

On higher-dimensional three-distance theorems

P. Arnoux, V. Berthé, D. H. Kim,
W. Steiner and J. Thuswaldner

IRIF-CNRS-Paris-France

<http://www.irif.fr/~berthe>

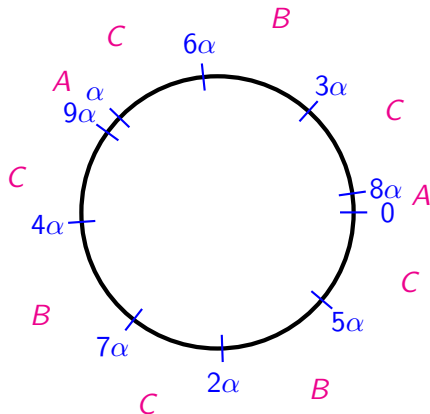


JCB 2019

Three-distance theorem

Let α be given and let us place the points $0, \alpha, 2\alpha, \dots, N\alpha$ on the unit circle

Theorem The points $0, \alpha, 2\alpha, \dots, N\alpha$ partition the unit circle into intervals having at most **three lengths**, one being the sum of the other two



Three-distance theorem

Let α be given and let us place the points $0, \alpha, 2\alpha, \dots, N\alpha$ on the unit circle

It is also called the Steinhaus theorem, the three length, the three gap, or else, the three step theorem

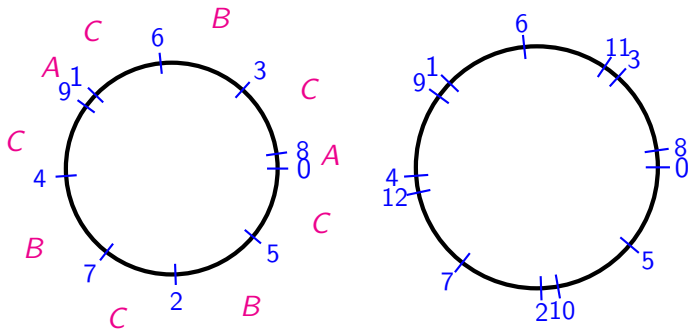
It was initially conjectured by Steinhaus, first proved by V.T. Sós'58, Surányi'58, Slater'64, Świerczkowski'59, Halton'65

There exist various types of proofs: arithmetical, combinatorial, dynamical, geometry of the space of $2d$ lattices, etc.

The lengths correspond to factors of a given length for Sturmian sequences

Three and two-distance cases

We assume α irrational



Two-distance case and Farey points

N - Farey point $0 \leq p/q \leq 1$, $\gcd(p, q) = 1$, $1 \leq q \leq N$

Let p_1/q_1 and p_2/q_2 be two successive N -Farey points s.t.

$$p_1/q_1 < \alpha < p_2/q_2$$

Case $N = q_1 + q_2$ There are q_2 intervals of length $q_1\alpha - p_1$ and q_1 intervals of length $p_2 - q_2\alpha$

Remark The closest points to 0 are $q_1\alpha - p_1$ and $p_2 - q_2\alpha$
One checks that

$$q_2||q_1\alpha|| + q_1||q_2\alpha|| = q_2(q_1\alpha - p_1) + q_1(p_2 - q_2\alpha) = 1$$

How to describe the succession of lengths?

Two-distance case and Farey points

N -Farey point $0 \leq p/q \leq 1$, $\gcd(p, q) = 1$, $1 \leq q \leq N$

Let p_1/q_1 and p_2/q_2 be two successive N -Farey points s.t.

$$p_1/q_1 < \alpha < p_2/q_2$$

Case $N = q_1 + q_2$ There are q_2 intervals of length $q_1\alpha - p_1$ and q_1 intervals of length $p_2 - q_2\alpha$

Remark The closest points to 0 are $q_1\alpha - p_1$ and $p_2 - q_2\alpha$
One checks that

$$q_2||q_1\alpha|| + q_1||q_2\alpha|| = q_2(q_1\alpha - p_1) + q_1(p_2 - q_2\alpha) = 1$$

How to describe the succession of lengths?

