

Asymptotic freeness over the diagonal of random matrices

Camille Male

*joint work with B. Au, G. Cébron,
A. Dahlqvist and F. Gabriel*

Université de Bordeaux

JCB 2019

The Problem: Numerical computation of eigenvalues

We are given a $N \times N$ Hermitian random matrix of the form

$$H_N = P(X_{1,N}, \dots, X_{L,N}),$$

- P is a non commutative polynomial in the matrices and their adjoint.
- $X_{1,N}, \dots, X_{L,N}$ are independent random matrices.

Question: How to compute the spectrum distribution of H_N ? (large scale)
Numerical method *bypassing diagonalization & based on fixed-point*.

Assumptions: The size N is large.

Assuming **distributional symmetries** on the random matrices: invariance in law by conjugation by permutation matrices

In this talk: consider only the sum

$$H_N = X_N + Y_N$$

Organization of the talk

I: presentation of the result

We consider the cases of

- 1 a single **Wigner** matrix X_N ,
- 2 $X_N + Y_N$ when X_N Wigner matrix,
- 3 $X_N + Y_N$ when one matrix is **unitary invariant**,
- 4 $X_N + Y_N$ when one matrix is **permutation invariant**,

II: Notions of independence

Involves several concepts

- 1 **Freeness** (non commutative analogue of independence)
- 2 **Freeness with amalgamation** (analogue of conditional independence)
- 3 **Traffic independence** (unifies Independence & freeness)

1. Encoding the spectrum into a probability measure

The **empirical eigenvalues distribution** of H_N : $\mu_{H_N} = \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i} \right]$, where the λ_i 's are the eigenvalues of H_N .

Definition

A (real) **Wigner matrix** is a symmetric random matrix $X_N = \left(\frac{x_{i,j}}{\sqrt{N}} \right)_{i,j}$ such that the $x_{i,j}, i \leq j$ are independent and identically distributed.

Assume: $\text{Var}(x_{i,j}) = \mathbb{E}[(x_{i,j} - \mathbb{E}[x_{i,j}])^2] = 1$

Wigner theorem: $\mu_{H_N} \xrightarrow{N \rightarrow \infty} \mu$ **Semicircular Distribution**: $\forall f$

$$\int f d\mu_{X_N} := \mathbb{E} \left[\frac{1}{N} \sum_{i=1}^N f(\lambda_i) \right] \xrightarrow{N \rightarrow \infty} \int f d\mu = \int_{-2}^2 f(t) \frac{\sqrt{4-t^2}}{2\pi} dt.$$

The even moments $\int t^{2n} d\mu(t)$ are the **Catalan numbers** $\frac{1}{n+1} \binom{2n}{n}$.

2. Perturbation of a Wigner matrix

A tool: The **Stieltjes transform** of a prob. measure μ on \mathbb{R} : $\forall \lambda \in \mathbb{C}^+$

$$g_\mu(\lambda) := \int_{\mathbb{R}} \frac{d\mu(t)}{(\lambda - t)}, \quad g_{H_N}(\lambda) := g_{\mu_{H_N}}(\lambda) = \mathbb{E} \left[\frac{1}{N} \text{Tr}[(\lambda \mathbb{I}_N - H_N)^{-1}] \right].$$

- **Generating function of moments:** $g_\mu(\lambda) = \sum_{n \geq 0} \frac{\int_{\mathbb{R}} t^n d\mu(t)}{\lambda^{n+1}}.$
- **Inversion formula:** $d\mu(t) = \lim_{\epsilon \rightarrow 0^+} \Im m \frac{-g_\mu(t+i\epsilon)}{\pi}.$

Theorem (Pastur 72)

X_N : Wigner with $\text{Var}(x_{i,j}) = \sigma^2$,

Y_N : independent of X_N , such that $\mu_{Y_N} \xrightarrow{N \rightarrow \infty} \mu_Y.$

Then $\mu_{X_N + Y_N} \xrightarrow{N \rightarrow \infty} \mu_{x+y}$, determined by the fixed-point equation

$$g_{x+y}(\lambda) = g_Y(\lambda - \sigma^2 g_{x+y}(\lambda)), \quad \forall \lambda \in \mathbb{C}^+.$$

2. Numerical Method

Spectrum of $X_N + Y_N$:

Density of $x + y$ at t : $-\lim_{\epsilon} \Im m g_{x+y}(t + i\epsilon)/\pi$. [Stieltjes inversion]

\Rightarrow With $\lambda_t = t + i\epsilon$, construct an approximation of $g_{x+y}(\lambda_t)$ for different values of t and a small ϵ .

