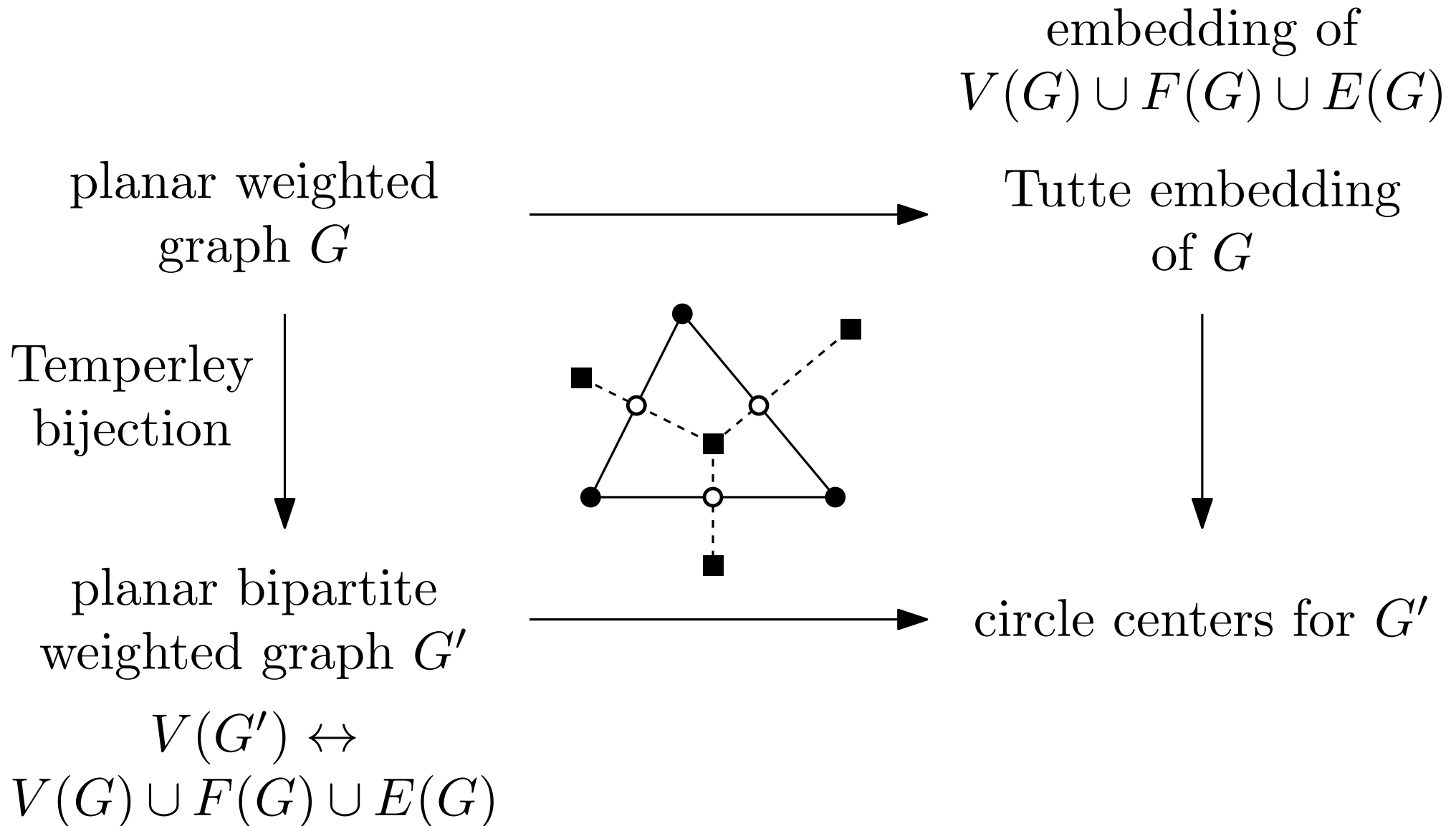
[Mathematica PDF]

- Notion of a discrete holomorphic function on such embeddings and conditions for converging to a continuous holomorphic function (Chelkak-Laslier-Russkikh, 2019).