Let us consider the map that sends a point $k\alpha$ to its neighbor point when placing $0, \alpha, \dots, (N-1)\alpha$

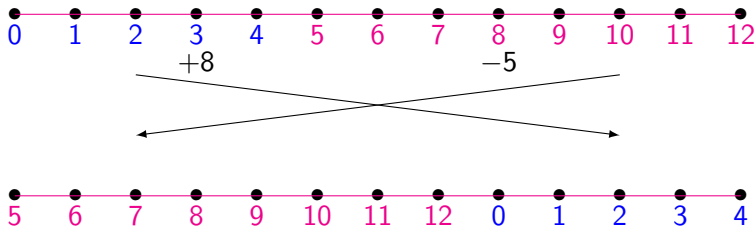
A dynamical description for the neighbor map

- We place the points $0, \alpha, \dots, (N-1)\alpha$
- We consider the map that sends a point $k\alpha$ to its neighbor
- The neighbor map can be described as a permutation over $\{0, \dots, N-1\}$



- We describe the permutation as the action of an exchange of two intervals.

$$q_1 = 8, \quad q_2 = 5, \quad N = 8 + 5 = 13, \quad 3/8 < \alpha < 2/5$$



Neighbor map and two-interval exchange

We express the permutation of the set $\{0, 1, \dots, N\}$ given by the neighbor map as an exchange of two intervals obtained by translating $[0, q_2)$ by $+q_1$, and $[q_1, q_1 + q_2)$ by $-q_2$.

This is a permutation since $\gcd(q_1, q_2) = 1$.

We take the orbit of 0 under the action of this exchange of two intervals.

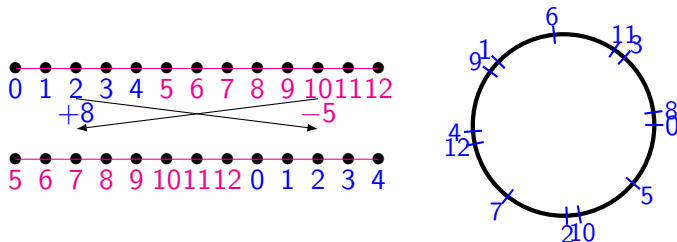
Neighbor map and two-interval exchange

We express the **permutation** of the set $\{0, 1, \dots, N\}$ given by the **neighbor map** as an **exchange of two intervals** obtained by translating $[0, q_2)$ by $+q_1$, and $[q_1, q_1 + q_2)$ by $-q_2$.

This is a permutation since $\gcd(q_1, q_2) = 1$.

We take the **orbit** of 0 under the action of this exchange of two intervals.

$$q_1 = 8, \quad q_2 = 5, \quad N = 8 + 5 = 13, \quad 3/8 < \alpha < 2/5$$



$$0 \mapsto 8 \mapsto 3 \mapsto 11 \mapsto 6 \mapsto 1 \mapsto 9 \mapsto 4 \mapsto 12 \mapsto 7 \mapsto 2 \mapsto 10 \mapsto 5 \mapsto 0$$

Two-distance case and Farey points

N - Farey point $0 \leq p/q \leq 1$, $\gcd(p, q) = 1$, $1 \leq q \leq N$

$$p_1/q_1 < \alpha < p_2/q_2$$

Case $N = q_1 + q_2 - 2$ There are q_2 intervals of length $q_1\alpha - p_1$ and q_1 intervals of length $p_2 - q_2\alpha$.

Proof

- $q_2||q_1\alpha|| + q_1||q_2\alpha|| = q_2(q_1\alpha - p_1) + q_1(p_2 - q_2\alpha) = 1$.
- The closest points to 0 are $q_1\alpha - p_1$ and $p_2 - q_2\alpha$.
- One gets a permutation of the set $\{0, 1, \dots, N\}$ by translating $[0, q_2)$ by $+q_1$, and $[q_1, q_2)$ by $-q_2$.