When X_N Wigner.

$$g_{x+y}(\lambda) = g_y(\lambda - \sigma^2 g_{x+y}(\lambda)) \quad [Pastur]$$

\Rightarrow Take $g^{(0)}$ arbitrary

Iterate $g^{(n)} = g_y(\lambda_t - \sigma^2 g^{(n-1)})$ until stationarity

Then $g^{(n)} \sim g_{x+y}(\lambda_t)$.

3. Definition of the \mathcal{R} -transform

X_N Wigner: $g_{x+y}(\lambda) = g_y(\lambda - \sigma^2 g_{x+y}(\lambda))$ [*Pastur*]

Remark: If X_N has variance σ^2 and $Y_N = 0$ then $g_y(\lambda) = \lambda^{-1}$ and

$$g_x(\lambda) = (\lambda - \sigma^2 g_x(\lambda))^{-1}, \quad \forall \lambda \in \mathbb{C}^+.$$

Another tool: For any y there is a unique \mathcal{R}_y such that

$$g_y(\lambda) = \left(\lambda - \mathcal{R}_y(g_y(\lambda)) \right)^{-1}, \quad \forall \lambda \in \mathbb{C}^+.$$

\mathcal{R}_y : the \mathcal{R} -transform of y .

y is a semicircular variable if and only if it is linear.

3. The free probabilist method

$$\mathcal{R}\text{-transform } \mathcal{R}_y \text{ of } y : g_y(\lambda) = \left(\lambda - \mathcal{R}_y(g_y(\lambda)) \right)^{-1}$$

We say that a random matrix X_N is **unitarily invariant** whenever

$$X_N \stackrel{\mathcal{L}aw}{=} UX_N U^* \text{ for any unitary matrix } U$$

Theorem (Voiculescu 86,91)

Let X_N and Y_N be independent random matrices, such that **one of the matrices is unitarily invariant** and $\mu_{X_N} \xrightarrow{N \rightarrow \infty} \nu_x$, $\mu_{Y_N} \xrightarrow{N \rightarrow \infty} \nu_y$. Then

$\mu_{X_N+Y_N} \xrightarrow{N \rightarrow \infty} \mu_{x+y}$ where

$$\mathcal{R}_{x+y} = \mathcal{R}_x + \mathcal{R}_y.$$

Voiculescu [86] says that x and y are free iff $\mathcal{R}_{x+y} = \mathcal{R}_x + \mathcal{R}_y$.

Voiculescu [91] says that X_N and Y_N converges to free variables x and y .

3. Numerical Method

When X_N and Y_N are asymptotically free.

\mathcal{R} -transform of $x + y$: $g_{x+y}(\lambda) = \left(\lambda - \mathcal{R}_{x+y}(g_{x+y}(\lambda)) \right)^{-1}$
 X_N, Y_N asymptotically free: $\mathcal{R}_{x+y} = \mathcal{R}_x + \mathcal{R}_y$ [Voiculescu]

Both \mathcal{R}_x and \mathcal{R}_y known: iterate $g^{(n)} = \left(\lambda_t - (\mathcal{R}_x + \mathcal{R}_y)(g^{(n)}) \right)^{-1}$.

Theorem (Voiculescu 93, Biane 98)

x and y are free iff $g_{x+y}(\lambda) = g_y \left(\lambda - \mathcal{R}_x(g_{x+y}(\lambda)) \right)$

Only \mathcal{R}_x known: iterate $g^{(n)} = g_y \left(\lambda_t - \mathcal{R}_x(g^{(n)}) \right)$.

None are known: use another fixed point for the **subordinator**

$\theta_{x,y}(\lambda) = \lambda - \mathcal{R}_x(g_{x+y}(\lambda))$ so that $g_{x+y}(\lambda) = g_y(\theta_{x,y}(\lambda))$.