Tutte embeddings and circle centers

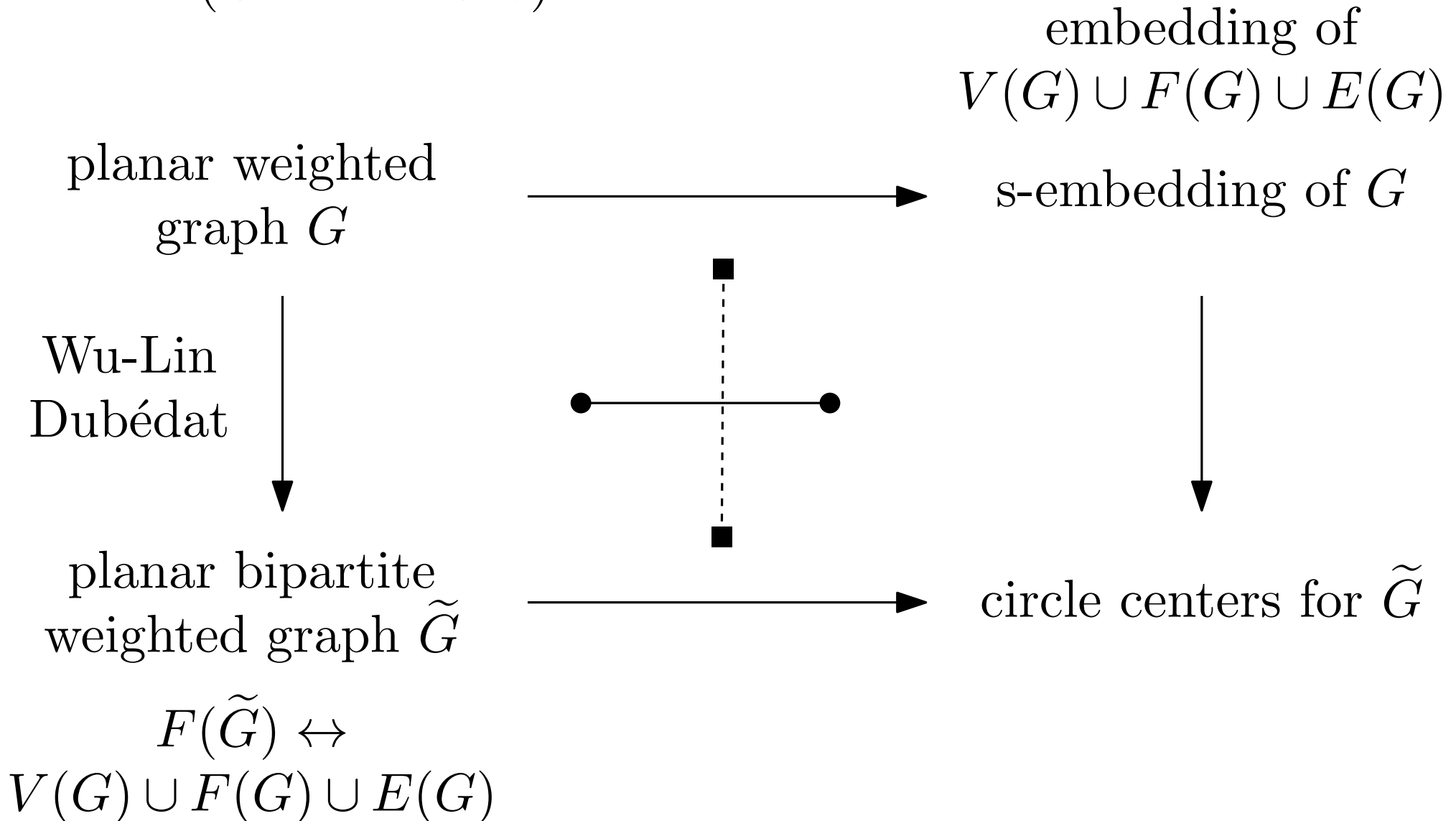


Tutte embeddings and circle centers