↪ The neighbor map that sends a point $k\alpha$ to its neighbor point when placing $0, \alpha, \dots, (N-1)\alpha$. 😊

Three-distance case

$$p_1/q_1 < \alpha < p_2/q_2$$

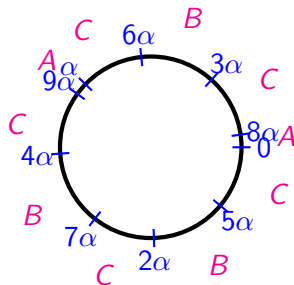
One has

- $N - q_1$ intervals of length $\|q_1\alpha\|$
- $N - q_2$ intervals of length $\|q_2\alpha\|$
- $q_1 + q_2 - N$ intervals of length $\|q_1\alpha\| + \|q_2\alpha\|$



$$N = 10, 3/8 < \alpha < 2/5$$

$$+8, +3, -5$$



Three-distance case

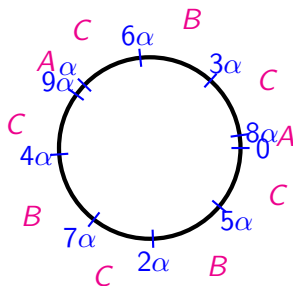
One has

- $N - q_1$ intervals of length $\|q_1\alpha\|$
- $N - q_2$ intervals of length $\|q_2\alpha\|$
- $q_1 + q_2 - N$ intervals of length $\|q_1\alpha\| + \|q_2\alpha\|$



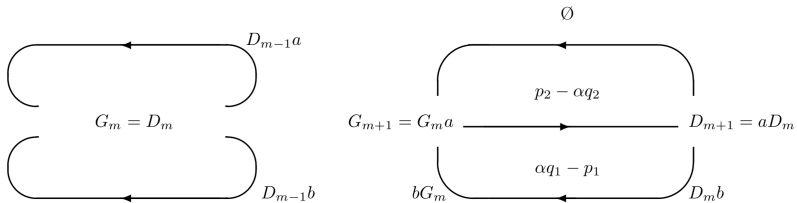
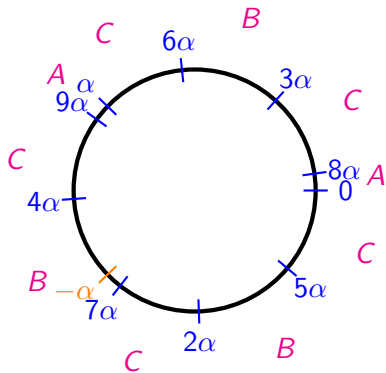
$$N = 10, 3/8 < \alpha < 2/5$$

$$+8, +3, -5$$

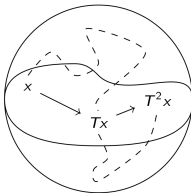
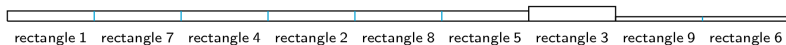
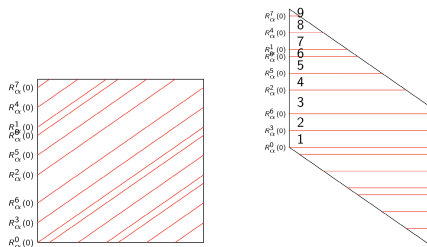


The coding over $\{A, B, C\}$ is obtained as the orbit of 0 under an
 cexchange of three intervals

$$\begin{array}{cccccccccc} 0 & \mapsto & 8 & \mapsto & 3 & \mapsto & 6 & \mapsto & 1 & \mapsto & 4 & \mapsto & 7 & \mapsto & 2 & \mapsto & 5 & \mapsto & 0 \\ & & A & & C & & B & & C & & B & & B & & C & & B & & C \end{array}$$



Toward interval-exchanges [D. Taha]



Three-distance and three-gap theorems

Three-length theorem [Sós'58] The points $0, \alpha, 2\alpha, \dots, N\alpha$ partition the unit circle into intervals having at most **three lengths**, one being the sum of the other two.

The sequence of lengths is a coding of a **three-interval exchange** on the set $\{0, \dots, N\}$.

Three-distance and three-gap theorems

Three-length theorem [Sós'58] The points $0, \alpha, 2\alpha, \dots, N\alpha$ partition the unit circle into intervals having at most **three lengths**, one being the sum of the other two.