4. \mathcal{R} -transform with amalgamation over the diagonal

We say that a random matrix X_N is **permutation invariant** whenever

$$X_N \stackrel{\mathcal{L}_{aw}}{=} UX_N U^*, \quad \forall U \text{ permutation matrix.}$$

Idea: to adapt, we replace

- the normalized trace $\frac{1}{N} \text{Tr}$ by the **diagonal operator** $A \mapsto \text{diag}(A)$,
- the scalars $\lambda \in \mathbb{C}$ by **diagonal matrices** $\lambda \in D_N(\mathbb{C})$.

Tool: **Stieltjes transform** \mathbf{g}_{X_N} with amalgamation over the diagonal is

$$\mathbf{g}_{X_N}(\lambda) = \text{diag}[(\lambda - X_N)^{-1}],$$

where λ is a diagonal matrix with $\lambda_{ii} \in \mathbb{C}^+$. The **\mathcal{R} -transform over the diagonal** is $\mathbf{R}_{X_N} : \mathcal{O}_N \subset D_N(\mathbb{C}) \rightarrow D_N(\mathbb{C})$ s.t.

$$\mathbf{g}_{X_N}(\lambda) = \left(\lambda - \mathbf{R}_{X_N}(\mathbf{g}_{X_N}(\lambda)) \right)^{-1}.$$

4. Consequence of the traffic approach

$$\mathbf{g}_{X_N}(\lambda) = \text{diag}[(\lambda - X_N)^{-1}] = \left(\lambda - \mathbf{R}_{X_N}(\mathbf{g}_{X_N}(\lambda)) \right)^{-1}.$$

Theorem (Au, Cébron, Dahlqvist, Gabriel, M. 18)

Let X_N and Y_N be independent random matrices, such that *one of the matrices is permutation invariant*. Assume the convergence of X_N and Y_N in *the sense of traffics* (generalization of the convergence of μ_{X_N}, μ_{Y_N}). Then $\mathbf{g}_{X_N+Y_N} \xrightarrow[N \rightarrow \infty]{} \mathbf{g}_{x+y}$ whose \mathcal{R} -transform over the diagonal is

$$\mathbf{R}_{x+y} = \mathbf{R}_x + \mathbf{R}_y.$$

Voiculescu [95]: x and y are free over the diagonal iff $\mathbf{R}_{x+y} = \mathbf{R}_x + \mathbf{R}_y$.

M. [11]: X_N and Y_N converges to *traffic-independent* variables x and y .

This theorem: *Traffic Independence implies Freeness with Amalgamation*

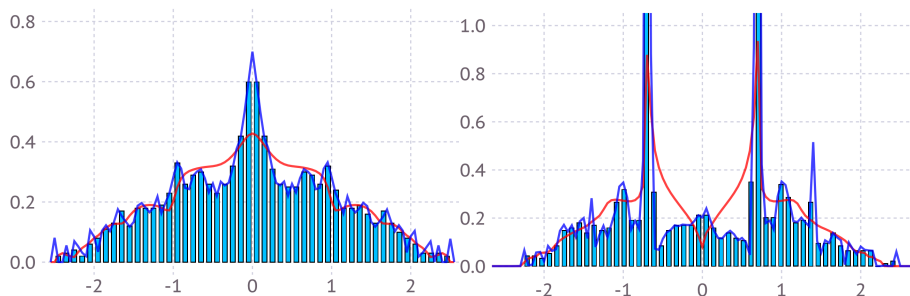
4. Numerical Method [Belinshi, Mai, Speicher 17]

When X_N and Y_N are asymptotically free over the diagonal.

Both \mathbf{R}_x and \mathbf{R}_y known: Take $\mathbf{g}^{(0)}$ an arbitrary diagonal matrix.

Iterate $\mathbf{g}^{(n)} = \left(\lambda_t \times \mathbb{1}_N - (\mathbf{R}_x + \mathbf{R}_y)(\mathbf{g}^{(n)}) \right)^{-1}$

For large enough n , then $\frac{1}{N} \text{Tr}[\mathbf{g}^{(n)}]$ is an approximation of $g_{x+y}(t + i\varepsilon)$.