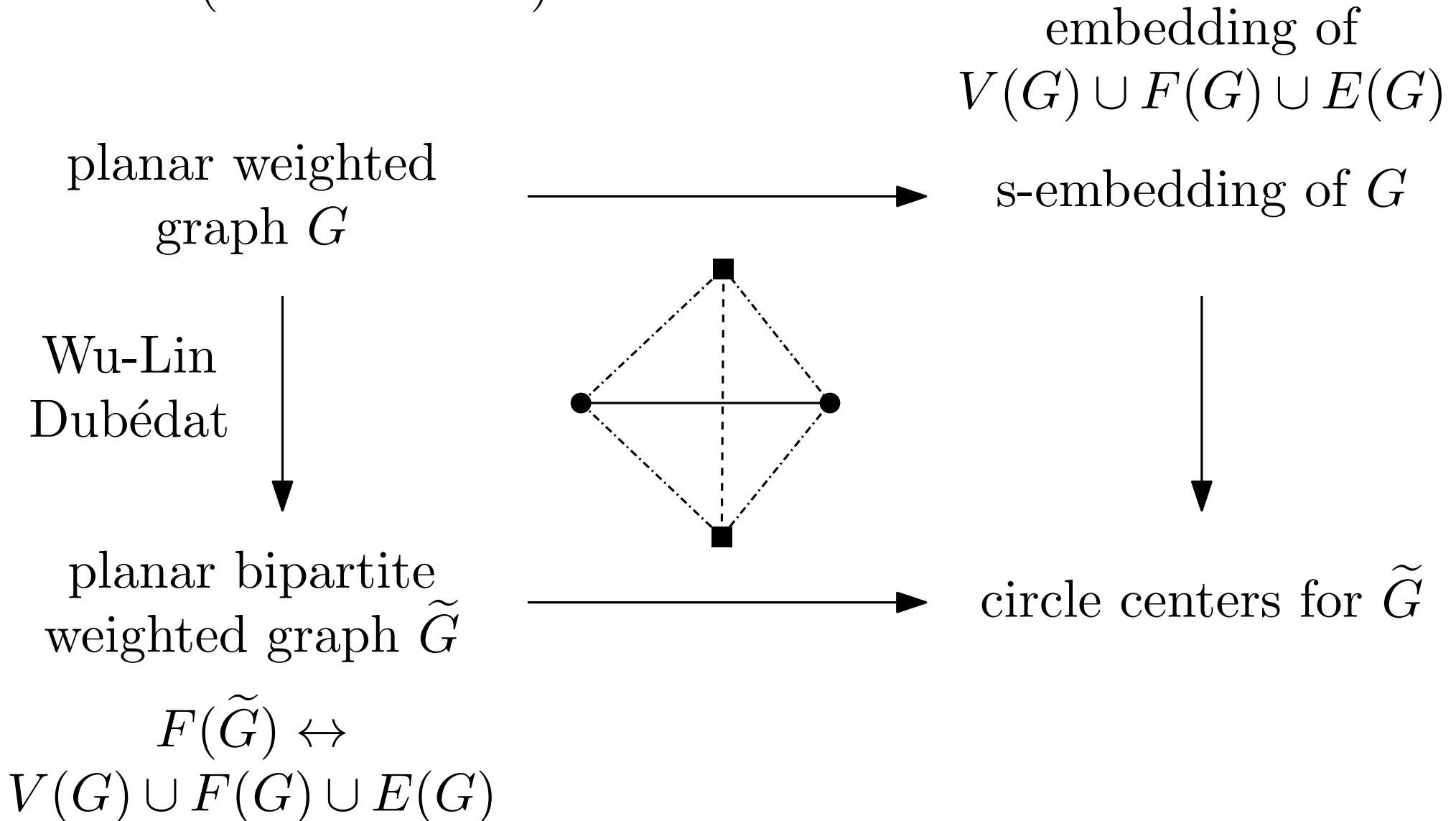
s-embeddings and circle centers

(Chelkak 2017)