The sequence of lengths is a coding of a **three-interval exchange** on the set $\{0, \dots, N\}$.

Three-gap theorem [Slater] Let I be an interval of \mathbb{R}/\mathbb{Z} . The **gaps** between the successive integers j such that $\{\alpha j\} \in I$ take at most **three values**, one being the sum of the other two.

The sequence of gaps is a coding of a **three-interval exchange** on I .

Higher-dimensional versions

- Linear forms in d variables on the one-dimensional torus \mathbb{T}
- Translations on the d -dimensional torus \mathbb{T}^d

Points $m\alpha + n\beta$ on \mathbb{T}

We consider the two-dimensional version of the three distance theorem obtained by placing on the unit circle the points

$$m\alpha + n\beta \quad \text{for } m, n \text{ in a finite set } \mathcal{S}_N$$

\mathcal{S}_N can be a square, a rectangle, or a more general shape

$$\text{Card } \mathcal{S}_N \rightarrow \infty \text{ with } N$$

We consider the two-dimensional version of the three distance theorem obtained by placing on the unit circle the points

$$m\alpha + n\beta \quad \text{for } m, n \text{ in a finite set } \mathcal{S}_N$$

\mathcal{S}_N can be a square, a rectangle, or a more general shape

$$\text{Card } \mathcal{S}_N \rightarrow \infty \text{ with } N$$

[Bleher et al., Erdős, Liang, Chung and Graham, Chevallier, Geelen and Simpson, Haynes and Marklof, etc.]

$$n\alpha \bmod 1 \Leftrightarrow \text{Sturmian words} \Leftrightarrow \text{Discrete lines} \rightsquigarrow \text{Quasicrystals, C\&P}$$

$$n\alpha + m\beta \bmod 1 \Leftrightarrow \text{2D Sturmian words} \Leftrightarrow \text{Discrete planes} \rightsquigarrow \text{QC, C\&P}$$

Square shapes

We place the points $m\alpha + n\beta$, for $0 \leq n, m \leq N$ on the unit circle.
We consider lengths between consecutive points.

Question Are there finitely many lengths?

Theorem [Holzman] There are finitely many lengths if $1, \alpha, \beta$ are \mathbb{Q} -linearly dependent.

Square shapes

We place the points $m\alpha + n\beta$, for $0 \leq n, m \leq N$ on the unit circle.
We consider lengths between consecutive points.

We assume $1, \alpha, \beta$ linearly independent.

Question Are there finitely many lengths?

Theorem [Boshernitzan and Dyson] Badly approximable vectors (α, β) produce a finite number of distances.

$$||m, n||^2 ||m\alpha + n\beta|| \geq C \text{ for all } (m, n) \neq 0$$

Theorem [Haynes-Marklof'17] The number of lengths is generically unbounded.

Proof An approach via homogeneous dynamics based on the ergodic properties of the diagonal action on the space of lattices

Effective realizations in the square case

Consider the points $n\alpha + m\beta \in \mathbb{T}$, for $0 \leq n, m < N$, and the cardinality of distances $\Delta_N(\alpha, \beta)$ between consecutive points.

[Berthé-Kim'18] We provide effective constructions for the following existence results.

- There exist pairs (α, β) , with $1, \alpha, \beta$ rationally independent and (α, β) **not badly approximable** producing at most

7 lengths.

- There exist pairs (α, β) , with $1, \alpha, \beta$ rationally independent, such that

$$\limsup_{N \rightarrow \infty} \# \Delta_N(\alpha, \beta) = \infty.$$

Effective realizations in the square case

Consider the points $n\alpha + m\beta \in \mathbb{T}$, for $0 \leq n, m < N$, and the cardinality of distances $\Delta_N(\alpha, \beta)$ between consecutive points.

[Berthé-Kim'18] We provide effective constructions for the following existence results.

- There exist pairs (α, β) , with $1, \alpha, \beta$ rationally independent and (α, β) **not badly approximable** producing at most

7 lengths.