X_N matrix of Erdos-Renyi graph ($p_N = \frac{1}{N}$), $Y_N = VDV^{-1}$ for D diagonal and V FFT matrix & Unitary BM (Method: via $\mathbf{g}_{x+y} = \mathbf{g}_y(\theta_{x,y})$)

Link between Independence and Freeness

Recall: The **Fourier transform** $f_x(t) = \mathbb{E}[e^{itx}] = \sum_{n \geq 0} \frac{(it)^n \mathbb{E}[X^n]}{n!}$.

x and y are independent iff $\ln f_{x+y} = \ln f_x + \ln f_y$

Expansion $F_x(t) = \sum_{n \geq 0} \frac{(it)^n c_n(x)}{n!}$: $c_n(x)$ is the n -th cumulant of x .

Claimed previously: The **\mathcal{R} -transform** characterizes freeness

x and y are free iff $R_{x+y} = R_x + R_y$

Expansion $\mathcal{R}_x(t) = \sum_{n \geq 0} k_n(x) t^n$: $k_n(x)$ is the n -th free cumulant of x .

The following **relations characterize** $(c_n(x))_{n \geq 1}$ and $(k_n(x))_{n \geq 1}$: $\forall n \geq 1$

$$\mathbb{E}[x^n] = \sum_{\pi \in \mathcal{P}(n)} \prod_{B \in \pi} c_{|B|}(x),$$

$$\mathbb{E}[x^n] = \sum_{\pi \in \mathcal{NCP}(n)} \prod_{B \in \pi} k_{|B|}(x)$$

Moment Formulation of Freeness

Recall: x_1, \dots, x_L are independent iff $\mathbb{E}[x_1^{n_1} \dots x_L^{n_L}] = \prod_{\ell=1}^L \mathbb{E}[x_\ell^{n_\ell}]$.

Freeness: x_1, \dots, x_L are free if and only if: $\forall n \geq 2, \forall \ell_1 \neq \ell_2 \neq \dots \neq \ell_n$ in $\{1, \dots, L\}$ and $\forall P_1, \dots, P_n$ polynomials

$$\mathbb{E} \left[\left(P_1(x_{\ell_1}) - \mathbb{E}[P_1(x_{\ell_1})] \right) \dots \left(P_n(x_{\ell_n}) - \mathbb{E}[P_n(x_{\ell_n})] \right) \right] = 0.$$

In words: An alternated product of centered elements is centered

Random matrices $X_{N,1}, \dots, X_{N,L}$ are asymptotically free whenever for any

$$\varepsilon_N = \left(P_1(X_{N,\ell_1}) - \mathbb{E} \frac{1}{N} \text{Tr} P_1(X_{N,\ell_1}) \right) \dots \left(P_n(X_{N,\ell_n}) - \mathbb{E} \frac{1}{N} \text{Tr} P_n(X_{N,\ell_n}) \right),$$

alternated product, we have $\mathbb{E} \frac{1}{N} \text{Tr}[\varepsilon_N] \xrightarrow{N \rightarrow \infty} 0$.

Freeness with Amalgamation

x and y are independent conditionally on z : $\forall f, g$,

$$\mathbb{E}[f(x)g(y)|z] = \mathbb{E}[f(x)|z] \times \mathbb{E}[g(y)|z].$$

Freeness with amalgamation over (\sim conditionally on) diagonal matrices. The map $\Delta : A \mapsto \text{diag}(A)$ is a conditional expectation in the sense that

$$\Delta(D_1 A D_2) = D_1 \Delta(A) D_2, \quad \forall D_1, D_2 \in D_N(\mathbb{C})$$

Random matrices $X_{N,1}, \dots, X_{N,L}$ are asymptotically free over the diagonal whenever

$$\Delta \left[\left(P_1(X_{N,\ell_1}) - \Delta[P_1(X_{N,\ell_1})] \right) \dots \left(P_n(X_{N,\ell_n}) - \Delta[P_n(X_{N,\ell_n})] \right) \right]$$

tends to zero. (Actually the P_i 's are non commutative polynomials whose coefficients are diagonal matrices...)