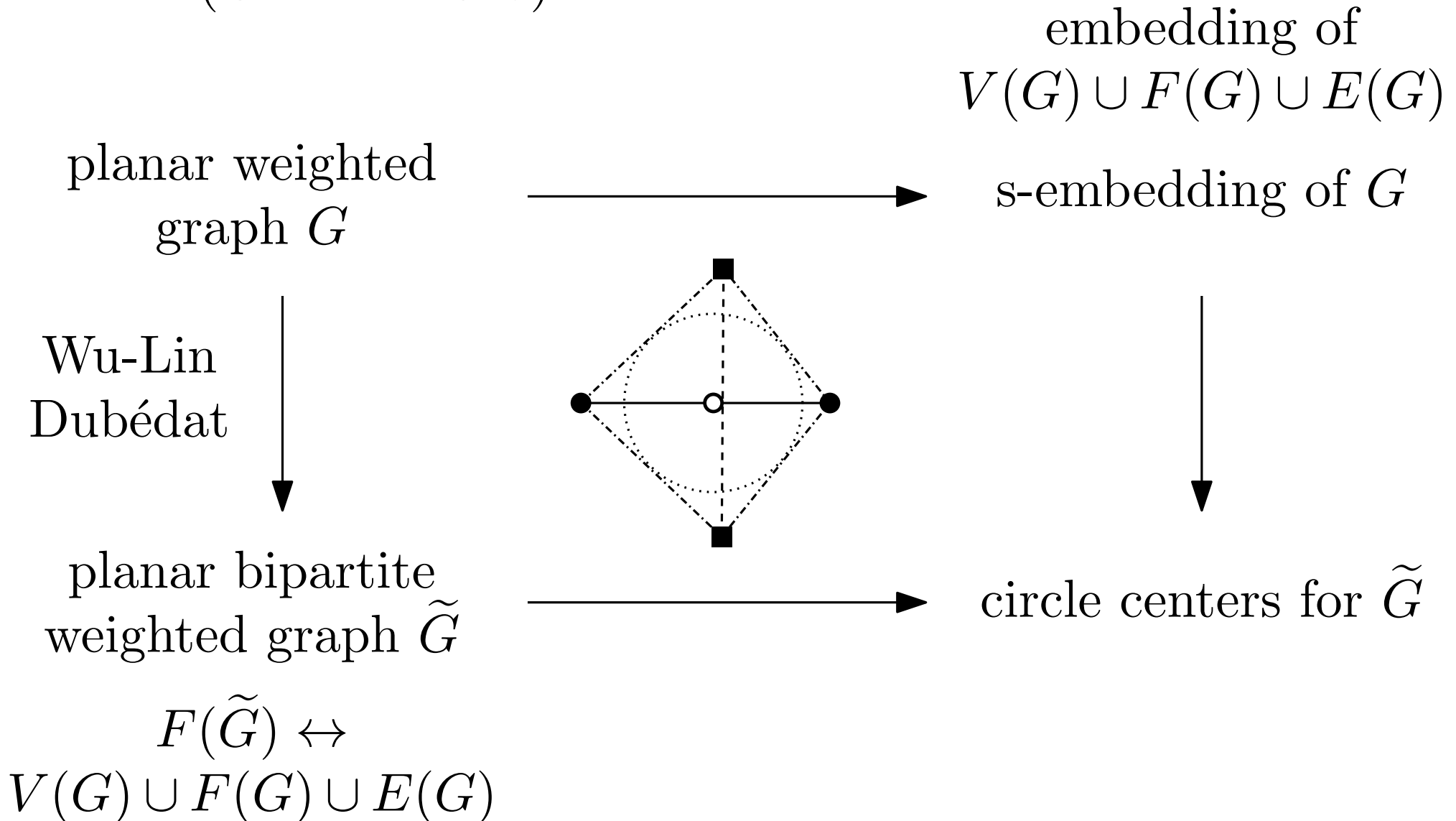
s-embeddings and circle centers

(Chelkak 2017)



s-embeddings and circle centers

(Chelkak 2017)



THANK YOU !