- There exist pairs (α, β) , with $1, \alpha, \beta$ rationally independent, such that

$$\limsup_{N \rightarrow \infty} \# \Delta_N(\alpha, \beta) = \infty.$$

Proof We use **regular continued fraction expansions** for α and β with intertwined sequences of partial quotients. We combine several 'rectangular' levels of points of the form $n\alpha + m\beta$, for $0 \leq n < N$ and $0 \leq m < M$, where N or M is a denominator of a principal convergent of α or β .

A dynamical approach: the neighbor map as an exchange of pieces

Notation $|n|_q \equiv n \pmod{q}$, and $0 \leq |n|_q < q$.

Let

$$E_{q_i, q'_j}(\alpha, \beta) = \{n\alpha + m\beta \in \mathbb{T} : 0 \leq n < q_i, 0 \leq m < q'_j\},$$

with

$$q'_j = b' q_i + 1 \quad b' > 0.$$

A **permutation** acting on $E_{q_i, q'_j}(\alpha, \beta)$:

$$\phi_{q_i, q'_j}: n\alpha + m\beta \mapsto |n - (-1)^i q_{i-1}|_{q_i} \alpha + |m + (-1)^j b' q'_{j-1}|_{q'_j} \beta.$$

We assume

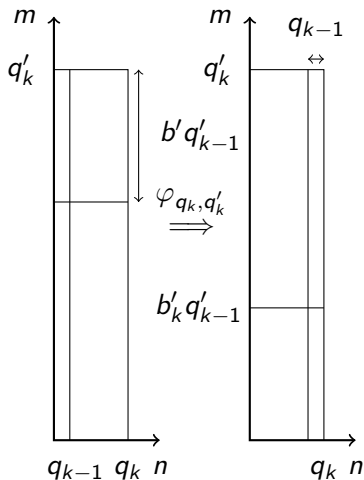
$$\|q'_j \beta\| < \|q_{i-1} \alpha\| - b' \|q'_{j-1} \beta\|.$$

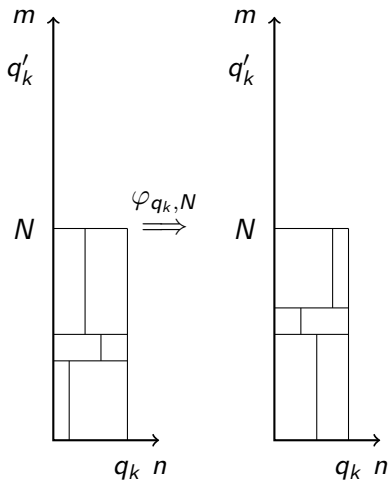
Lemma The neighbor point of $n\alpha + m\beta$ is given by

$$\phi_{q_i, q'_j}(n\alpha + m\beta).$$

This provides a rectangular case where there are only **4 possible lengths**.

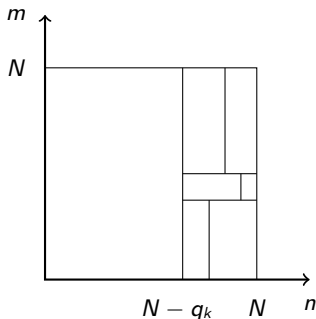
An analog of the two-distance case





From 4 to 7 lengths $q_k < N \leq q'_k$

- There are 4 lengths for (q_k, q'_k) .
- We reduce the set of points to (q_k, N) . We make an **induction** and go from 4 lengths to 6 lengths.
- We insert points to reach (N, N) . We perform an **'exduction'** and get 7 lengths.



Square case

We place on the unit circle the points $m\alpha + n\beta$, for $0 \leq n, m < N$.
We consider lengths between consecutive points.

Question Is it possible to find (α, β) , with $1, \alpha, \beta$ rationally independent, with at most 6 lengths for the square case?

d -dimensional parallelepipedic case

Let $d \geq 3$.

Let $\alpha_1, \dots, \alpha_d \in \mathbb{T}$ and $2 \leq n_1 \leq \dots \leq n_d$ be integers.