Traffic Probability

Eig. distr.: $\int t^n d\mu_A(t) = \frac{1}{N} \sum_{i=1}^N \lambda_i^n = \frac{1}{N} \text{Tr}(A^n).$

N. C. distr.: $\frac{1}{N} \text{Tr}(A_1 \dots A_n) = \frac{1}{N} \sum_{i_1, \dots, i_n=1}^N A_1(i_1, i_2) \dots A_n(i_n, i_1).$

Define: A test graph T in matrices A_1, \dots, A_K :

- a finite connected oriented graph (V, E) ,
- a labeling $\gamma : E \rightarrow \{1, \dots, K\}$ of edges by matrices.

For N by N matrices A_1, \dots, A_K , the trace of test graphs

$$\frac{1}{N} \text{Tr}[T(A_1, \dots, A_K)] = \frac{1}{N} \sum_{\phi: V \rightarrow \{1, \dots, N\}} \prod_{e=(v,w) \in E} A_{\gamma(e)}(\phi(w), \phi(v)).$$

Injective trace of test graphs

Define: the **injective version** of the trace of test graphs

$$\frac{1}{N} \text{Tr}^0[\mathcal{T}(\mathbf{A}_N)] = \frac{1}{N} \sum_{\substack{\phi: V \rightarrow \{1, \dots, N\} \\ \text{injective}}} \prod_{e=(v,w) \in E} A_{\gamma(e)}(\phi(w), \phi(v)).$$

Then the following relation is similar to the moment-cumulant relations

$$\frac{1}{N} \text{Tr}[\mathcal{T}] = \sum_{\pi \in \mathcal{P}(V)} \frac{1}{N} \text{Tr}^0[\mathcal{T}^\pi],$$

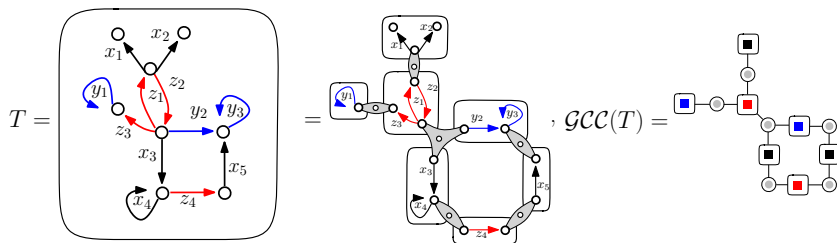
where \mathcal{T}^π is the quotient graph where vertices in a same block are identified.

Traffic-independence

The matrices $A_{N,1}, \dots, A_{N,L}$ are **asymptotically traffic independent** \Rightarrow

$$\mathbb{E} \frac{1}{N} \text{Tr}^0 [T(A_{N,1}, \dots, A_{N,L})] \xrightarrow{N \rightarrow \infty} 0$$

if the **Graph of Colored Components** $\mathcal{GCC}(T)$ is not a tree.



Traffic-independence \Rightarrow freeness over the diagonal

We prove that

$$\varepsilon_N = \Delta \left[\left(P_1(X_{N,\ell_1}) - \Delta[P_1(X_{N,\ell_1})] \right) \cdots \left(P_n(X_{N,\ell_n}) - \Delta[P_n(X_{N,\ell_n})] \right) \right]$$

tends to zero in the sense that

$$\mathbb{E} \frac{1}{N} \text{Tr}(\varepsilon_N \varepsilon_N^*) = \sum_{\substack{T \text{ satisfying} \\ \text{some condition}}} \alpha_T \mathbb{E} \frac{1}{N} \text{Tr}^0[T(X_{N,1}, \dots, X_{N,L})] + o(1),$$

where the condition is incompatible with the fact that $\mathcal{GCC}(T)$ is a tree.

Next challenging question: compute the \mathcal{R} -transform for interesting models ! Find \mathbf{R}_N such that for any diagonal matrix λ

$$\text{diag}[(\lambda - X_N)^{-1}] \sim \left(\lambda - \mathbf{R}_N(\text{diag}[(\lambda - X_N)^{-1}]) \right)^{-1}.$$

Thank you for your attention !