We place the points

$$\sum_{i=1}^d k_i \alpha_i \quad \text{for } 0 \leq k_i < n_i, \ i = 1, \dots, d.$$

Theorem [Chevallier] The number of lengths is at most

$$\prod_{i=1}^{d-1} n_i + 3 \prod_{i=1}^{d-2} n_i + 1.$$

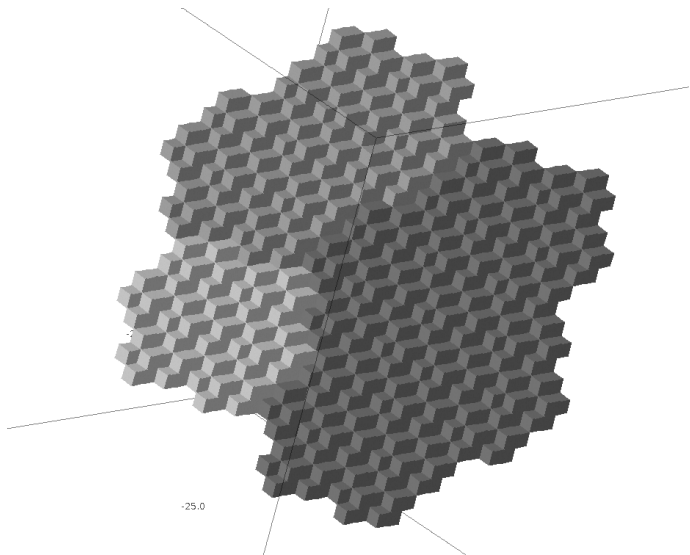
$d = 2 \rightsquigarrow$ The upper bound is $N + 3$ for $\{m\alpha + n\beta, 0 \leq n, m < N\}$

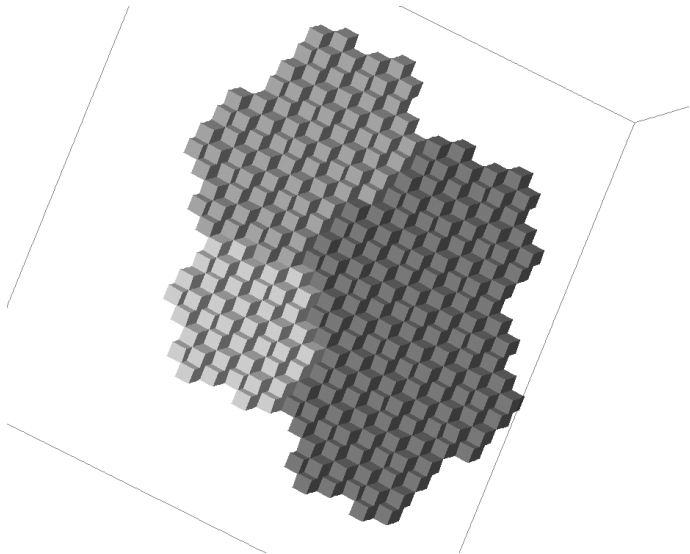
Our strategy for Brun shapes

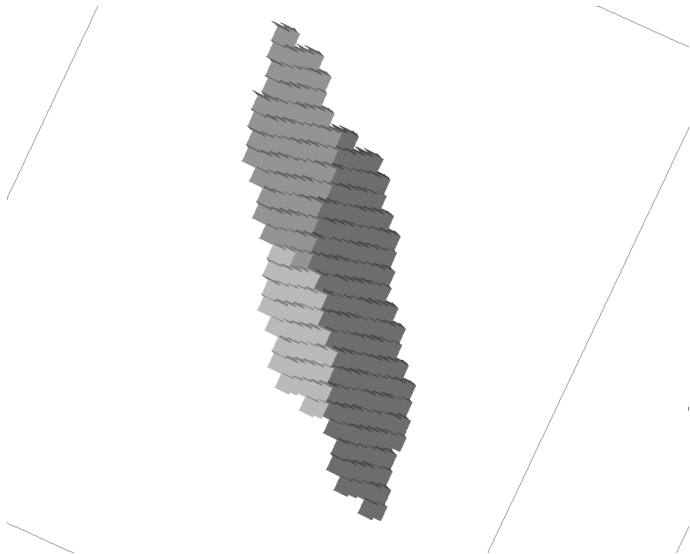
- We consider the points $n\alpha + m\beta$, for $n, m \in \mathcal{S}_N$, for a family of shapes $(\mathcal{S}_N)_N$.
- Theorem [Arnoux-B.-Kim-Steiner-Thuswaldner] Brun algorithm produces shapes $(\mathcal{S}_N)_N$ for which there are 3 lengths. These shapes are no more square or rectangular ones (they are approximations of Rauzy fractals).
- The cardinality of points in \mathcal{S}_N tends to infinity and the shapes \mathcal{S}_N contain a ball of radius tending to infinity with N .

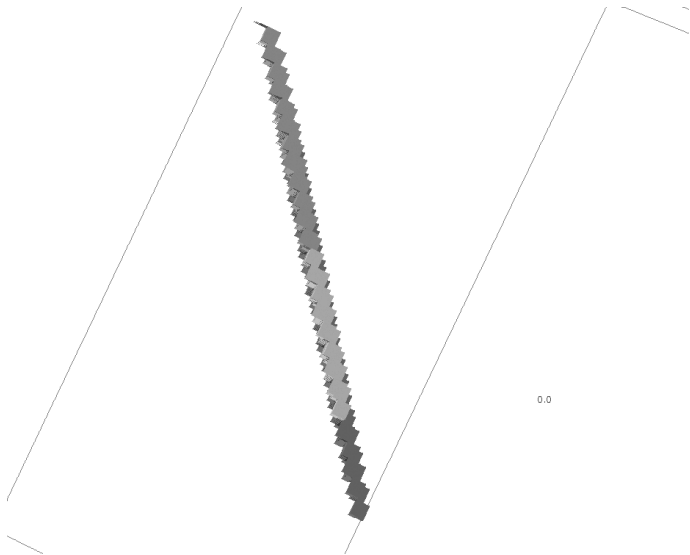
Our strategy for Brun shapes

- We consider the points $n\alpha + m\beta$, for $n, m \in \mathcal{S}_N$, for a family of shapes $(\mathcal{S}_N)_N$.
- Theorem [Arnoux-B.-Kim-Steiner-Thuswaldner] Brun algorithm produces shapes $(\mathcal{S}_N)_N$ for which there are 3 lengths. These shapes are no more square or rectangular ones (they are approximations of Rauzy fractals).
- The cardinality of points in \mathcal{S}_N tends to infinity and the shapes \mathcal{S}_N contain a ball of radius tending to infinity with N .
- Patterns are obtained by iterating a geometric, higher-dimensional substitution $E_1^*(\sigma)$ and then, taking a projection from \mathbb{Z}^3 to \mathbb{Z}^2 .
- The resulting three-dimensional patterns obtained by iterating $E_1^*(\sigma)$ lie within a discrete plane, i.e., a discretized version of a Euclidean plane with normal vector $(1, \alpha, \beta)$.

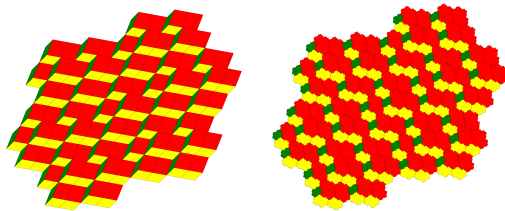








Translations on \mathbb{T}^2



Translations on \mathbb{T}^2

[Vijay] 11 mutual distances Eleven Euclidean distances are enough

[Chevallier] A triangulation of \mathbb{R}^2 with at most 10 triangles whose set of vertices is $\{0, \alpha, \dots, N\alpha\} + \mathbb{Z}^2$, invariant under \mathbb{Z}^2 -translations, and the maximum of the diameters tends to 0 with N . Based on Ferguson-Forcade and Brun algorithm.

[Arnoux-B-Kim-Steiner-Thuswaldner] For a.e. (α, β) , a tessellation of \mathbb{R}^2 with at most 6 shapes whose set of vertices is $\{0, \alpha, \dots, N\alpha\} + \mathbb{Z}^2$, invariant under \mathbb{Z}^2 -translations, and the maximum of the diameters tends to 0 with N .

- Based on Brun algorithm.
- Pieces are given by subpieces of the Rauzy fractal.

And now

- Beyond dimension 2 (strong convergence)
- Beyond fractal shapes to recover boxes

